MMA memo 188:
Another look at anomalous refraction on Chajnantor

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Abstract

In MMA memo #186 (Holdaway 1997), Mark Holdaway pointed out the important contribution of atmospheric anomalous refraction to the pointing error budget for the proposed MMA antennas. He derived expressions for the magnitude of the effect of anomalous refraction on the pointing of antennas of different sizes, and calculated values for proposed MMA antenna sizes for the particular atmospheric characteristics of the Chajnantor site. I think Mark got the story right, in that his conclusion was that the pointing error due to anomalous refraction is smaller in absolute magnitude for larger antennas, but larger as a fraction of beamwidth. However, it is not clear that he got the absolute values of the numbers right in the non–zenith case. He derived a pretty unwieldy equation for the anomalous refraction for this case (his equation 12), which has some very non–physical terms in it, and I am not convinced that this equation is correct. I have derived what I *think* is the correct formula for the non–zenith case. This formula is much simpler, and is (see below for notation):

$$\sigma_{\overline{\psi}} = \frac{\sqrt{D_l(d \sec z)}}{d} \sqrt{\sec z} = C_l \, d^{\beta/2-1} \, [\sec z]^{(\beta+1)/2} \quad . \tag{1}$$

I'll present this derivation and some numerical examples in this memo. It turns out that my numbers have a slightly weaker dependence on elevation than Mark's do, e.g., at 20° my numbers are about 10% smaller than his (I derive a smaller rms pointing error due to anomalous refraction). The intent of this memo is *not* to imply that Mark did something egregious, but merely to present this simpler formula, and the new values obtained from it. Again, he got the basic story right in memo 186.

Geometry

Turbulent fluctuations in the troposphere give rise to differences in electrical path length across an antenna surface. These differences can then be thought of as wedges of physical material with an index of refraction greater than the surrounding air between the points on the antenna surface. The excess electrical path length (Δp) and the physical wedge height (Δh) are related via:

$$\Delta h = \frac{\Delta p}{n' - 1} \quad , \tag{2}$$

where n' is the relative index of refraction of the wedge (the ratio of the true index of refraction of the wedge to the index of refraction of the surrounding medium [air]). This is the way that Mark approached the problem, and I use the same physical model here. These virtual wedges of material cause plane waves to be redirected as they travel through, and the net effect over the entire antenna surface is to produce a pointing error. Another effect is a broadening of the antenna primary beam. Such effects have been noted for several antennas on different sites (Altenhoff *et al.* 1987; Downes & Altenhoff 1990; Church & Hills 1990; Coulman 1991; Zylka *et al.* 1992; Zylka *et al.* 1995).

Zenith case

Figure 1 shows a upwardly propagating plane wave incident on a wedge of length Δx and height Δh, both in physical units. The propagation direction vector of the wave is anti–parallel to the normal of the wedge lower edge, appropriate for the zenith direction. Passage through the wedge, which has index of refraction n' relative to the surrounding medium, causes the wave phase velocity to decrease. The different thicknesses through the wedge traversed by different portions of the wave then cause a bending of the wave front, which is equivalent to a change of direction of the propagation vector of the wave. The angle between the original propagation direction vector and the redirected one (ψ) is the effective pointing error. This angle can be found by considering the time it takes a wave crest to cross the distance Δh at the thickest point of the wedge:

$$t_1 = \frac{\Delta h}{v_1} \quad , \tag{3}$$

where v_1 is the phase velocity of the wave in the wedge. This phase velocity is given by:

$$v_1 = \frac{v_o}{n'} \quad , \tag{4}$$

with v_o the phase velocity in the surrounding medium. In time t_1, the wave crest just beyond the point of the wedge travels a distance:

