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The Design of Large Steerable Antennas

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Abstract

The design of large, steerable, single reflectors is investigated in full generality in order to find the basic principles involved and the most economical solutions. (Let λ = shortest wavelength, D = antenna diameter, W = antenna weight; ρ = material density, S = maximum stress, E = modulus of elasticity, C = thermal expansion coefficient; p_s = survival surface pressure, p_o = observation wind pressure, Δ = temperature differences; γ = dimensionless constants). There are three natural limits for tiltable antennas. First, the thermal deflection limit, $\lambda \geq \gamma C \Delta D = 2.4$ cm (D/100 m), for $D \leq 45$ m. Second, the gravitational deflection limit, $\lambda \geq \gamma D^2 \rho / E = 5.3$ cm $(D/100\text{ m})^2$. Third, the stress limit, $D \leq \gamma S/\rho = 620$ m. Let each antenna be a point in a D,λ-plane. The part of the plane permitted by the three limits can be divided into four regions according to what defines the antenna weight. First, the gravitational deflection region (W governed by ρ/E); second, the wind deflection region (governed by p_o/E); third, the survival region (governed by p_s/S); fourth, the minimum structure region (stable self-support). Formulae are derived for the regional boundaries, and for W(D,λ) within each region. Some aspects of economy are considered. There is a most economical λ for any D. Radomes give no advantage for $D \geq 50$ m. The azimuth drive should be on normal railroad equipment for $D \geq 100$ m. An economical antenna with D = 150 m and λ = 20 cm should cost about four million dollars.

There are three means for passing the gravitational limit. First, avoiding the deflections by not moving in elevation angle (fixed-elevation transit telescope). Second, fighting the deflections with motors (Sugar Grove). Third, guiding the deflections such that they transform a paraboloid into another paraboloid (homology deformation). It is proved that homology deformation has solutions, and an explicit solution is given for two dimensions.

Introduction

There is a growing need in radio astronomy for building very large antennas for 10 or 20 cm wavelength, and another need for observing as short a wavelength as possible with antennas of moderate size. Since both these demands soon run into structural as well as financial limitations, a general survey of the whole problem seems indicated.

Before the astronomer can ask the engineer to design a telescope of diameter D and for wavelength λ, he ought to know which D,λ-combinations are possible at all, which are at the limit of his funds, and which combinations might be considered the most economical ones. On the other side, it might help the engineer to know that antennas with certain D,λ-combinations are completely defined by survival conditions and nothing else, others by gravitational deflections and nothing else, and so on. Furthermore, it might give the engineer a helpful challenge if he knows with what weight a near-to-ideal design is supposed to meet the specifications.

The present investigation is held as general as possible in order to make it applicable to telescopes of any diameter. It asks for the natural limits of antennas, for the most economical type of design, and for useful approximation formulae giving the weight as function of diameter and wavelength. With the help of these formulae, one then can ask for the most economical D,λ-combinations. It turns out, for example, that a tiltable antenna of conventional design, with a given diameter of 100 m, cannot be built for wavelengths below 5.3 cm; but this limit is reached only with infinite weight. Between 5.3 cm and 7.3 cm, the weight is entirely defined by keeping the gravitational deflections down, whereas the structure has more strength than needed for survival. Above 7.3 cm, the weight is entirely defined by survival, whereas the structure is more rigid than needed for observation. It follows that 7.3 cm is the most economical wavelength, any other wavelength giving a waste of either strength or rigidity.

Finally, we ask whether the gravitational limit can be passed by designing a structure which deforms as a whole, but still gives always some exact paraboloid of revolution.

I. Natural Limits

1. Gravity and Elasticity

Even with gravity as the only force (no load or wind) one could not build indefinitely high structures. A limit is reached when the weight of the structure gives a pressure at its bottom equal to the maximum allowed stress of the material used. We call S = maximum allowed stress of material, ρ = density of material, h_o = maximum height of structure, and γ_1 = geometrical shape factor ($\gamma_1 = 1$ for standing pillar or hanging rope). The maximum height of a structure, no matter what its purpose, is then

$$h_o = \gamma_1 S/\rho \; . \qquad (1)$$

A second limit applies to any structure which, while being tilted, shall maintain a given accuracy, defined in our case by the shortest wavelength to be used. Even a standing pillar gets compressed under its own weight, the lower parts more than the upper ones. We call E = modulus of elasticity, h = height of structure, Δh = change of height under its own weight, and γ_2 = geometrical shape factor ($\gamma_2 = 1/2$ for standing pillar or hanging rope). Integrating the compression from bottom to top yields

$$\Delta h = \gamma_2 h^2 \rho/E. \qquad (2)$$

The deformations increase with the square of the size. For antennas of given wavelength and increasing size, this second limit is reached much earlier than the first one. Both limits are easily understood, since the weight goes with the third power of the size, but the strength only with the second power. Both limits are not ultimate but can be surpassed with certain tricks. As to the first limit, one could start at the bottom with a large cross section and taper it toward the top, but this structure cannot be tilted. Passing the second limit will be discussed later.

Both limits depend on the combination of only three material constants: maximum stress, density and elasticity. Table 1 gives four examples, together with the coefficient of linear thermal expansion, and with a rough estimate of price including erection. The largest structure can be made from aluminum, about two miles high. All four materials give the same order of magnitude for this maximum height, which could be increased only by tapering, and we understand why even mountains cannot be higher than a few miles. Steel, aluminum and wood are about equal with respect to deflections under their own weight, while concrete is worse by a factor of three. Thermal deflections are worse for aluminum and best for wood (but wood has too much deformation with humidity). Since the second limit will be reached first, there is no need to go to the more expensive aluminum, and we arrive at normal steel as the best material. The largest block of steel could be a mile high, but a block only 400 feet high is already compressed under its own weight by 3 mm.

Table 1
Some Material Constants

	density ρ	maximum stress S	elasticity E	maximum height S/ρ	gravitat. deflection ρ/E	thermal expans. C_{th}	price p
	g/cm^3	kg/cm^2	$10^3 kg/cm^2$	km	$cm/(100\ m)^2$	$10^{-6}/°C$	\$/kg
steel	7.8	1400	2100	1.79	0.37	12	1.5
aluminum	2.7	910	700	3.37	0.38	24	4.5
wood	0.7	130	120	1.86	0.58	3.5	0.5
concrete	2.4	200	200	0.83	1.20	8	0.08

2. The Octahedron

What should be the over-all shape of a large structure for minimizing the deflections from its own weight and from wind, if the structure is to be held at a few points and to be turned in all directions? If a structure is cantilevered with length a and width b, a lateral force will give a deflection proportional to $(a/b)^2$. Since this is a rapid increase with decreasing width, and since any external force can become lateral for a turning structure, we get the requirement:

Equal diameters in any direction. (3)

Small deviations from this rule do not matter much, but for greater deviations the deflections increase with the square of the diameter ratio.

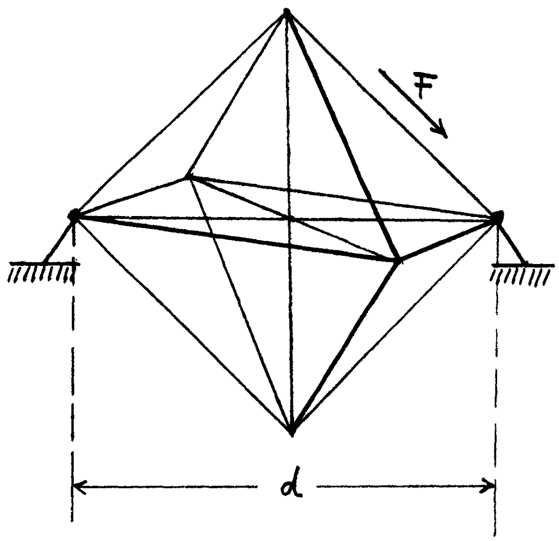


Figure 1. Octahedron with diagonals

The simplest structure we can think of, approaching requirement (3), which can be held at two points and turned from a third one, and which provides a flat surface through its center with a point normal to it for the focus, is the octahedron. Furthermore, its deflections are easily calculated. Thus, we adopt the octahedron as a near-to-ideal model for the basic structure of an antenna. Compared with usual designs, it gives more depth, and it includes the feed support as part of the basic structure.

