REPORT 6

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PROJECT: LFST

SUBJECT: Homology Method

Approximation to the Inverse of a Matrix

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In my Report No. 4 (Calculating Method ...), I used an approximation method for the inverse of a slightly changed matrix, when the inverse of the unchanged matrix is given. Call A the matrix and A^{-1} its (known) inverse; add a matrix a to it which has elements much smaller than those of A, and call

$$B = A + a . \tag{1}$$

The approximation used in Report 4, then, is

$$B^{-1} \approx A^{-1} - A^{-1}a\,A^{-1}. \tag{2}$$

R. Jennings suggested to investigate this approximation somewhat closer. Thus, in this present report I show that equation (2) is the first term of an infinite series; I give a condition for the convergence of this series, and another condition for the "usefulness" of equation (2).

Statement: (with E = unit matrix)

$$B^{-1} = A^{-1}\left\{E + \sum_{\nu=1}^{\infty}(-aA^{-1})^{\nu}\right\}, \text{ if the series converges.} \tag{3}$$

Proof:

Multiply (3) from right side with $A+a$

$$E = E + A^{-1}a + A^{-1}\sum_{\nu=1}^{\infty}(-aA^{-1})^{\nu}A + A^{-1}\sum_{\nu=1}^{\infty}(-aA^{-1})^{\nu}a \tag{4}$$

The term $A^{-1}a$ cancels the first term of the first series, which then starts with $\nu=2$. We then rewrite this series, using $(\ )^{\nu} = (\ )^{\nu-1}(\ )$, and $A^{-1}A=E$, and obtain

$$E = E - A^{-1}\sum_{\nu=1}^{\infty}(-aA^{-1})^{\nu} + A^{-1}\sum_{\nu=1}^{\infty}(-aA^{-1})^{\nu}, \tag{5}$$

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Both series in (5) cancel each other, and we are left with an identity, E=E, which proves that statement (3) is correct.

Convergence:

I have not found a neccessary condition for the convergence of (3). But if we define a matrix α by

$$\alpha = aA^{-1} \tag{6}$$

and call its largest element

$$\alpha_m = \max|\alpha_{ij}| \quad \text{for all} \begin{cases} i = 1 \ldots n \\ j = 1 \ldots n \end{cases} \tag{7}$$

then the following condition obviously is sufficient for convergence:

$$\alpha_m < 1/n \; . \tag{8}$$

The Error of (2):

Equation (2) which is used in Report 4 contains only the first term of the series of equation (3). We call this approximation

$$B_o^{-1} = A^{-1} - A^{-1}a\,A^{-1}, \tag{9}$$

and its error

$$\Delta B^{-1} = B^{-1} - B_o^{-1} \; . \tag{10}$$

As the relative error, $\mathcal{E}$, we will define ΔB^{-1} relative to B^{-1}, which means ΔB^{-1} multiplied with B:

$$\mathcal{E} = B\,\Delta B^{-1} = B\,(B^{-1} - A^{-1} + A^{-1}a\,A^{-1}) \; . \tag{11}$$

A different way of defining the same quantity $\mathcal{E}$ is, as can be shown,

$$\mathcal{E} = E - B\,B_o^{-1}. \tag{12}$$

From (11) we obtain

$$\mathcal{E} = (aA^{-1})^2 = \alpha^2 \tag{13}$$

In order to make (2) a good approximation, it is sufficient to demand

$$\alpha_m \ll 1/n \; . \tag{14}$$