ALMA MEMO 586

Walsh Function Choices for 64 Antennas

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Summary

ALMA Memo #537 studied the loss of orthogonality between pairs of Walsh functions, when one function had undergone a small time shift with respect to the other. It also showed that 5461 of the 8128 possible cross-product pairs from a set of N=128 Walsh functions remain perfectly orthogonal in the presence of a time shift. This memo shows that in general a fraction $N^2/3$ of all possible cross-products of a set of N Walsh functions remain orthogonal in the presence of a relative time slip.

This memo investigates the optimum choice of a subset of M functions, corresponding to M antennas, from a complete set of N, with the aim of minimizing crosstalk between antennas in the presence of electronic timing errors. ALMA has already adopted N=128, and it is found here that there would be relatively little gain for additional effort required to implement N=256 or greater. This Memo concentrates in particular on arrays of M=64 antennas, but the same approach may be used for any number of antennas.

Different optimization strategies are examined, aimed at maximizing the number of zero cross-products, at minimizing the cumulative crosstalk level, and at minimizing the loss of sensitivity. Minimum crosstalk is not obtained by maximizing the number of zero cross-products, nor does the set of 64 functions chosen from 128 having lowest cumulative crosstalk include the highest number of zero cross-product pairs. Specific results and recommendations are summarized in Table 2; the favored subset of functions for 64 antennas is WAL 0-31, 47-63 & 113-127.

Contents:

1 Background and Aim

ALMA has adopted a 128-element Walsh function set. The functions are used in ALMA (1) for sideband separation and (2) to reduce the impact of spurious signals in the receiver IF, or of DC offsets in the digitizers. This memo only considers (2); in ALMA, the time of the shortest element within the Walsh set is 125 microseconds, giving a total cycle length of 125*128=16000 microseconds or 16 milliseconds. ALMA Memo #537 "Walsh Function Demodulation in the Presence of Timing Errors, leading to Loss and Crosstalk" included calculation of the precise level of crosstalk between all pairs of Walsh functions with a relative time slip between functions of 1% of the shortest element, or in this case 1.25 microseconds. The level of crosstalk is linearly proportional to this time slip. In ALMA Memo #581, "Selection of Walsh Functions for ACA" a set of functions was found optimized specifically to give minimum loss of sensitivity for the ACA, with crosstalk level a secondary consideration.

1.1 Criteria for optimization

There are different ways of optimizing the choice of functions, including any one of the following separate strategies:

1) Minimizing loss of sensitivity, regardless of the level of potential crosstalk
2) Minimizing the RSS (root sum of squares) of crosstalk amplitude summed over all products, with sensitivity a secondary consideration
3) Minimizing crosstalk by first maximizing the number of zero amplitude cross-products, then minimizing the remaining RSS crosstalk over all products

In a particular context, choosing which criterion or combination of criteria to adopt - achieving the lowest overall summed RSS crosstalk, maximizing the number of zero crosstalk products, or simply choosing a set of functions giving the minimum loss of sensitivity - is a choice of the overall system design.

Memo #537 found that the lowest loss of sensitivity in the presence of a relative time slip is obtained with the lowest WAL indices; see Figure 1. This is not surprising, since by definition the lowest WAL indices have the lowest number of state transitions.

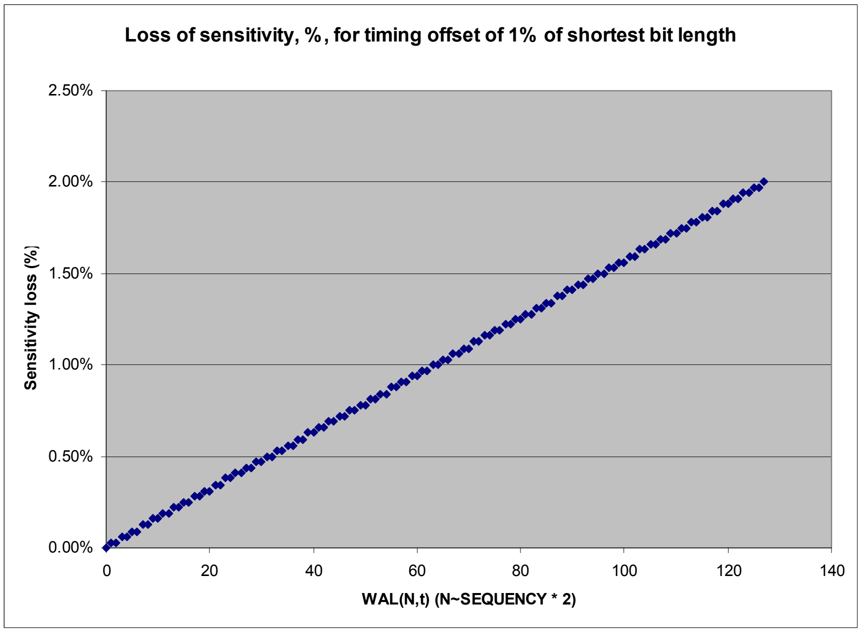

Figure 1 Loss of sensitivity versus WAL index (see Memo #537)

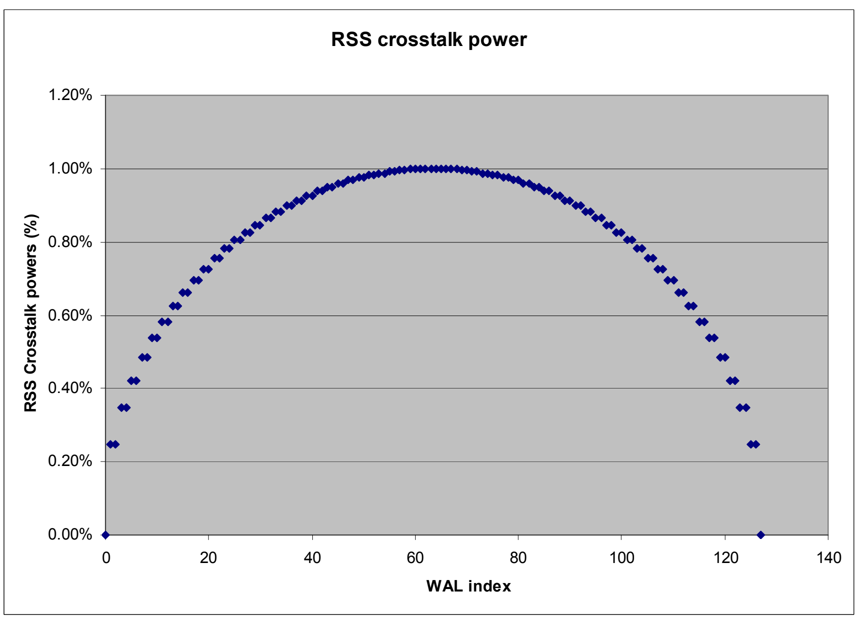

Figure 2 Cumulative crosstalk versus WAL index (see Memo #537)

It was found that, of the **N.(N-1)/2**=8128 possible cross-products when **N=128**, 5461 of these remained completely orthogonal in the presence of a small[1] time shift. It was noted in particular that the products of functions with odd PAL indices with functions having even indices guaranteed continued orthogonality in the presence of a timing slip.

On the other hand, at least statistically, lowest crosstalk was obtained with products involving WAL indices of the extreme lowest or of the extreme highest values, avoiding Walsh indices near the middle of the set; see Figure 2. However, no strategy was given for choosing the optimum subset of **M** functions from the total set of **N**, where cross-products only from within the subset are taken.

In the Appendix below, it is shown that for an **N**-element Walsh function set, in general, for large N, a fraction:

$$\frac{N^2}{3}$$

of all possible cross-product pairs remain orthogonal in the presence of a relative time slip. For 128-element Walsh sets, this would be precisely 5461, exactly as was found earlier (Memo #537) empirically. Compare this to the total number of cross-products

$$\frac{N.(N-1)}{2}$$

which for large **N** becomes

$$\frac{N^2}{2}$$

That is, for large **N,** a fraction (2/3) or 66.67% of all possible products remain identically zero in the presence of a relative time slip.