If all members shown in Figure 1 have equal cross section Q, the weight of the whole structure gives rise to the force $F = 2.88\ dQ\rho$ in one of the outer members, and from equation (1) we find the diameter of the largest possible octahedron from steel as

$$d_o = \frac{1}{2.88}\ S/\rho \quad = \quad 620 \text{ meter.} \qquad (4)$$

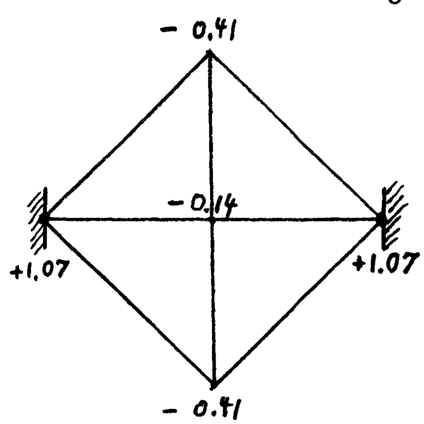


Figure 2. Deflections (in cm) in the horizontal plane of an octahedron of 100 meter diameter, as seen from the top point.

The numerical value of γ_2 depends on where we measure the deflection and with respect to which point. With respect to the focal point at the top, we get the values shown in Figure 2 for a diameter of 100 meter. The rms deflection in the horizontal plane, as seen from the top, is 0.34 cm; but since we have neglected any lateral sagging of the members, we multiply by a safety factor of 1.5 in order to be on the safe side, and we obtain for the rms deflection

$$\text{rms } (\Delta h) \quad = \quad 0.51 \text{ cm } (d/100 \text{ m})^2 \ . \qquad (5)$$

We call D the diameter of the antenna surface, and we call C = D/d the cantilevering factor. The latter should not be much larger than unity because of requirement (3) and should be chosen such that strong torques around the basic structure are avoided. The best value of C will depend on the actual design, and after some estimates we adopt

$$C = D/d = 1.25 . \tag{6}$$

Calling λ_{gr} the shortest possible wavelength with respect to gravitational deflections, we demand rms(Δh) = $\lambda_{gr}/16$. The gravitational limit of a telescope then is

$$\lambda_{gr} = 5.3 \text{ cm } (D/100 \text{ m})^2. \tag{7}$$

3. Active and Passive Weight

Since the next point is a crucial one for large antennas, and since no suitable terminology seems to exist, I shall introduce my own, calling:

Active weight = W_{ac} = weight of those parts of the structure which oppose deflections to the same extent that they add weight. In our case, only the main chords of the octahedron members are active. If we have nothing but active weight, the gravitational deflections are completely independent of the cross section of the members, and thus, for a given diameter, the deflections of the structure are given by (5) and do not depend on the weight.

Passive weight = W_{ps} = weight of everything else, such as braces and struts in the octahedron members, the antenna surface and the structure beneath it, and any part of the drive mechanism being fixed to the octahedron. Passive weight adds to the total weight without opposing the deflections it causes.

Total weight = $W_{ps} + W_{ac}$.

Passivity factor = K = (total weight)/(active weight) = $1 + W_{ps}/W_{ac}$.

With the help of this terminology we obtain a very quick estimate for the deflections of any given structure; because if any passive weight is present and is distributed about evenly, we simply have to multiply both equations (5) and (7) with K. The connection between antenna diameter and shortest wavelength with respect to gravitation then is

$$\lambda = 5.3 \text{ cm } K\ (D/100 \text{ m})^2. \qquad (8)$$

Since passive weight always is present, at least in the antenna surface and its holding structure, K can approach unity only if the active weight approaches infinity. Practically, K will be between, say, 1.2 and 1.8. Table 2 gives some examples for the utmost limit, K = 1.

Table 2

Shortest wavelength λ for an antenna of diameter D. λ_{gr} with respect to gravitational deflections if tilted by 90°; see (7); λ_{th} with respect to thermal deflections in sunshine, see (10).

D	λ_{gr}	λ_{th}
m	cm	cm
25	0.33	0.60
50	1.32	1.2
75	2.98	1.8
100	5.3	2.4
150	11.9	3.6
200	21.2	4.8
300	47.7	7.2

4. Thermal Deflections

If an outer member of the octahedron is ΔT degrees (centigrade) warmer than the rest of the structure, its length will increase by $C_{th}\ \Delta T\ d/\sqrt{2}$. A rough estimate gives for the rms deflection of the antenna surface, with C_{th} from Table 1 for steel,

$$\text{rms}(\Delta h) = 0.03 \text{ cm } \Delta T\ D/100 \text{ m}. \qquad (9)$$

A large antenna will most probably stand in the open; ΔT then is given by sunlight and shadow but is independent of the antenna diameter. The thermal deflections then increase with D and will dominate in small antennas, while the gravitational deflections, increasing with D^2, dominate in large antennas.

ΔT is negligible during nights and cloudy days, and a good reflecting paint keeps it rather low even in sunshine. Measurements at Green Bank on sunny summer days give an average difference of 8°C between a painted metal surface in full sunshine, and some structure in the shadow. Since this is the most extreme case (and since the surface itself should "float" on the structure), the average difference in the main structure will be considerably less; adopting ΔT = 5°C should be safe enough. Calling λ_{th} = rms(Δh)/16 the shortest wavelength to be used, with respect to thermal deflections alone, we have from (9)

$$\lambda_{th} = 2.4 \text{ cm } (D/100 \text{ m}). \tag{10}$$

Comparing (10) with (8), we find:

$$\text{Gravitational deflections} \geqslant \text{thermal deflections, if } D \geqslant 45 \text{ m } /K. \tag{11}$$

Around an antenna in a radome, a vertical temperature gradient builds up, and ΔT will increase with D. The thermal deflection then is proportional to D^2, just as the gravitational one, and the question of which one is larger depends only on the gradient, but not on the diameter. An estimate shows that both deflections are equal if the gradient is about 15°C/100 m. A cooling system must keep it below this limit.

In summary we have three natural limits for the size of steerable antennas if the shortest wavelength is given. First, antennas below 45 m diameter are limited by thermal deflections according to (10). Second, the diameter of larger antennas is limited by gravitational deflections according to (8). Third, the largest tiltable structure has about 600 m diameter according to (4), independent of wavelength.

For antennas between 45 and 600 m diameter, the second limit applies as given in Table 2. This limit is not final. It can be pushed a little by adjusting the surface at an elevation angle of 45°, for example. But it cannot be surpassed considerably without applying special tricks (to be discussed in Section IV).

II. Some Formulae for Estimates

After having derived the limit of an antenna, we next want to know its weight. There are four items that can define the weight: first, gravitational deflections; second, wind deflections; third, survival conditions; fourth, the minimum stable structure. We need general formulae in order to learn which of these items is the defining one, and in order to estimate the resulting weight. One could use a model design which can be scaled up and down, but if we do not ask for more accuracy than, say, ± 30%, these formulae can be derived on general grounds without a special model.