$$l_o = v_o t_1 = v_o \frac{\Delta h}{v_1} = \Delta h \, n' \quad . \tag{5}$$

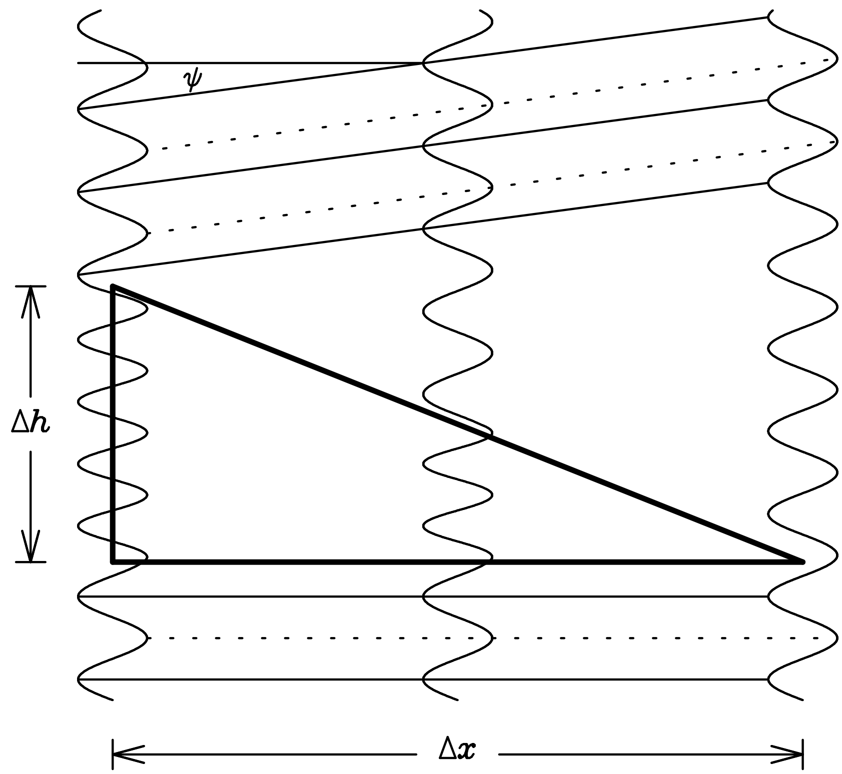

Figure 1. Geometry of virtual wedge for anomalous refraction (zenith)

The angle ψ is given by:

$$\tan\psi = \frac{\Delta l}{\Delta x} \quad , \tag{6}$$

where $\Delta l = l_o - \Delta h$. So,

$$\tan\psi = \frac{\Delta h\ (n' - 1)}{\Delta x} \quad , \tag{7}$$

which for small values of ψ reduces to:

$$\psi = \frac{\Delta h\ (n' - 1)}{\Delta x} = \frac{\Delta p}{\Delta x} \quad . \tag{8}$$

At any one time, there will be a complicated variation of excess path length vs position across the projected surface of the antenna. If the excess path length correlation function is only a function of radial distance from point to point in the atmosphere, then a 1–D cut through the 2–D distribution across the projected surface of the antenna is a good statistical representative of the entire distribution. This 1–D cut can then be broken up into small intervals (Δx), each of which has an effect which is approximated by the wedge treatment above. The propagation direction vector for the net wave which emanates from the antenna is then given by a vector sum of each of the propagation direction vectors for the individual wedges. For each small wedge, the propagation vector is decomposed into 2 orthogonal components, one along the direction parallel to the antenna surface (the x–axis), and the other perpendicular to it (the y–axis). For an antenna of diameter d, we have $N = d/\Delta x$ small wedges, and there are values of the excess electrical path length Δp_i at $N+1$ locations. For the i^{th} wedge, the two components of the propagation direction vector are:

$$k_{x_i} = \sin\psi_i \quad , \tag{9}$$

and

$$k_{y_i} = \cos\psi_i \quad . \tag{10}$$

ψ_i is the angle given in equation 8, i.e., $\psi_i = (\Delta p_i - \Delta p_{i+1})/\Delta x$. The propagation direction vector for the net wave ($\overline{\psi}$) is then (for small $\overline{\psi}$):

$$\overline{\psi} = \frac{\sum_{i=1}^{N} \sin\psi_i}{\sum_{i=1}^{N} \cos\psi_i} \quad . \tag{11}$$

If the ψ_i are small angles (which they should be), then

$$\overline{\psi} = \frac{\sum_{i=1}^{N} \psi_i}{\sum_{i=1}^{N} 1} = \frac{\Delta p_1 - \Delta p_{N+1}}{N\,\Delta x} = \frac{\Delta p_1 - \Delta p_{N+1}}{d} \quad . \tag{12}$$