2 Sensitivity Loss

Figure 1, taken from ALMA Memo #537, shows the sensitivity loss as a function of WAL index IW. This loss is antenna-based, not dependent on cross-products, and for a t % time slip the loss of sensitivity (%) using a given Walsh function with WAL index IW, chosen from a total set of N (here N=128) is given by

[1]In this context, "small" means smaller than the shortest element length in the complete Walsh set. In the context of ALMA, this means smaller than 125 microseconds. The effect on orthogonality of even larger relative time slips, as will often occur for example in the formation of lag functions for high spectral resolution observations, has not been investigated.

$$\mathrm{Loss}(\mathrm{IW}) := \frac{2 \cdot t \cdot \mathrm{floor}\left(\frac{\mathrm{IW}+1}{2}\right)}{\left(\frac{N}{2}\right)}$$

where floor() truncates to the lower integer value. For an array of M antennas, the total loss of sensitivity is given by the average of Loss(IW) over all functions IW in use. In particular, if the WAL indices 0 to 63 were used (M=64) from a set of N=128 functions, with equal weight given to all antennas, the loss for a slip of t=1% would be given by

$$\sum^{M-1} \left[\frac{\left[\frac{2 \cdot \mathrm{floor}\left(\frac{\mathrm{IW}+1}{2}\right)}{\left(\frac{N}{2}\right)}\right]}{M}\right] \quad \mathbf{=0.5\ \%\ .}$$

This is the best possible case from the point of view of sensitivity loss.
Similarly, if only the highest Walsh indices were used, the loss of sensitivity with the 1% time slip would be:

$$\sum_{\mathrm{IW}=64}^{127} \left[\frac{\left[\frac{2 \cdot \mathrm{floor}\left(\frac{\mathrm{IW}+1}{2}\right)}{\left(\frac{N}{2}\right)}\right]}{M}\right] \quad \mathbf{=1.5\ \%\ .}$$

From the point of view of sensitivity loss, this would be the worst possible case.

In practice, the loss of sensitivity for an arbitrary 64 functions chosen from a set of 128 will be intermediate between these extremes; see Section 5 below, where a minimum-crosstalk set of functions giving a sensitivity loss of only 0.8% is found.

3 Increasing the size of the Walsh function set?

From a given set of N functions, two thirds of all cross-products remain orthogonal in the presence of a time shift. Is it possible to choose a subset M of these N functions so that all cross-products within that subset remain orthogonal?

The current plan within ALMA is to provide a 128-element set of Walsh functions, from which a set of M (say, 64) functions are chosen for the M antennas. Not all products from any set of 64 functions chosen from 128 will be orthogonal. So, how much better might it be to start from an initial set of, say, 256 functions from which we choose 64, rather than choosing from 128 as planned? Would this enable a significantly higher proportion of cross-products to remain completely orthogonal?

In Figure 3 below is a plot of the maximum possible number of cross-products that can remain completely orthogonal in the presence of relative time shifts, for (blue line) M antennas chosen from N=128 Walsh functions, for (purple line) M antennas chosen from N=256 Walsh functions, and finally (green dashed line) a crude prediction for "M from N=256", derived simply by doubling the X-scale on the N=128 plot.

With 64 antennas but limiting the choice of function to an initial 64-element Walsh set, precisely 1365 of the 2016 useful cross-products remain orthogonal. This is 67.7%.

Choosing the best sub-set of M=64 functions from a Walsh set of N=128, it is found that at best 1621 of the 2016 useful cross-products, or 80.4%, can remain orthogonal. Choosing the best sub-set of M=64 functions from a Walsh set of N=256, at best 1717 of the 2016 useful cross-products can remain orthogonal. This is 85.2%, only a very small further improvement.

One strategy. which would ensure that 100% of all cross-products remain orthogonal for M antennas, would be to chose the M functions from an initial Walsh set of $N=2^{(M-1)}$ functions; this would allow each chosen Walsh function to be a different Rademacher function. So, to have all products orthogonal with 8 antennas, we need an initial Walsh set of $2^7=128$ functions, which is seen in the Figure 3. (With 9 antennas and an initial set of 128 Walsh functions, 35 of the possible 36 cross-products remain orthogonal.) Extrapolating, this implies that to have all products orthogonal with M=64 antennas, we would need to choose from an initial Walsh set of $N=2^{63} \sim 10^{19}$ functions. This is clearly not feasible.

With M=64 antennas chosen from N=128 functions, although the best possible choice of functions allows 1621 of the 2016 products to remain orthogonal in the presence of a time slip, there are many different combinations that give that same optimum count of 1621. (Note that there are $\sim 10^{37}$ ways of choosing a combination of 64 functions from a total set of 128, where order is disregarded.)

No rigorous proof is presented for the conclusions presented here. However, different, independent algorithms have been used to derive the numbers, which all give consistent results. This gives some confidence in the results, without being in any sense a proof.

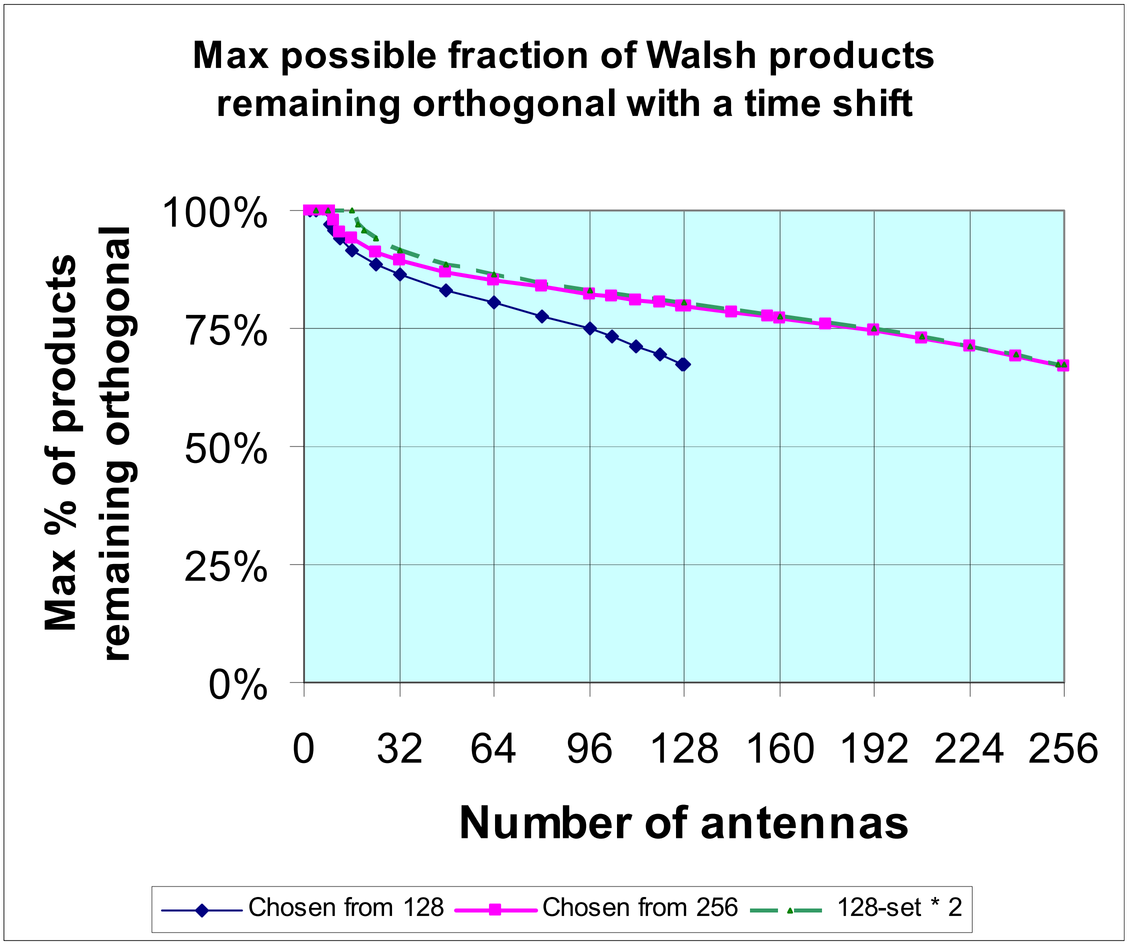

Figure 3 A plot of the maximum fraction of orthogonal products, in the presence of relative time slips, of M functions, corresponding to M antennas, chosen from initial sets of 128 and 256-element Walsh functions. The figures were calculated using the "alternative algorithm" described below.