1. Weight of Members

Each member needs a certain minimum diameter in order to prevent buckling and sagging. There are two opposing criteria for the design of a member: the passivity factor should be close to unity in order to keep the deflections down, but the total weight should be low to keep the costs down. Starting with the first extreme, we could avoid any passive weight for the octahedron if we build each of its members from a single steel pipe of proper diameter and wall thickness. But then an octahedron of 400 feet diameter would weigh over 3000 tons, much more than we want to pay for, and much more than is needed against wind loading. This means we must split up the members into three or four main chords connected by braces; for very long members and small forces even a multiple splitting is necessary, where the main chords again are split up into three thinner chords. Going again to the extreme, we arrive at a certain minimum structure just for stable self-support, no matter what its purpose. A rough estimate shows that if we do not care at all about deflections and wind forces, an octahedron of 400 feet diameter would have a minimum structure of about 130 tons (but would deform under its own weight by about 5 cm).

This calls for a careful compromise between the two opposing criteria. Since the same type of problem must arise in communication towers, we have taken the data quoted for 10 towers with a non-guyed length between 40 and 140 feet, and with longitudinal forces between 7 and 120 tons; in addition, some examples with double splitting were calculated for a length up to 300 feet and forces up to 1500 tons. The result can roughly be approximated by the formula (W = weight in tons, F = force in tons, ℓ = length in 100 meter):

$$W = 0.06\ F\ell + 8\ell^2 \text{ (for normal steel).} \tag{12}$$

Struts perpendicular to the main chords are passive, diagonals at 45° are half passive and half active. As an approximation we will assume that the first term plus 1/3 of the second term is active, while 2/3 of the second term is passive weight.

2. Weight of Surface

For wavelengths above 5 cm, we do not need a closed surface and adopt a simple galvanized wire mesh (transmission 15 - 20 db down). The weight then shows only a very slow variation with wavelength which we neglect. Some available examples of wire mesh show a weight of 0.3 lb/ft^2 which we should multiply by 2 or 3 to allow for some light aluminum frames. To be on the safe side, we adopt 1.2 lb/ft^2 = 5.8 kg/m^2 for the weight of the surface; this allows for a closed aluminum skin of 2.15 mm thickness if $\lambda < 5$ cm. A circular surface of diameter D then has the weight

$$W_{sf} = 46 \text{ tons } (D/100\text{ m})^2. \tag{13}$$

3. Survival Condition, and Forces during Observation

We adopt a maximum wind speed of 110 mph, giving a pressure of 50 lb/ft^2 = 242 kg/m^2. In stow position (looking at zenith) the antenna projects perpendicular to the wind only a fraction of its surface, for which we adopt 1/4. Some experiments with wire mesh show that the wind force varies roughly as $1/\lambda$ and, compared to a closed surface, is down by a factor 5.5 for $\lambda = 20$ cm, if the surface is perpendicular to the wind. Since this is not the case in stow position, and since we have neglected wind forces on the structure, we multiply by a safety factor of 2.9 and adopt for the maximal wind force 500 tons (10 cm/λ) $(D/100\text{ m})^2$ for $\lambda \geq 5$ cm, and

1000 tons $(D/100\ m)^2$ for $\lambda \leq 5$ cm. This would allow for a solid layer of ice 6.4 cm thick, or for a snow load of 13 lb/ft^2, if the antenna is built for $\lambda = 10$ cm. This seems to be enough from our present experience at Green Bank, where large accumulations of snow always can be avoided by carefully tilting the dishes, and where ice can be melted off with a jet engine from the ground. In order to be a little more safe, we increase the maximum snow load to 20 lb/ft^2 for $\lambda = 10$ cm (or 40 lb/ft^2 for $\lambda \leq 5$ cm) and obtain as the survival force in stow position

$$F_{sv} = \begin{cases} 760 \text{ tons } (10\ cm/\lambda)\ (D/100\ m)^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 1400 \text{ tons } (D/100\ m)^2 & \text{for } \lambda \leq 5 \text{ cm.} \end{cases} \qquad (14)$$

If built for this specification defined by snow loads, the antenna will withstand a wind velocity of 136 mph in stow position, and of 85 mph in observing positions.

If we observe only in winds up to 25 mph (going to stow position for higher winds), we derive, with a safety factor of 1.5 to include the forces on the structure, the maximum horizontal force in observing positions, F_{oh}, as

$$F_{oh} = \begin{cases} 75 \text{ tons } (10\ cm/\lambda)\ (D/100\ m)^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 150 \text{ tons } (D/100\ m)^2 & \text{for } \lambda \leq 5 \text{ cm.} \end{cases} \qquad (15)$$

The maximum uplifting force, F_{ou},(at elevation angle 45°) is about $2^{-3/2}$ of the maximum horizontal force, and we adopt

$$F = \begin{cases} 25 \text{ tons } (10\ cm/\lambda)\ (D/100\ m)^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 50 \text{ tons } (D/100\ m)^2 & \text{for } \lambda \geq 5 \text{ cm.} \end{cases} \qquad (16)$$

4. Wind Deflections

Let a structure of length ℓ be built for survival under force F_{sv}. If it is used under force F_{oh}, its length will change by

$$\Delta \ell = \frac{S}{E} \frac{F_{oh}}{F_{sv}} \ell. \qquad (17)$$

Here we have another important combination of material constants, S/E, which for normal steel is 6.7×10^{-4}. We demand $\Delta \ell = \lambda/16$ for the shortest wavelength, and we assume $\ell = D/\sqrt{2}$ as the average distance a force has to travel from the surface to a main bearing. If an antenna is built for survival, the shortest wavelength as defined by wind deflections then is

$$\lambda = 7.5 \text{ cm } (D/100 \text{ m}). \tag{18}$$

This limit can be surpassed by multiplying the active weight of the structure by a factor α_w, and the shortest wavelength is then

$$\lambda = \frac{1}{\alpha_w} 7.5 \text{ cm } (D/100 \text{ m}). \tag{19}$$

A comparison of equations (18) and (8) shows that wind deflections are more important for small antennas than for large ones; they can be neglected for diameters above about 100 m. A comparison of (18) and (10) shows that wind deflections always become important before the thermal limit is reached.

5. Framework between Surface and Octahedron

A three-dimensional framework connecting the surface to a main support can be designed in many ways, but it will always connect many (N) structural surface points to few (2) main bearings. We imagine the surface as being the base of a quadratic pyramid whose top represents the holding point or bearing. We divide this pyramid into layers by horizontal planes; the first plane at 1/2 the full height, the second plane at 1/4, the third at 1/8, and so on. At the top we have 1 structural point, in the first plane we assume 4 points, in the second plane 16 points, and so on. If plane j is the surface, we have $N = 4^j$ surface points. In the layer between plane i and i + 1 we need $n_i = 2 \times 4^i$ main members (i = 1, 2, ... j-1), and we assume a maximum force along each member of $F_{sv}\sqrt{2}/n_i$. From (12) we obtain the weight of each member, and we multiply by two to include horizontal members and bracing diagonals. We neglect the four members in the first layer as being part of the octahedron. In this way the weight of the whole framework turns out to be (for $\lambda \geqslant 5$ cm)

$$W_{fr} = 46 \text{ tons } (10 \text{ cm}/\lambda)\ (D/100 \text{ m})^3 + 16.6 \text{ tons } (j-1)\ (D/100 \text{ m})^2. \quad (20)$$

How many (j) planes do we need? We include the structure of surface panels (if any) in our estimate, and we demand that the distance between neighboring surface points, $\ell_o = D/2^j$, can be covered by a straight line without deviating at its center more than $\lambda/16$ from the ideal surface, which means $\ell_o{}^2 = D\lambda/2$. Both equations for ℓ_o give together

$$j = 5.48\ \{1 + 0.303 \log (\frac{10 \text{ cm}}{\lambda} \frac{D}{100 \text{ m}})\}. \quad (21)$$