If there are values of the Δp_i at M locations ($M > N+1$), then the time averaged behavior of the atmosphere flowing over the antenna can be simulated by virtually sliding the values of the Δp_i over the antenna, i.e., there are then $M - N + 1$ values of the net propagation direction vector, each with pointing error:

$$\overline{\psi_j} = \frac{\Delta p_j - \Delta p_{j+N}}{d} \quad . \tag{13}$$

The rms value of $\overline{\psi}$ over the $M - N + 1$ distributions is:

$$\sigma_{\overline{\psi}} = \sqrt{\frac{\sum_{j=1}^{M-N+1} \overline{\psi_j}^2}{M-N} - \left(\frac{\sum_{j=1}^{M-N+1} \overline{\psi_j}}{M-N}\right)^2} \quad . \tag{14}$$

The mean value of the $\overline{\psi_j}$ should be 0 for atmospheric turbulence, which leaves:

$$\sigma_{\overline{\psi}} = \sqrt{\frac{\sum_{j=1}^{M-N+1} \overline{\psi_j}^2}{M-N}} = \sqrt{\frac{\sum_{j=1}^{M-N+1} (\Delta p_j - \Delta p_{j+N})^2}{M-N}\,\frac{1}{d^2}} \quad . \tag{15}$$

This is directly related to the excess path length *structure function*, which is defined as the mean–squared difference of path length over some distance r (e.g., Tatarski 1961):

$$D_l(r) = < (\Delta p(r_o) - \Delta p(r_o + r))^2 > \quad . \tag{16}$$

Let $r = d$, and assume a discrete distribution for Δp with $M - N$ values at intervals $\Delta x = d/N$, then the discrete form of the structure function is:

$$D_l(d) = \frac{\sum_{j=1}^{M-N} (\Delta p_j - \Delta p_{j+N})^2}{M - N} \quad , \tag{17}$$

which is the form in equation 15. Substituting this in gives:

$$\sigma_{\overline{\psi}} = \frac{\sqrt{D_l(d)}}{d} \quad . \tag{18}$$

This then is the rms pointing error due to anomalous refraction in an atmosphere with the specified excess path length structure function, and is the same as what Mark derives in his equation 9.

The structure function may be written:

$$D_l(d) = C_l^2 \ d^\beta \quad , \tag{19}$$

where C_l and β are measured quantities which characterize the atmosphere for a given location. Given this substitution, the rms pointing error at zenith for an antenna of diameter d due to anomalous refraction is given by:

$$\sigma_{\overline{\psi}} = C_l \ d^{\beta/2-1} \quad . \tag{20}$$

This means that the absolute value of the anomalous refraction rms pointing error gets *smaller* as antenna size gets larger for all values of $\beta < 2$. However, since the width of the primary beam is proportional to d^{-1}, the rms pointing error *as a fraction of primary beam width* gets *larger* as antenna size gets larger (for $\beta > 0$). Mark correctly pointed this out.

Non–zenith case

Consider now the case where the same structure of excess path length exists above the antenna, but it is observed at some angle from the zenith z. In this case, the time to travel through the thickest part of the wedge (analagous to equation 3) is given by:

$$t_1' = \frac{\Delta h'}{v_1} \quad , \tag{21}$$

where the path length through the wedge material is now increased to:

$$\Delta h' = \Delta h \ \sec z \quad . \tag{22}$$

Proceeding just as in the zenith case gives for the net pointing error:

$$\overline{\psi_j} = \frac{(\Delta p_j - \Delta p_{j+N'}) \sec z}{d} \quad , \tag{23}$$

where $N' = d \sec z / \Delta x$, reflecting the fact that the projected size of the antenna on the lower edge of the turbulent layer is increased by $\sec z$ over its intrinsic size. Again, proceeding as in the zenith case, the rms pointing error is then:

$$\sigma_{\overline{\psi}} = \frac{\sqrt{D_l(d \sec z)}}{d} \sqrt{\sec z} = C_l \, d^{\beta/2-1} \, [\sec z]^{(\beta+1)/2} \quad . \tag{24}$$

This has the same dependence on antenna size as the zenith case, so the conclusions about the absolute and relative value of the pointing error vs antenna size in the zenith case also hold here.