It is noted that the curve for antenna functions chosen from 256 could have been predicted fairly well by taking the 128-element curve, and expanding the horizontal scale by a factor of two; this is shown by the dashed line in the figure. Extrapolating, it is seen that it is a law of diminishing returns: even providing a choice from a 512-element Walsh function set would give only a marginal improvement.

4 Optimizing for highest number of zero cross-products

For 64 antennas, there are (64.63)/2 = 2016 possible cross-products; for 50 antennas (50.49)/2=1125 cross-products.. From the complete set of 128 functions, there are 5461 cross-products from the possible 8192 pairs that result in a zero cross-product , remaining orthogonal in the presence of a relative time shift. However, it is still not possible to find a set of 64 functions, chosen from 128, where all possible cross-products within that set simultaneously remain orthogonal.

The initial criteria adopted for finding an optimized set of M (e.g. 64 or 50) functions chosen from 128 are:

(1) Identify a large number of sets of M functions chosen from 128 where the maximum possible number (i.e. 1621 for M=64, or 1014 for M=50) of the internal cross-products remain orthogonal to each other, in the presence of the small time slip.
(2) From these sets of M functions, calculate the summed squares of crosstalk from all available cross-product within each set – i.e. contributed by those cross-products within the set that are not perfectly orthogonal anyway.

The optimum set of M functions is then the one, containing the maximum number of orthogonal cross-products that simultaneously shows the smallest summed squares of crosstalk. (It is shown in Section 5 below that these are not necessarily the best set of criteria.)

More than 10^6 such sets, all independent, have been identified and analyzed.

4.1 Algorithm for finding sets of functions with the maximum number of orthogonal cross-products.

The number of combinations to be available is ${}_nC_r$, or n!/{(n-r)!.r!}. For n=128 and r=64, this is ~2.4×10^{37}. For 50 antenna array, r=50 and the number of combinations is ~10^{36}. It is clearly not feasible to examine every single combination of 64 (or 50) functions chosen from the total set of 128. In the absence of a theory to guide the optimum choice of 64 or 50 functions from 128, a Monte Carlo optimization algorithm is adopted.

First, a lookup table is computed to identify whether a given cross-product has zero crosstalk or not. The principle summarized in the Appendix is used to do this, although the results could also have been taken directly from ALMA Memo 537.

Then, a random set of 64 indices, corresponding to Walsh functions for 64 antennas, is computed. The 64 indices range from 0 to 127, and there are no duplicates within the set. All possible (2016) cross-products are examined, and with the help of the lookup table the number of non-zero cross-products is noted.

In the random set of 64, for each antenna in turn, the Walsh index for that antenna is scanned to find a new index giving the lowest number of zero cross-product for all possible 2016 products within this modified set. The index value giving the highest count of zero cross-products is chosen. If there are several index values for this antenna giving the same maximum count, then that index closest in value to the original random index is chosen.

After this optimization has been followed for each antenna, the steps are repeated once more for this initial, optimized set of 64 functions, looking for further optimization, on this antenna-by-antenna basis. Finally, this entire process is repeated many times, starting each iteration with another, different set of random indices

After running this entire optimization process for $>10^6$ iterations, we have obtained $\sim 10^6$ optimized sets of 64 functions. It is found that **every optimized set of 64 shows exactly the same number of zero crosstalk products.** For the case of 64 functions chosen from 128, this is always precisely 1621. Since precisely the same number is always found after $\sim 10^6$ iterations of optimization, it is postulated without proof that this is the maximum value possible. The $\sim 10^6$ sets represent of course a very small fraction of the $\sim 10^{37}$ sets available. Nevertheless, when the same algorithm is applied to special cases where all combinations can be examined individually – e.g. choosing functions for 8 antennas from a set of 16 functions, or even choosing functions for 127 antennas from 128 functions, the comprehensive count of non-zero products gives an identical maximum count to this Monte Carlo optimization process. Although by no means a proof, this does give some confidence to the result.

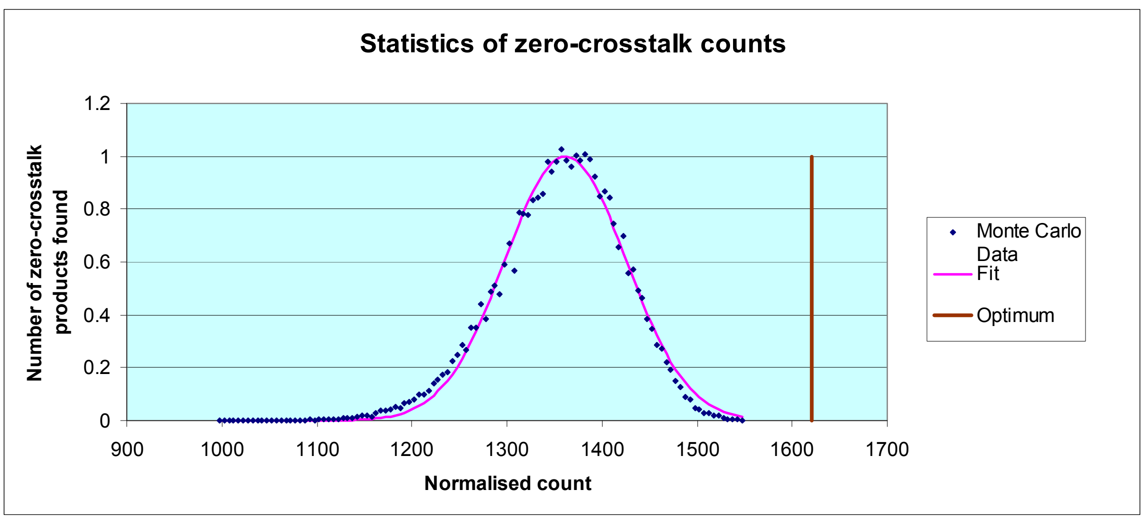

Figure 4 A histogram showing the relative number of times that a given count of zero cross-products has been found, using non-optimized random sets of 64 functions chosen from 128. The solid line shows a Gauss fit to the distribution. The optimized count of 1621 zero cross-products is shown by the vertical line

Figure 4 shows a statistical result of how many zero cross-products occur in totally random, non-optimized sets of 64 functions chosen from 128. This plot was derived from $\sim 10^5$ random sets of 64 functions. The central peak gives a most probable value of 1362 non-zero products. Note that the **total** number of zero-crosstalk pairs available after choosing N=64 functions from an N=64-element Walsh set is $(N^2)/3= 1365$. In other words, had we used an original 64-element Walsh function instead of 128, meaning that all possible cross-products would be needed and used, the resulting count of zero crosstalk pairs, would be essentially the same as using our initial 128-element Walsh function, but having chosen the subset of 64 functions from 128 purely randomly with no optimization.