The logarithmic term can be neglected for a wide range of D and λ, and we obtain for the weight of framework plus panel structure, as defined by survival,

$$W_{fr} = \begin{cases} 46 \text{ tons } (10 \text{ cm}/\lambda)\ (D/100 \text{ m})^3 + 75 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 92 \text{ tons } (D/100 \text{ m})^3 + 75 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \leq 5 \text{ cm}. \end{cases} \quad (22)$$

6. Octahedron and Total Weight for Survival

The forces acting on the octahedron are the survival force, the weights of framework and surface, and the octahedron's own weight; in order to be safe, we add all contributions directly. We multiply by $\sqrt{2}$ since the octahedron legs are tilted, we divide by four since the forces are distributed over four legs, and we obtain the force along one leg. Formula (12) then gives the weight of one leg, which we multiply by 12 (diagonals already being represented by the framework). We solve the resulting equation for the weight of the octahedron, W_{os}, as defined by survival conditions, and as a good approximation for the range $o \leq D \leq 300$ m we obtain

$$W_{os} = \begin{cases} 170 \text{ tons } (10 \text{ cm}/\lambda)\ (D/100 \text{ m})^3 + 39 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 340 \text{ tons } (D/100 \text{ m})^3 + 39 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \leq 5 \text{ cm}. \end{cases} \quad (23)$$

Including the surface from equation (13), the framework from (22) and the octahedron from (23), we obtain for the total weight of the moving structure, as defined by survival

$$W_{ms} = \begin{cases} 216 \text{ tons } (10 \text{ cm}/\lambda)\ (D/100 \text{ m})^3 + 160 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 432 \text{ tons } (D/100 \text{ m})^3 + 160 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \leq 5 \text{ cm}. \end{cases} \quad (24)$$

The passive part includes the surface plus 2/3 of the D^2-terms of octahedron and framework, the rest being active:

$$W_{ac} = \begin{cases} 216 \text{ tons } (10 \text{ cm}/\lambda)\,(D/100 \text{ m})^3 + 38 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \geqslant 5 \text{ cm} \\ 432 \text{ tons } (D/100 \text{ m})^3 + 38 \text{ tons } (D/100 \text{ m})^2 & \text{for } \lambda \leqslant 5 \text{ cm} \end{cases} \quad (25)$$

$$W_{ps} = 122 \text{ tons } (D/100 \text{ m})^2 \, . \quad (26)$$

and the passivity factor is derived as

$$K = \begin{cases} 1 + \dfrac{\lambda/10 \text{ cm}}{1.77\, D/100 \text{ m} + 0.31\, \lambda/10 \text{ cm}} & \text{for } \lambda \geqslant 5 \text{ cm} \\ 1 + \dfrac{1}{3.54\, D/100 \text{ m} + 0.31} & \text{for } \lambda \leqslant 5 \text{ cm.} \end{cases} \quad (27)$$

7. Increased Rigidity

In equation (24) the weight of the antenna is defined entirely by survival conditions. This is all we need for longer wavelengths, but for shorter ones we must increase the active weight for reducing the deflections caused by gravitation and wind.

First, the gravitational deflections. Given D and λ, we check whether K obtained from (27) is smaller than or equal to K needed for (8); if it is, the gravitational deflections of the survival structure are small enough, and the total weight is given by (24). If not, we must multiply the active weight of (25) by a factor α_{gr} so that by multiplying the denominator of (27) by α_{gr} we make K from (27) equal to K from (8), which gives

$$\alpha_{gr} = \frac{\lambda_{gr}}{\lambda - \lambda_{gr}} \; \frac{W_{ps}}{W_{ac}} \quad (28)$$

where λ_{gr} is the gravitational limit of (7), and W_{ps} and W_{ac} are given by (26) and (25). The total weight, as defined by gravitational deflections, then is

$$W_{mg} = \frac{\lambda}{\lambda - \lambda_{gr}} \; 122 \text{ tons } (D/100 \text{ m})^2 \, . \quad (29)$$

Second, the wind deflections. Given D and λ, we check whether α_w obtained from

(19) is smaller than or equal to one. If it is, the wind deflections of the survival structure are small enough. If not, we multiply the active weight of (25) by α_w, add the passive weight of (26), and obtain the total weight of the moving structure, as defined by wind deflections (W_{mw} measured in tons, D in 100 m, and λ in 10 cm):

$$W_{mw} = \begin{cases} 162\ D^4/\lambda^2 + 29\ D^3/\lambda + 122\ D^2 & \text{for } \lambda \geq 5 \text{ cm} \\ 324\ D^4/\lambda + 29\ D^3/\lambda + 122\ D^2 & \text{for } \lambda \leq 5 \text{ cm.} \end{cases} \tag{30}$$

8. Regions of Different Weight Definitions

We have now derived the weight of an antenna, as defined by survival in equation (24), as defined by gravitational deflections in (29), and by wind deflections in (30). Each of these equations holds within a certain region of a D,λ-diagram (see Figure 3), and we now ask for the boundaries between the different regions.

The boundary between the wind deflection region and the survival region was already given in (18) as λ = 7.5 cm (D/100 m). The boundary between the survival region and the gravitational deflection region we obtain by letting $\alpha_{gr} = 1$ in (28) and solving the resulting quadratic equation for λ. The boundary between the wind deflection region and the gravitational deflection region we obtain by letting α_{gr} from (28) equal α_w from (19) and solving for λ. Finally, we observe that only the D^3-term in (24) is defined by survival, while the D^2-term is the weight of the minimum structure (for $\lambda \to \infty$); as the boundary between survival region and the minimum structure region we define (arbitrarily) the value of λ where the D^3-term is 2/3 of the D^2-term.

The results are shown in Figure 3, together with the three natural limits derived in Section I, and some values are given in Table 3. For example, an antenna of 50 m diameter (164 feet) cannot be built for wavelengths below 1.32 cm (gravitational limit); within the region 1.32 cm $\leq \lambda \leq$ 1.59 cm, the weight of the antenna is defined by keeping the gravitational deflections down; within the region

1.59 cm $\leq \lambda \leq$ 3.75 cm, the weight is defined by fighting the wind deflections; from 3.75 cm to 10 cm, the weight is defined by survival conditions, and above 10 cm the minimum structure dominates.

Table 3
Characteristic points for antennas of diameter D.
λ = wavelength, W = weight of moving structure.

limits:

gr = gravitational limit ($W \to \infty$)

th = thermal limit (ΔT = 5°C)

mn = minimum structure ($\lambda \to \infty$)

boundaries:

gw = gravitational deflections and wind defl.

ws = wind deflections and survival

gs = gravitational defl. and survival

D	λ_{gr}	λ_{th}	λ_{gw}	W_{gw}	λ_{ws}	W_{ws}	λ_{gs}	W_{gs}	W_{mn}
m	cm	cm	cm	tons	cm	tons	cm	tons	tons
25	0.33	0.60	0.39	52	1.88	16.8			10
50	1.32	1.20	1.59	181	3.75	94			40
75	2.95	1.80	3.63	383	5.63	252			90
100	5.30	2.40	7.30	460	7.30	460	7.30	460	160
150	11.9	3.60					18.9	740	360
200	21.2	4.8					38.5	1090	640
300	47.7	7.2					105	2020	1440