The physical basis for the $(\beta+1)/2 = \beta/2 + 1/2$ dependence on $\sec z$ in equation 24 can be understood as the combination of two effects. The first is that the physical path length through the turbulent atmosphere has increased by $\sec z$, and hence the accumulation of excess path length is larger by that same amount. This means that the amplitude of the structure function is increased by that amount, and hence the rms increases by $\sqrt{\sec z}$ giving rise to the $1/2$ term. The second, as mentioned above, is that the projected size of the antenna on the lower edge of the turbulent layer is larger by a factor of $\sec z$, so the structure function needs to be evaluated on that larger spatial scale, giving rise to the $\beta/2$ term. The fact that both of these effects must be taken into account was noted (and derived) by Taylor (1975). The increase in the amplitude of the structure function by $\sec z$ (and hence an increase in the rms by $\sqrt{\sec z}$) has been noted by many previous workers (see e.g., Lutomirski & Buser 1974; Tatarski 1961; Kolchinskii 1957). Kolchinskii (1957) also noted that when different sets of actual observed variations were fit to a power law in $\sec z$ there were many which had a power law exponent > 0.5, which was unexpected by him. This was most likely the manifestation of the $\beta/2$ term from the argument of the structure function. Note also that as pointed out by Treuhaft & Lanyi (1987), the $\sec z$ increase in the amplitude of the structure function is only valid for baseline lengths much less than the height of the troposphere (the baseline length is equivalent to the antenna diameter here). For larger baseline lengths, the increase is $\sec^2 z$. This has no bearing on the problem of anomalous refraction for millimeter antennas, however, since the antenna diameters are always much smaller than the height of the troposphere.

Numerical values

For Chajnantor, the median value of β is 1.2, from Mark's memo. Using this value, and the values of $\sqrt{D_l(d)}$ for median conditions from Table 1 of that memo, the values for the rms pointing error due to anomalous refraction can then be calculated. These values are shown in Table 1 for different antenna sizes and different elevations at Chajnantor. The rms pointing errors relative to the primary beam size are not shown in Table 1, nor are the values for the other quartiles of atmospheric conditions. The equivalent values from Mark's memo are shown in parentheses in Table 1, for comparison. It seems that my numbers are very slightly smaller than his, as the dependence on elevation I've derived is somewhat weaker than his.

Table 1. Anomalous refraction pointing error for median atmospheric conditions at Chajnantor (in arcsec; values in parentheses are from Holdaway [1997]).

ant diam (m)	elev angle 90°	50°	30°	20°	10°
8	0.46 (0.47)	0.62 (0.64)	0.99 (1.07)	1.51 (1.65)	3.18 (3.55)
10	0.42 (na)	0.57 (na)	0.91 (na)	1.38 (na)	2.92 (na)
12	0.39 (0.39)	0.53 (0.55)	0.84 (0.90)	1.28 (1.40)	2.70 (3.02)
15	0.36 (0.36)	0.48 (0.49)	0.77 (0.82)	1.17 (1.27)	2.47 (2.76)
50	0.22 (0.22)	0.30 (0.31)	0.48 (0.51)	0.72 (0.79)	1.53 (1.70)

Notes on assumptions

The derivation presented here assumes that geometric optics is appropriate to describe the propagation of the wave through the turbulent atmosphere. When the wave optics treatment is included, the dependence is roughly as I've derived here, but is more complicated (see e.g., equation 24 of Taylor [1975] [where he uses the Rytov method, which should be valid for mm-submm wavelengths at Chajnantor] and note that the pointing error can be directly related to the phase structure function [or phase correlation function] as easily as to the path length structure function, i.e., via equation 13.3 of Tatarski [1961] or equation 73 of Fante [1975]). There is also an implicit assumption that the turbulence is in a plane parallel slab which is above the antennas. If the turbulence extends down to the antenna surface, then the $\beta/2$ part of the $\sec z$ term goes away, since it is due strictly to the assumed geometry. This plane parallel assumption will also break down at very low elevations.

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