It is clear that a purely random selection criterion, without any optimization, is statistically unlikely to stumble on to a combination with the optimum combination of 64 functions to give the maximum number of zero cross-products.

4.2 An Alternative Algorithm for optimizing the number of zero cross-products

The above procedure has started with a random choice of 64 functions for 64 antennas, without duplication, and then optimized the function corresponding to each antenna in turn, to maximize the total number of zero cross-products within the complete set of 64 functions.

An alternative procedure was also tried, but found an identical maximum number of zero cross-products. This starts with N=128 functions, but in random order, and then successively rejects those functions contributing the smallest number of zero cross-products, until only M (e.g. 64) functions remain.

It has been found that this process also gives sets of functions with the same maximum number of zero cross-products as had been found before, i.e. 1621 if choosing for 64 antennas from a set of 128 functions, 1014 choosing for 50 antennas from 128 functions, and 1365 non-zero functions choosing for 64 antennas from 64 functions. This gives some additional confidence that the original algorithm described in section may also be valid.

Note that the result found is not the same as the rather simpler operation of just removing the worst 64 functions from the 128, in one step. The intermediate recalculation of numbers of zero cross-products for all remaining functions, as the total number of functions being considered changes, is essential. Note also that in most steps, there is no unique “worst” function to be removed, but several that are equally bad. The choice of which “equally bad” function to remove is arbitrary, but in all trials undertaken the process does nevertheless lead to the same optimum number of totally orthogonal cross-products.

4.3 Optimum Sets of Functions for M=64 Antennas

An optimized set of 64 functions, chosen from the full Walsh set of 128, is required. The algorithm described in section 4.1 was used. Figure 5 is a histogram showing the relative likelihood of a given level of RSS crosstalk occurring from $\sim 10^6$ sets of functions that have been preselected to give the maximum number of zero crosstalk products. With this preselection of sets, the most probable value of crosstalk is 3.79%, but the best set (in the left wing of the figure) shows only 3.41%, and the worst at the right is 4.3%.

The worst single cross-product of any two functions is WAL(63) with WAL(64), which for a 1% time shift gives 1% crosstalk. Interestingly, every one of the ~10^6 sets of 64, each optimized to give the same maximum number (1621) of zero cross-products within the set, contain both WAL(63) and WAL(64). (Although WAL(63)*WAL(64) is the worst single pair at 1%, there are other pairs of products nearly as bad, such as .97%, .94%, .91%, .88% and so on.)

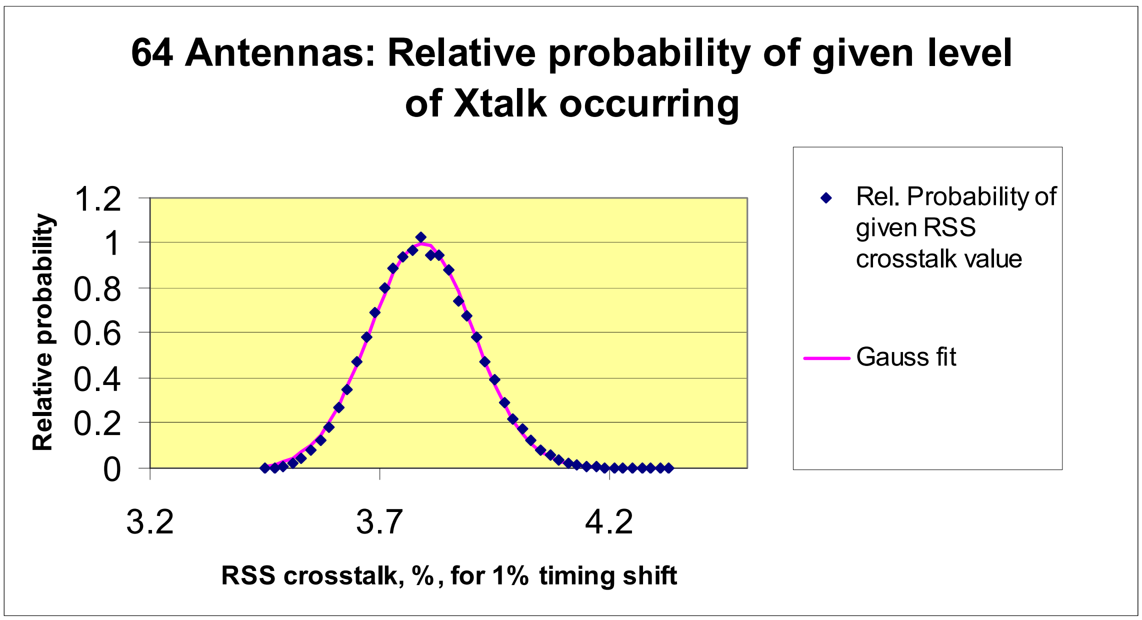

Figure 5 From about 10^6 successful tries with sets of 64 functions selected to contain the maximum possible number of zero crosstalk products, the relative occurrence of RSS crosstalk values from the remaining non-zero products. The most probable value of crosstalk is 3.79%, but the optimum set found shows an RSS crosstalk of only 3.41%, given the 1% relative timing slip.

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A similar process was followed for M=50 antennas; it was possible to find a set of 50 functions, optimum for M=50 (RSS crosstalk 2.6%), that is for a subset of the optimum set found for M=64.

For the case where optimization is chosen to give the maximum number of zero cross-products, the recommended list of functions for 64 antennas is given in Table 1. The 14 functions which should be omitted for the case of M=50 antennas are shown in an italic font.

Table 1

0	**1**	*2*	**3**	**4**	**7**	**8**	*11*	*12*	**15**	**16**	*22*	**23**	**24**	**31**	**32**	**34**	**35**	**37**	*39*	**40**	**44**
47	**48**	**51**	**52**	*55*	**56**	**59**	**61**	*62*	**63**	**64**	**67**	**69**	*71*	**72**	**79**	**80**	**81**	**84**	*87*	**88**	**89**
91	**94**	**95**	**96**	**103**	*104*	**111**	**112**	**114**	*115*	*116*	**119**	**120**	**121**	**122**	**123**	*124*	**125**	*126*	**127**	**-**	**-**

These form an optimized set of WAL indices for 64 functions chosen from a full set of 128, to give the maximum number of zero cross-products. For the best 50 functions, omit those given in italics.

Nevertheless, although the functions listed in Table 1 provide the largest number of zero cross-products with a reasonably low cumulative crosstalk, they are not the best choice. Having the maximum number of zero cross-products gives a higher than necessary value for cumulative crosstalk, as demonstrated below in Section 5.

5 Optimizing for lowest crosstalk, ignoring the number of zero cross-products

The probability distribution shown Figure 6 was derived, by generating ~10^7 sets of 64 randomly chosen functions but without duplicates, each function chosen from the full set of 128. Note that there are ~2 x 10^{37} ways of choosing 64 items from 128 where ordering is not important, so inevitably this represents a small sample of the total number of