III. Economy

1. The Most Economical Wavelength

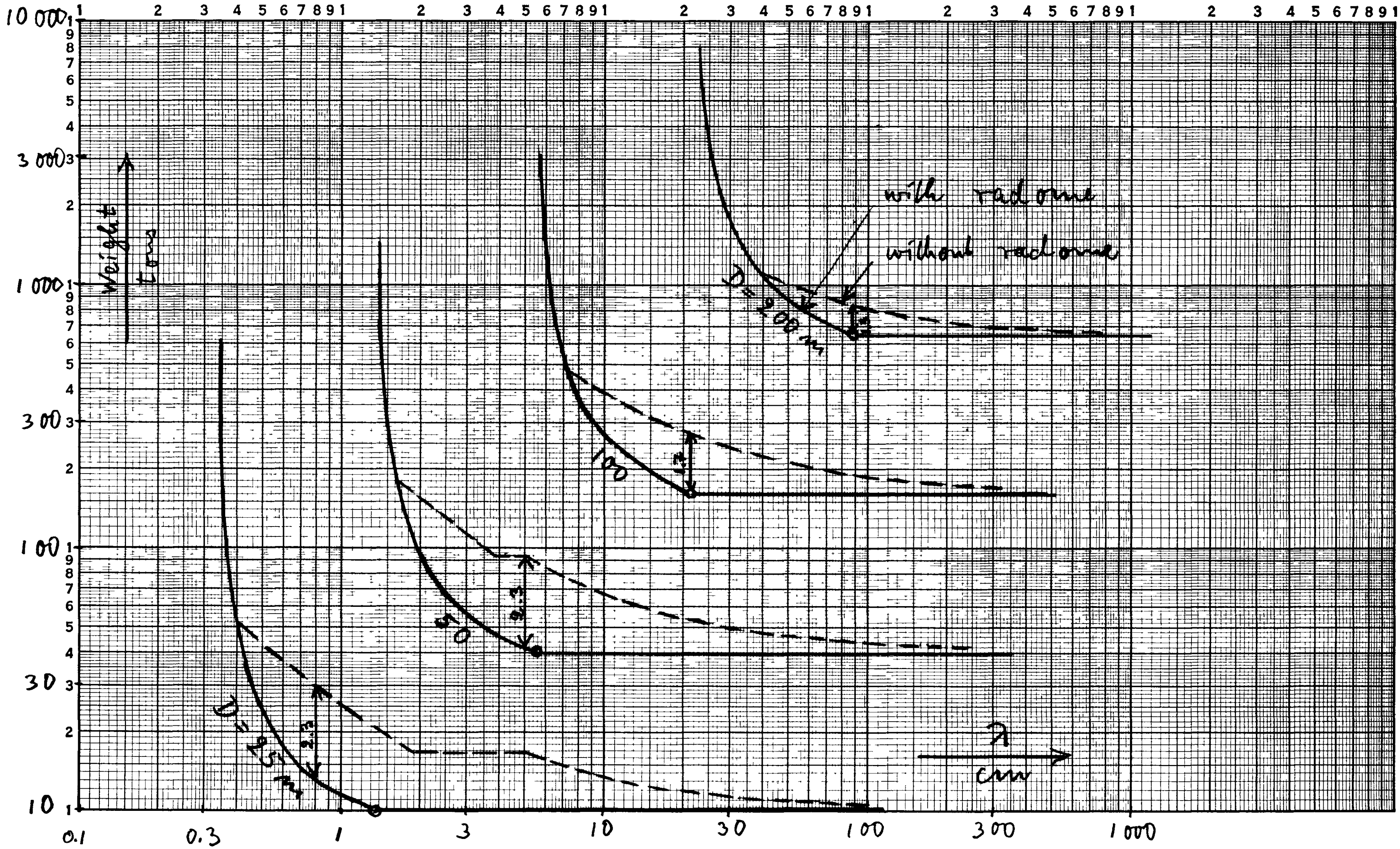
In Figure 4 we give the weight of an antenna as function of its diameter and of the shortest wavelength to be used. The weight of an actual antenna will depend on its special design, but any type of design will show natural limits and characteristic boundaries qualitatively similar to those of Figure 4. The quantitative values of Figure 4 belong to a "near-to-ideal" design and depend, first, on using an octahedron as the basic structure (more depth than usual, feed supports are part of main structure); second, on taking simple wire mesh with low wind resistance (instead of expanded metal); third, on the validity of equation (12) for long members, which actually means: many trials, even for details, until the best solution is found. Under these conditions, Figure 4 is supposed to be a realistic estimate.

As to the choice of D and λ, an economical antenna should be close to the boundary of the gravitational deflection region, because of the steep increase of the weight in this region. This is especially obvious for $D \geqslant 100$ m, where the boundary is given by λ_{gs} in Table 3. Below λ_{gs}, the weight is entirely defined by the rigidity needed for observation (governed by the quantity ρ/E), while the structure is stronger than needed for survival. Above λ_{gs}, the weight is entirely defined by survival (governed by p/S, with p = wind pressure on the wire mesh), while the structure is more rigid than needed for observation. For $\lambda \gg \lambda_{gs}$, the stable self-support of the structure becomes more important than its purpose (governed by Parkinson's Law). It follows that λ_{gs} is the most economical wavelength, any other wavelength giving a waste of either strength or rigidity.

2. Radome or Not?

Observing in severe weather is not important in radio astronomy. The 300-foot telescope at Green Bank is not used above 25 mph wind and in heavy snowfall, which gives a completely negligible loss of 43 hours per year. The only advantage of a

Figure 5. Decreasing the weight by using a radome. If observation is limited to 25 mph wind velocity without radome, the decrease of the antenna weight is not enough to pay for the radome, especially for antennas above 50 meter diameter. The arrow gives maximum weight decrease.



radome is the suppression of the wind force, but counteracting this force with a radome is not much cheaper than counteracting it with a telescope (you just need a certain amount of metal, no matter where you put it). Furthermore, the radome does not help in the gravitational deflection region, nor in the minimum structure region. Since radomes are expensive, they should be used only if they cut down the weight of the telescope by a factor two or three, and from Figure 5 we find the result:

$$\text{No radomes for diameters above 50 m.} \tag{31}$$

3. Foundations

Most economical seems to be an alt-azimuth mount with two towers standing on wheels on a circular track on the ground. The tower legs should be wide astride in order to decrease the uplifting forces at the ground, the best basic shape being a regular tetrahedron (slightly modified for more clearance for the rotating dish). In order to have only one circular track, we put one leg of each tower at the center of the circle on a strong pintle bearing. The track, then, has no lateral force, which is a great advantage.

Foundations are very expensive if made for a special purpose. A single steel track embedded in concrete, taking 300 tons downward and 80 tons laterally and upward, would cost $700,000/mile. Thus, we recommend using standard railroad equipment, with normal roadbed, ties and rails; this costs $80,000/mile; the maximum load is 30 tons per axle or 450 tons per 100 feet. An estimate shows for $500 \leq D \leq 600$ feet, that two large steel gondolas per tower leg are sufficient for load and counterweight during observation, if filled with rock and gravel. A small piece of fortified foundation is needed for the stow position.