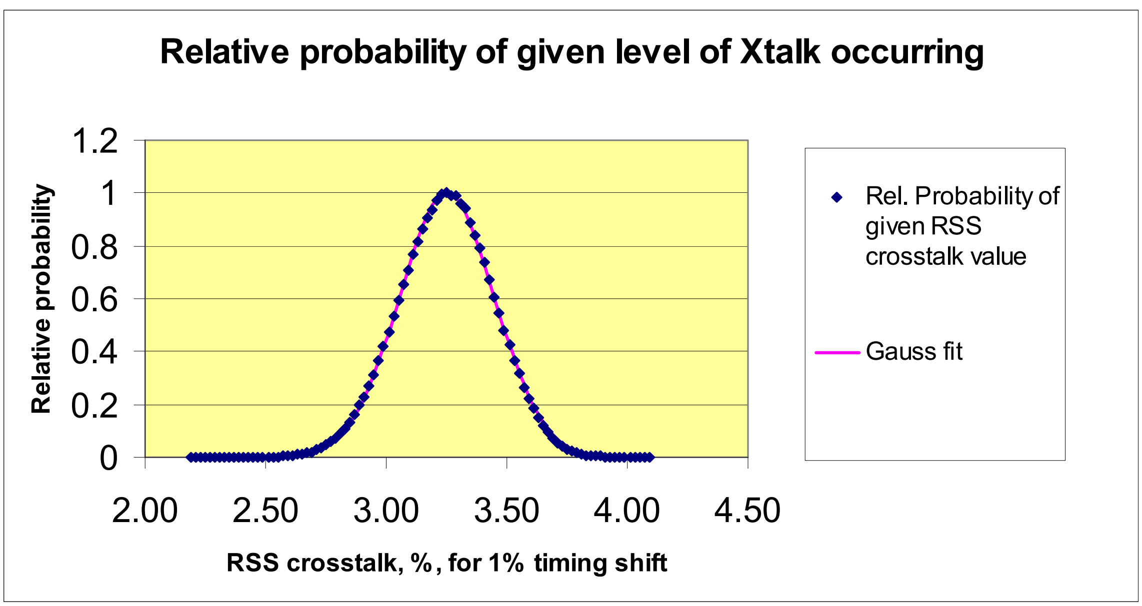

Figure 6 Relative number of random sets producing a given level of crosstalk

possibilities available. Nevertheless, a clear Gaussian-like probability distribution is seen, with the most probable level of crosstalk being 3.25%; the half-width of the probability distribution is +/-0.23%. This is more favorable than the most probable value of 3.79% +/-0.13 found in Section 4.3 and seen in Figure 5, where the choices were preselected to sets containing the maximum number of zero cross-products.

Choosing the best (i.e. lowest RSS crosstalk) 0.25% of the 64-function subsets, Figure 7 is a histogram showing the relative occurrence of specific Walsh (WAL numbering) functions. The overall shape of the distribution is reminiscent of the distribution of total RSS crosstalk of products derived from individual functions combined with all other functions, shown in Figure 2. Clearly subsets including preferentially the lowest WAL indices and the highest WAL indices are statistically more favorable than those containing WAL indices from the middle of the available range. Note however that there is a weak but very broad peak in the occurrences, at the trough in the middle of Figure 7 of WAL indices between about 50 and 80, so the situation is a little more complicated. However, the results seen later in Figure 11 below show, qualitatively, a rather similar tendency.

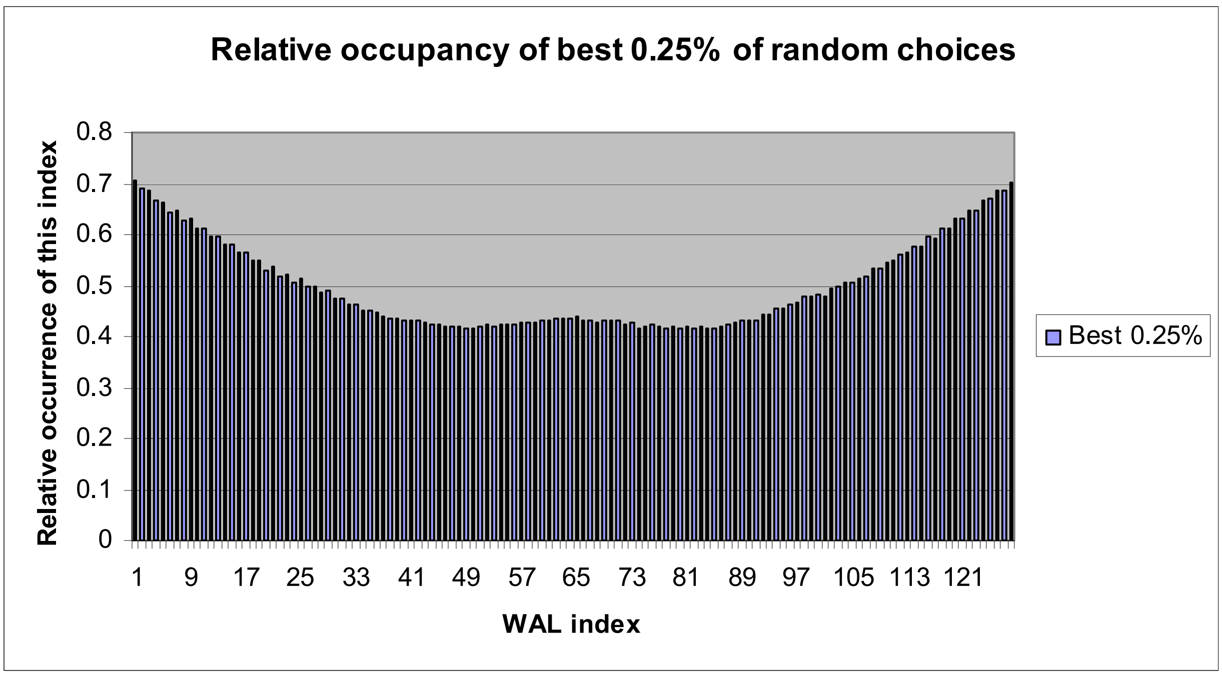

Figure 7 Relative occurrence of particular Walsh indices, best 0.25% of RSS crosstalk

Figure 8 below is similar to Figure 7, but chosen from the worst (highest RSS crosstalk) 0.25% of the randomly chosen subsets of functions. As anticipated, the highest, and the lowest, WAL indices occur relatively less often at the higher levels of total RSS crosstalk.

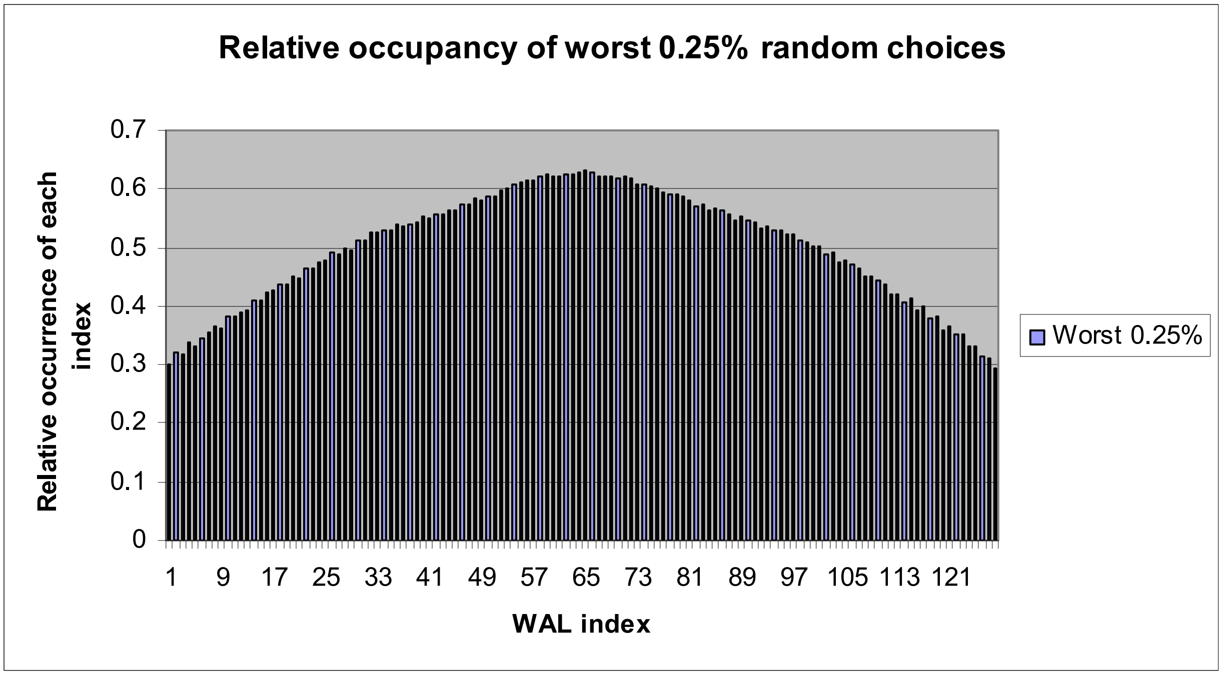

Figure 8 The worst 0.25% sample of random sets of 64 functions

The following optimization process was adopted:

1. Start with a randomly chosen subset of 64 functions chosen from the full set of 128 functions, corresponding to the initial guess of functions for 64 antennas.
2. Calculate the RSS of crosstalk of all possible cross-products of the 64, using the method described in ALMA Memo #537.
3. For each antenna in turn, try a substitution of each of the other 127 functions, to see if a lower RSS for all products of the 64-element subset can be obtained. Where there are several possibilities giving equally good RSS crosstalk, adopt that function with the lowest WAL index for that given level of crosstalk. This ensures the lowest loss of sensitivity, making use of the trend shown in Figure 1.
4. After optimizing the individual functions for each of the 64 antennas in turn, repeat the process once more to see if an even lower RSS crosstalk can be obtained. The result is one 64-element subset optimized for lowest total RSS of all possible cross-products within that set, but also optimized for the lowest loss of sensitivity, subject to the lowest RSS crosstalk.
5. The entire process is repeated from step 1, with a new set of 64 randomly chosen functions.
6. The above process 1-5 was run for $\sim 10^7$ iterations. As well as starting with randomly chosen functions, the same procedure was also tried starting with obvious non-random subsets – such as starting with WAL functions 0 to 63, or with functions 64 to 127 inclusive, or with functions 0-31 and 96-127. **In all tries,** whether starting with a random seed or with a carefully defined non-random seed, the optimization process yielded the same best (lowest) RSS crosstalk of 1.824%, in the presence of the same 1% time slip. Many independent solutions were always found with the same low value of RSS crosstalk, but further, **in all**

tries, after choosing one solution with this lowest value of RSS crosstalk that also has the lowest loss of sensitivity, one of two specific final solutions appeared. This empirical approach does not guarantee the best solution, although the fact that in ~10^7 tries the convergence was always to the same solutions is suggestive. The two possible solutions are, in WAL indices:

0 – 31 , 47 – 63, 113 – 127

or equally,

0 – 31 , 47 – 62, 64, 113 – 127

The only difference between these two solutions is that WAL(63,t) has been replaced by WAL(64,t), or equivalently that SAL(32,t) has been replaced by CAL(32,t); recall that CAL or SAL functions with the same index are subject to the same loss of sensitivity in the presence of a time slip. Interestingly, WAL(63,t) and WAL(64,t) are the particular pair of Walsh functions whose product gives the worst individual crosstalk of 1%, in the presence of a 1% timing slip. This fact may be useful as a tool in adjusting hardware or software for a null in crosstalk, and so to optimize hardware and software timing.

For the 1% time slip, the optimum solutions yield an RSS crosstalk of 1.82%, an average sensitivity loss of 0.79%, and contain 1366 zero cross-products out of the total of [N.(N-1)/2] = 2016 cross-products, or 66.7% of the total number of products ($N^2/2$).

There are many other solutions that are equally good in terms of RSS crosstalk, but which may have worse loss of sensitivity. For example, the choice 0-14, 64-80, 96-127, which is a mirror image of the first optimum choice, has an identical RSS cumulative crosstalk but gives a worse overall sensitivity loss of 1.2%.

There is a tendency for the WAL indices in optimum sets of functions, defined as those giving the lowest RSS crosstalk, to cluster in blocks of 8 or 16 consecutive indices. This is not unexpected, especially considering the regular criss-cross patterns seen in Figure 3 from Memo #537, reproduced below as Figure 9. Although it is not feasible to examine all ~10^{37} possible selections of 64 functions chosen from 128, if the WAL indices are blocked into 16 groups each having 8 consecutive indices (i.e. 0-7,8-15… 120-127), the problem becomes the relatively simple one of choosing 8 blocks of functions from 16, giving only 12870 independent possibilities. From these 12870 choices, it was found that precisely 64 sets gave the same, low value of RSS crosstalk of 1.82%, given the same 1% time slip. The function also giving the lowest WAL indices, so the lowest sensitivity loss, consists of WAL 0-31, 48-63, 112-127. This is the same as was found via the random optimization process described above. All 64 solutions are shown graphically in Figure 10, where the different sets are ordered from bottom to top in order of increasing sensitivity loss. The lower set, with index "1", is identical to that already found from the optimization process described above.

The sets of functions shown in Figure 10 were summed vertically, to give the number of times that a given Walsh function occurs over all of these 64 optimum sets. For example, it is seen that WAL 0-15 and 112-127 are represented in every single set, while WAL indices 16 - 47, and 80 - 111, occur relatively less seldom. Figure 11 illustrates the result. It is interesting to compare this to Figure 7, showing a similar display for sets for 0.25% sets of functions, selected for lowest crosstalk. Qualitatively, both show the strong preference for the lowest and for the highest WAL indices, and both show a tendency for a broad central peak, with slightly enhanced representation from functions closer to WAL index N/2.