The deviation of a normal railroad track is about 0.5 inch after one year of normal use; comparing the speeds of trains and telescopes, we may safely assume half of this value. With the cantilevering from (6) and tetrahedral towers, one edge of the dish will deviate by ± 6.3 mm (relative to the opposite edge), independent of diameter. If we demand a pointing accuracy of $\pm$ 1/16 of a beamwidth, the deviations of the rails after one year of use will allow observation down to 8.2 cm. Comparing with λ_{gs} from Table 3, we get the result:

$$\text{Normal railroad for diameters above 100 m.} \quad (32)$$

4. Price Estimate (for λ = 20 cm and D = 500 feet)

The following estimate is certainly very approximate but still tries to be without bias. With respect to hydrogen-line observations, we choose λ_{gs} = 20 cm and obtain from Table 3 a diameter of 500 feet. For the azimuth drive, we assume friction wheels on the tracks. For the elevation drive, we definitely want to avoid any additional passive weight at the dish, and we recommend guiding the anti-focus point of the octahedron along a curved leg (1/4 circle) of a third tower. All three towers are connected close above ground by long horizontal members. After calculating wind, snow and dead loads for legs and connections, we find a total weight of about 600 tons for the azimuthal structure.

Table 4
Price Estimate

Item	Amount	Price (incl. erection)	Million Dollars
Dish structure	750 tons	2,000 \$/ton	1.50
Surface	200,000 ft^2	1.5 \$/ft^2	.30
Towers	600 tons	1,500 \$/ton	.90
Bearings + pintle b.			.10
Drives			.20
Controls			.10
Miscellaneous			.05
Railroad	0.9 miles	100,000 \$/mile	.09
Gondolas	10	35,000 \$	.35
Stow foundation			.15
Pintle foundation			.10
		total	3.84

The result of Table 4 might seem low. But since many safety factors have already been included in our estimates, we regard 4 Million Dollars as a realistic figure for the total price, provided that the design really uses optimization in every detail, and that economy in fact is desired.

IV. Passing the Gravitational Limit

1. In General

Equation (7) gives the limit of a tiltable telescope for diameters above 45 m according to (11). This limit is derived from equation (2) giving the compression of a structure under its own weight. Details of the design do not matter much (as long as they don't make it worse); any structure must be compressed under its own weight, and by changing amounts if tilted. For a diameter of 150 m, the largest deformation at the rim is somewhere around 1.5 cm, no matter whether we hold the structure at two bearings in average height, or support it at many points from the bottom; even a floating sphere will deform by a similar amount.

If we want to pass this limit, we have three possibilities:

1. Avoiding the deformations by not moving in elevation angle,
2. Fighting the deformations with strong servo motors in the structure,
3. Guiding the deformations so they do not hurt the performance.

The first possibility is verified at the Arecibo dish in Puerto Rico, where a spherical reflector of 1000 feet diameter is fixed to the ground in a round valley; observation there is limited to within $\pm$ 20° from the zenith. Any fixed-elevation telescope will have a limited sky coverage but still can be a very valuable instrument for radio astronomy, and some solutions will be briefly discussed shortly. The second possibility was tried at Sugar Grove; it will always be very complicated and expensive, and we will not include it here. Our main emphasis will be on the third possibility, with special regard to those observations where full sky coverage is needed.

Suppose we pass the gravitational limit in some way or other. The next natural limit then is the thermal limit from equation (10), assuming ΔT = 5°C in full sunshine with a good protecting paint. We might go one step further and adopt a second thermal limit, without sunshine, for ΔT = 2°C. In Figure 6 we show the weight of a telescope as function of diameter and wavelength, if the gravitational deflections are omitted in some way. Since we have used for Figure 6 the same formulae as before, the values given in Figure 6 will hold for structures not too different from the previous one, which means mainly that the height of the structure is comparable to its diameter. The most economical wavelength now is given by the boundary between the wind deflection region and the survival region; especially for antennas above 100 m diameter, the weight increases very steeply to the left of this boundary. Here, a radome might be reconsidered, or placing the antenna in a valley shielded by mountains against wind, or limiting the observation to lower wind velocities, or designing a structure which sits flatter to the ground.

2. Fixed-Elevation Transit Telescopes

Although most radio astronomers would prefer full steerability about two axes, they would be sufficiently satisfied with a transit instrument if it gives them one or two hours observing time per transit of a source; this means full steerability about one axis, and a very limited steerability about a second one. Since gravitational deflections are our main problem, and since movement in azimuth does not change these deflections, we arrive at a telescope turning 360° in azimuth but only about 10° in elevation. The beam then describes a circle around the zenith, and a radio source of proper declination will give two transits per day through this circle. The choice of the best elevation angle is a compromise between two opposing demands: we want a large sky coverage (low elevation), but we should not observe too close to the horizon (high elevation). The best way might be to build two such telescopes with elevation angle 45°, one situated at +45° and one at -45° geographical latitude. The mirror could either be a paraboloid which turns by 10° in elevation, or it could be a fixed-elevation sphere, where a small secondary mirror and feed move 10° in elevation. The azimuth movement in any case would be on circular tracks. The feed could be either fixed to the dish with feed supports, or it could be on a separate, non-moving tower.

The least expensive telescope of this kind would be the fixed-elevation sphere with separate feed tower. The most flexible telescope would be the one from the previous paragraphs, with full steerability in elevation; we adjust the surface for a given elevation angle and use it only within ± 5° of this elevation; whenever desired later on, the surface can be adjusted to any other elevation angle. The weight of this telescope is given by Figure 6 as function of diameter and wavelength, while the fixed-elevation sphere might be lower by 30 or 40 per cent (height somewhat lower, no elevation drive).

Large mirror flat on the ground. If we want a large telescope of say 200 m diameter, and want to use it for as short a wavelength as possible, we see from Figure 6 that the weight increases very rapidly with decreasing wavelength for $\lambda < 15$ cm. This increase is due to the D^4/λ^2-term in equation (30) where the weight is defined by wind deflections. The increase is so steep that we should look for a better solution, where the antenna sits flat on the ground and does not pick up so much wind force.

A possible solution is sketched in Figure 7, using a parabola at 45° elevation with its focus at F. In a usual design, we would use part AB, where a large surface is high above ground. Now, we use part CD which is 40% larger but is never more than 40 m above ground. This parabolic mirror P (282 m long and 200 m wide) is mounted on wheels on a flat cylindrical trough GHI (343 m long) with its center line through M. Moving the mirror in this trough around axis M gives the 10° of movement in elevation. The trough sits on wheels on horizontal circular tracks around center point Z, giving the 360° of movement in azimuth. The feed is mounted movable along a track T about 50 m long, which is 10° of a circle around M. The track can be turned by 360° around a vertical axis and is mounted on a tower 200 m high.

Having the feed at the primary focus gives the following disadvantage. In order to illuminate the antenna beam symmetrically, the feed must have an asymmetric pattern. This could be done, but then the feed annot be rotated for polarization measurements. This problem is resolved by using a small, tilted secondary mirror of Gregorian type. Feed and secondary mirror then are moved together along track T.

3. Homology Deformation, in General

Although radio astronomers would be satisfied with a transit instrument, they still would prefer full steerability if it can be achieved (within reasonable costs). There is one observing technique, lunar occultations, where full steerability is crucial in order to obtain the brightness distribution across a source in many directions and for very accurate position determinations. This technique yields the highest resolution. It is worthwhile to design a telescope with special regard to lunar occultations; half of the time (moon below horizon) it can be used for other observations. But can we pass the gravitational limit, with 90° tilt in elevation, and without strong servo motors in the structure?

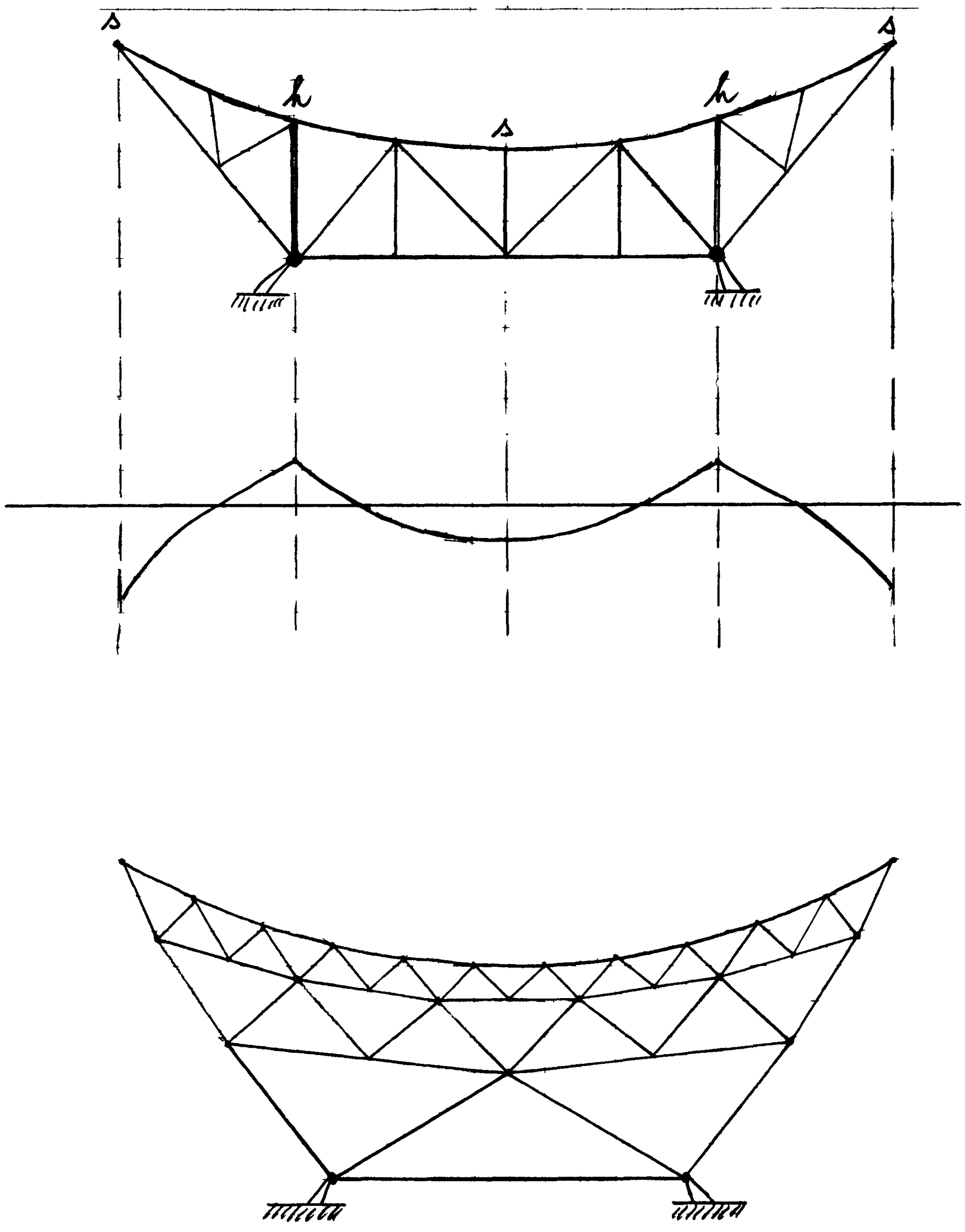
The laws of physics tell us that a structure, under the influence of gravitation, deforms into a state of minimum energy; the center of gravity must move down. The material constants ρ and E tell us the amount by which it must move down. But there is no law of nature telling us that a parabolic surface must deform into something different from a parabolic surface. We thus look for a structure which deforms down whatever it must, but still gives a perfect paraboloid of revolution for any elevation. The focal length might change a bit, but this can be taken care of by servo-adjusting the feed according to elevation. A deformation of this kind, deforming a surface of given type into another surface of same type, we call "homology deformation".

Equal softness. As a first approach, let us consider in Figure 8,a the cross section through a large telescope of conventional design. The heavy vertical lines above the main bearings represent the main frame which usually is a heavy, flat square box. Figure 8,b shows the deformation of the surface; the two "hard" points above the bearings stick out considerably, while the "soft" points at the rim hang down. This disadvantage of having soft and hard surface points can be avoided by a structure shown in Figure 8,c, where the way from each surface point to the next bearing is of the same length, giving equal softness to all surface points. A structure of this type is not an exact solution of our problem, but it might be a fairly good first approximation.

a)

b)

c)



<u>Figure 8</u>. Equal softness.

a) Conventional design, with hard (h) and soft (s) surface points.

b) Deformation of this telescope, looking at zenith.

c) Structure where all surface points have equal softness.

Exact solutions. We chose N structural points at the surface such that the deformation between neighboring points can be neglected, and we demand that all N points lie on an exact paraboloid of revolution for any elevation angle. Since we care only about deformations normal to the surface but not tangential to it, we must fulfill N conditions per elevation angle (more exactly, N-6, since six points define a paraboloid). Since any tilted force can be split up into an axial and a normal component, the problem is solved for any elevation if it is solved for two, for example looking at the zenith and at the horizon. This means that the structure must fulfill 2N conditions.

Next, we count the degrees of freedom, because the problem is mathematically solvable if the number of degrees of freedom equals or exceeds the number of conditions. A stable structure connecting i points has a minimum number of 3(i-2) members. Our structure has N surface points, and we assume at least the same number of intermediate joints between surface and bearings. The minimum number of members is thus $3(N-2) \approx 6N$, which means 6N degrees of freedome for the cross sections of the members. We regard the positions of the surface points as given, but we have 3N degrees of freedom for the positions of the intermediate joints. Altogether, we have 9N degrees of freedom. Subtracting 2N conditions leaves 7N; this means the problem is solvable, and there is a family of solutions with 7N free parameters.

Of course, a mathematical solution is not always a useful one. In order to make it a physical solution, each cross section must be within $0 < Q < \infty$. Also, what we might call a "practical solution" must fulfill some additional conditions, for example: the total weight should not be much greater than needed for a normal telescope; the structure should not be too soft against wind deflections; it should be not critical against small deviations of cross sections from the theoretical value; the bearings should have clearance for rotation, and so on. But, on the other hand, having 7N free parameters to play with should give us enough freedom for practical solutions, too.

Layers and cells. Even if there are solutions, how do we get one? We certainly cannot play at random with 7N free parameters (not even on a fast computer) until we hit something useful. We need some logical principle to guide us during the design. For this purpose we suggest dividing the space between bearings and surface into layers of decreasing thickness with increasing number of joints, each layer being divided into cells by the joints, and we make all cells topologically identical (they may have different sizes, proportions and cross sections, but all cells have the same basic structure). Figure 8,c gives an example with three layers. Going from the bearings to the surface, each 3-dimensional cell should quadruple the number of joints such that each layer has four times more cells than the previous one, until we arrive at the surface with N cells. In order to keep the structure more compact and balanced, one or more of the layers could be "folded backwards".

The basic idea of this arrangement is to let the single cell fulfill a certain set of conditions such that the structure as a whole fulfills the same conditions. One might formulate, for example: provided that layer i is on a surface of given type for any elevation angle, layer i+1 then must be on a surface of the same type for any elevation, too. Solving the problem for one cell then would give a solution for the whole telescope.