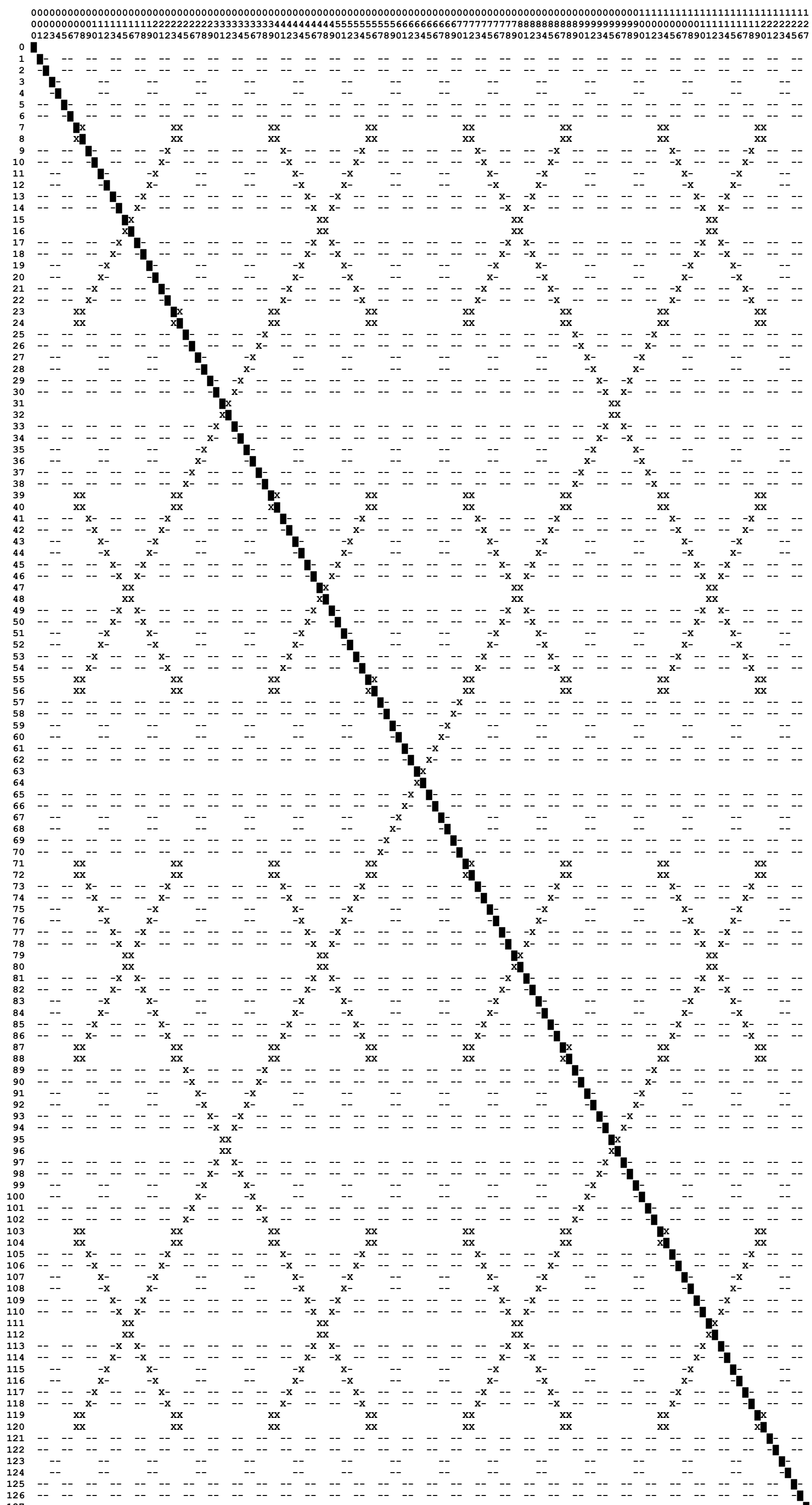

Figure 9 (from ALMA Memo #537). Amplitudes of cross-products of different 1% time shifted Walsh functions, inWAL order. Amplitudes of >-0.1 dB, -20 to -0.1 dB, and -30 to -20 dB are shown with "▮", "x" and "-". Weaker than -30 dB is left blank.

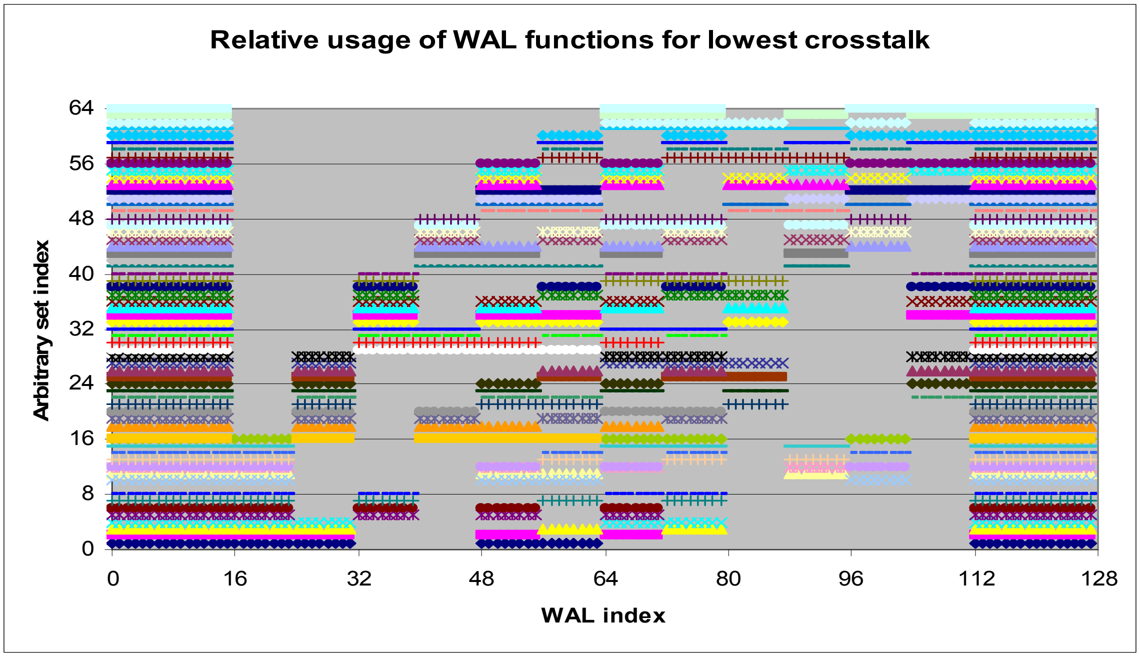

Figure 10 A graphical representation of all choices giving the minimum value of RSS crosstalk, when blocks of functions are put into 16 groups each of 8 consecutive indices. (The different plotted colors correspond to different set indices of the vertical axis.)

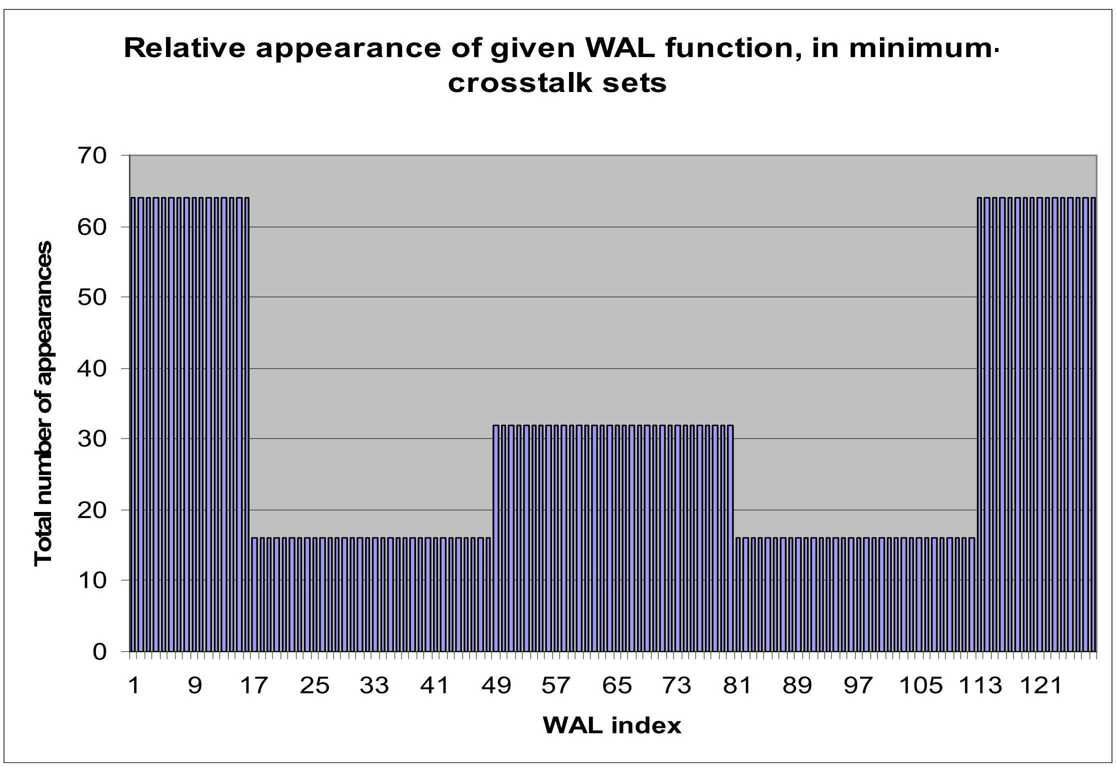

Figure 11. Number of occurrences of a given Walsh function in all 64 of the sets shown in Figure 10

6 Conclusions

In the presence of a relative time slip between the Walsh modulation and demodulation signals, there is both a loss of sensitivity and loss of immunity to crosstalk. The particular case of 64 functions chosen from a set of 128 has been studied.

1. For a time slip of 1% of the smallest Walsh interval (1.25 microseconds in the ALMA context), the total loss of sensitivity is between 0.5% and 1.5%, depending on the specific Walsh functions selected.