4. Homology, Solutions in Two Dimensions

In order to learn whether the suggested procedure works and whether there are reasonable solutions, we try it for two dimensions; and for simplicity we replace parabolas by straight lines. The simplest cell we could find is shown in Figure 9,a where load w replaces the weight of the structure in all higher layers. We let cross sections $Q_a = Q_b$ and call $U = Q_a/Q_d$. Given w, ρ, Q_d and e, the cell has three degrees of freedom: c, q, and U. For solving our homology problem, we impose two conditions:

1. If gravity has direction y, point P_3 shall not move in x-direction. (33)

2. If gravity has direction x, point P_3 shall not move in y-direction. (34)

With three degrees of freedom and two conditions, the cell has one free parameter left. But we might go one step further and use it up for fulfilling a third condition:

3. For any direction of gravity, point P_3 shall move the same amount. (35)

Under conditions (33) and (34), P_3P_4 is parallel to P_1P_2 for any elevation angle, and P_3P_4 keeps constant length if P_1P_2 does. (Members f and g are redundant but should be included). Under the addition of condition (35), the cell can easily be applied to a curved line, too. If layer i keeps its shape under any elevation, layer i+1 keeps its shape, too, and the gravitational deformation just gives it a parallel shift. Applied to a parabola, even the focal length would stay constant for varying elevation.

We omit the tedious calculations and just quote the results. The cell of Figure 9,a has two types of solutions, a general solution and a particular one. First, we give the general solution. Calling

$$\eta = \frac{q}{4}\,\frac{2e-c}{ep-cq}, \tag{36}$$

conditions (33) and (34) take the form

$$U = \frac{c}{qd^3}\,(a^3 + \eta b^3) \tag{37}$$

and

$$\frac{2w}{d\rho Q_d} = U \left\{ \frac{(c+q)\ (a+b)}{4cd(\eta - 1)} - \frac{a}{d} \right\} - 1 , \tag{38}$$

while condition (35) leads to

$$e^2 = qc. \tag{39}$$

Imposing all three conditions leaves no freedom, but the solution depends on $w/\rho Qd$. Calling W the weight of the cell, and $\Omega = 2w/W$ the weight ratio of all upper cells to this one, we find the full range $o \leq \Omega \leq \infty$ is already covered if c/e varies over the narrow range $1.62 \leq c/e \leq 1.66$, and if $q/c \approx 0.371$. For various Ω, the solution for U is as follows:

Ω	0	0.5	1.0	2.0	∞
U	8.53	5.90	5.48	5.20	4.91

(40)

We see that not only are there solutions, they are physical solutions ($o < U < \infty$); but we hesitate to call them practical because the values obtained for U are so large that our homology principle would introduce too much extra weight. Even by dropping the third condition, we could not find any geometrically reasonable solution with U below 3.

Second, there is a particular solution for

$$ep = cq \qquad \text{and} \qquad 2e = c, \tag{41}$$

shown in Figure 9,b, which does not fulfill the third condition, but where the geometry does not depend on the weight ratio. For the particular solution we obtain

$$\Omega = 0.167 \ \frac{-U^2 + 1.68U + 7.75}{U^2 - 3.57U - 2.60} \tag{42}$$

The full range $o \leq \Omega \leq \infty$ is now covered if U varies within

$$3.75 \leq U \leq 4.19, \tag{43}$$

but these values, again, are too high. Two things could be considered: by introducing an additional degree of freedom, $Q_a \neq Q_b$, we might bring the cross sections down, also a structure with smaller U might be a good-enough approximation. But there is the following additional difficulty.

Even if we had a good solution for the single cell, it would be a solution for the whole structure only if the structure were indefinitely long. In a structure of finite size we get <u>boundary distortions</u>. They result mainly in pressure (or tension) along line g, and partly in torques such that the forces at P_3 and P_4 have different directions. It seems possible to counteract or smooth these distortions by varying all cross sections within a layer from the center to the end of the structure, especially since the layers must keep their shape but can deviate from being parallel to each other. An investigation of this type would go far beyond the scope of this paper, however. We therefore look for another type of cell where at least the influence of pressure is removed.

The simplest <u>pressure-stable cell</u> we could find is given in Figure 9,c. Redundant members like f and g can be included in three dimensions (as diagonals) although not in two. Under various tilts, P_3P_4 and P_4P_5 will not keep their lengths constant, but this does not matter since the next layer is pressure-stable, too. Conditions (33) and (34) are fulfilled automatically by reasons of symmetry for P_3, P_4 and P_5. We add condition (35) for P_4 in order to guarantee application to a curved structure. Given w, ρ, Q_b and e, the cell has five degrees of freedom: c, q, Q_a, Q_c and Q_d. Homology is reached by imposing three conditions:

1. If P_1P_2 is compressed, P_4 shall not move in y-direction. (44)
2. If gravity has direction y, point P_3 shall move as much as point P_4. (45)
3. For any direction of gravity, point P_4 shall move the same amount. (46)

This leaves two free parameters, and we use both for demanding:

4. The additional weight introduced by homology shall be a minimum. (47)

Condition (44) is fulfilled if

$$\frac{cdeq}{Q_d} + \frac{a^3c}{Q_a} = \frac{b^3q}{Q_b} . \tag{48}$$

The following treatment is restricted to $\Omega \gg 1$ for simplicity. Condition (45) then gives

$$\frac{2c^3}{Q_c} = \frac{b^3}{Q_b} . \tag{49}$$

Conditions (44) and (46) give

$$2e^3 = c(e^2 + bc) \tag{50}$$

which is an equation of sixth order for c/e and has only one physical solution:

$$c/e = 0.903. \tag{51}$$

We let $U_a = Q_a/Q_b$, $U_c = Q_c/Q_b$, $U_d = Q_d/Q_b$ and $\varphi = q/e$, then conditions (44), (45) and (46) lead to

$$U_c = 0.603 \tag{52}$$

and

$$U_a = \frac{1}{\varphi} \ (1+\varphi^2)^{3/2} \ \frac{U_d}{2.70\ U_d - \varphi - 0.903} \tag{53}$$

where the two free parameters are φ and U_d.

In order to fulfill condition (47), we call $W = \rho(2Q_a a + 2Q_b b + Q_c c + Q_d d)$ the total weight of the cell, and $W_o = 2\rho Q_b(b + c + e)$ the weight if we do not ask for homology, letting q = o and letting all Q be equal to Q_b. Now, we tak the ratio $\tau = W/W_o$, and we let $\partial\tau/\partial\varphi = o$ and $\partial\tau/\partial U_d = o$. The solutions are $\varphi = 0.441$ and $U_d = 1.44$. With these values, we obtain $\tau = 1.37$, which is a very fortunate result. Compared to a normal structure, our homology condition is fulfilled with only 37% increase of the total weight. Altogether, the cell is now completely defined and we have:

$$c/e = 0.903 \qquad q/c = 0.489 \qquad Q_a/Q_b = 1.67 \qquad Q_c/Q_b = 0.603 \qquad Q_d/Q_b = 1.44 \tag{54}$$

With the values (54), the cell of Figure 9,c seems to be a very good, practical solution. The geometrical shape is convenient, and a weight increase of only 37% is a very low price to be paid for homology deformation. This cell is an exact solution for an indefinite structure; and for a finite structure, the main part of the boundary distortion is removed. The remaining distortion by torques should be removable, too, by varying the cross sections toward the boundary. The application to a 3-dimensional cell is easy; for example, let three lines f be the three sides of an equilateral triangle, and let member d go perpendicular through the center of the triangle. This cell will double the number of joints from one layer to the next. If we want to quadruple the number of joints, we let three members d be perpendicular on the center of each side of the triangle. Instead of a triangular structure, we could also take a quadratic structure with the same two possibilities.

In summary, we have proved that homology deformation must have mathematical solutions, and we can show that practical solutions also exist. With this finding, the present investigation has reached its goal. The next steps call for a good program on a fast computer: first, investigating simple, multi-cell structures in two dimensions for solving the boundary problem; second, repeating the same in three dimensions; third, designing an actual telescope.