2. Two different approaches to minimizing crosstalk have been examined:

    a. First selecting functions that retain the maximum number of cross-product pairs with zero crosstalk in the presence of a time slip, and then using a Monte Carlo method to identify sets of these preselected functions that give the lowest crosstalk amplitude. With 64 functions chosen from 128, a maximum of 1621 of the total of 2016 cross-products can remain orthogonal. The best set of functions found satisfying that criteria give a cumulative crosstalk RSS amplitude of 3.4%. A set of functions has been identified where the optimum set of 50 (suitable for a 50-antenna array) is a subset of this set of 64. However, this pre-selection by maximizing the number of zero cross-products does not lead to the lowest cumulative crosstalk.
    b. Without pre-selection, starting with a random choice of 64 functions chosen from 128, then optimizing to find the lowest RSS cumulative crosstalk, many sets of functions giving identical total RSS values of 1.82% have been identified. The lowest sensitivity loss from these sets is 0.8%. This proposed, optimum set of 64 functions chosen from 128 consists of **WAL indices 0-31, 47-63** and **113-127.** This set includes 1366 zero cross-products out of 2016 cross-products.

3. If Walsh functions are chosen at random with no pre-selection or optimization, the most probable loss of sensitivity with this time slip is 1%, with a most probable RSS crosstalk value of 3.25%.

4. Although the set of functions suggested above may be the most appropriate, the final choice of function – whether to optimize based primarily on sensitivity, or whether it is important to have maximum the number of zero cross-products – is a broader, systems-level decision. Reassuringly, degradation in performance is relatively slight even with a poorly chosen subset of functions. It may nevertheless be worth using those Walsh functions with the poorest average spurious suppression performance on the longest baselines, where the natural higher fringe and delay rates will somewhat make up for the reduced suppression.

The results of different selection strategies are summarized in Table 2 . The most recommended choice overall is given in the last row of the table.

Table 2: Summary of Results

Criteria for choosing the subset of 64 functions from the total set of 128 Walsh functions	RSS Crosstalk level	Number of zero products	Total # of products (excluding self-products)	Total Sensitivity Loss	The set of functions: WAL indices	Notes
Randomly chosen, no optimization, most probable result	3.25%	1362	2016	1%	Most subsets of 64 functions randomly chosen from 0-127	1
Random seed, selecting only sets having the maximum number of zero cross-products	3.79%	1621	2016	1%	(Not useful)	1,2
Random seed, then optimize for max number of zero products, then minimize RSS crosstalk	3.41%	1621	2016	1%	See Table 1	3
Random seed, then optimize for max number of zero products. **Worst** crosstalk then could be:	4.3%	1621	2016	1%	(Not useful)	4
Lowest possible sensitivity loss, ignoring crosstalk	2.31%	1365	2016	0.50%	WAL 0-63	5
Worst possible sensitivity loss, ignoring crosstalk	2.31%	1365	2016	1.50%	WAL 64-127	5
Random seed, then optimize for minimum RSS crosstalk, then minimize sensitivity loss	1.82%	1366	2016	0.80%	WAL indices 0-31,47-63,113-127	6

Table 2: Summary of results

1. Results given are the most probable result of a random choice of functions. See Figure 4 & Figure 6.
2. See Figure 5. RSS crosstalk is the most probable value within the given criteria. Sensitivity loss is the most probable value.
3. See Figure 5. RSS crosstalk is the lowest found according to the given criteria after ~10^7 tries
4. See Figure 5. Worst RSS crosstalk found in ~10^7 tries.
5. See Figure 6 Sensitivity loss calculated according to section 2.
6. This is the recommended set of functions for M=64 antennas chosen from the full set of N=128 functions. See Section 5.

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APPENDIX

Derivation of the fraction of Walsh-function pairs remaining orthogonal in the presence of a small[2] relative time shift, choosing all possible pairs from a set of N functions

It was found in ALMA Memo #537 that all combinations of an odd PAL index with an even PAL index remain orthogonal in the presence of a small[2] relative time shift. It was also found that no odd-odd pairs remain orthogonal. The following is a non-rigorous derivation of the fact that from a set of **N** Walsh functions, $\mathbf{N^2/3}$ of the possible cross-product pairs remain orthogonal in the presence of a small time shift.

The odd-even orthogonality can be understood by considering the component Rademacher functions whose products generate any set of Walsh functions. Any Walsh function with an odd PAL index includes the product with the Rademacher function R(1,t), which is identical to PAL(1,t). This means that the second half of the complete cycle of that function is precisely the negative of the first half. For an even PAL index, R(1,t) is not a component product, so the second half of the complete cycle of the even function is a precise copy of the first half.

When an even function is multiplied by an odd function, but including a small relative time shift, taking just the first half of the function there may be a finite residual in the averaged cross-product caused by the time shift. However, in the second halves of the functions, the odd function now has now changed sign compared to the first half of itself. So, the finite residual in the averaged cross-product of the second halves of the two functions will now be equal in magnitude but opposite in sign to that in the first half. So, the residual in the averaged cross-product of the complete functions, for any even with any odd PAL index, will be identically zero. The two functions remain orthogonal in the presence of the relative time shift.

If two functions are both even, they may nevertheless still be orthogonal in the presence of a time shift. Still considering the cross-products of the first halves of complete function, it is noted that (e.g.) the first half of PAL(31,t) chosen from a set of 32 Walsh functions is identical to PAL(15,t) chosen from a set of 16 Walsh functions. The separate halves of any Walsh function make another Walsh function. So, the odd-even criterion can be applied to each half of the complete functions. If the cross-products of each half

[2] In this context "small" means smaller than the smallest element length of the Walsh function. In the context of ALMA, this means smaller than 125 microseconds. The effect on orthogonality of even larger relative time slips, as will often occur for example in the formation of lag functions for high spectral resolution observations, has not been investigated.

of a given pair of functions are orthogonal, then the cross-product of the complete functions will also be orthogonal.

The total number of orthogonal products of N functions can then be derived as follows. First, count product A*B as being distinct from product B*A, where A and B represent arbitrary Walsh functions. Possible products are then odd-odd, odd-even, even-odd and even-even. There are then N^2 possible products, including self-products.

1. All odd-even and even-odd pairs are orthogonal. Half all possible products are a mix of odd and even, so $N^2/2$ of the N^2 possible products are accounted for.
2. Odd-odd pairs are never orthogonal. This applies to $N^2/4$ of possible products, so these pairs can be rejected.
3. Even-even pairs may or may not still be orthogonal. There are now $N^2/4$ products not accounted for by criteria 1 or 2 above, which should be examined further.
4. Divide these remaining $N^2/4$ functions into half-functions, taking the Walsh functions derived from the first and second halves separately. If we can show that first halves of the original functions are orthogonal with each other, then the second halves must also be orthogonal, and so the entire functions are orthogonal. (Since we are only considering even functions, the second half of each function is always an exact repeat of the first half.)
5. Apply the criteria 1, 2 & 3 to these half-functions. One half of the $N^2/4$ functions will satisfy the odd-even criterion, and one quarter of the $N^2/4$ can be rejected by the odd-odd rule, leaving one quarter of the $N^2/4$ to be considered further.
6. From the $N^2/16$ remaining functions, apply the steps 4 and 5. One half of the $N^2/16$, i.e. $N^2/32$, will be identified as orthogonal.

Continue the above sequence 4-5-6 until the Walsh functions can no longer be subdivided into two halves. From successive steps, we will have identified

$$N^2/2 + N^2/8 + N^2/32 + \ldots$$

orthogonal pairs. This is a geometric series with successive terms being multiplied by (1/4). In the limit, as N becomes large, using the well known formula for the sum of an infinite geometric series, the sum becomes $(N^2/2).1/(1-\{1/4\})$ which simplifies to $(N^2/2).(4/3) = 2.N^2/3$.

Now in the above we have counted a product A*B as distinct from B*A, so the total number of orthogonal product pairs remaining from a set of N Walsh functions, in the presence of relative time slips, now becomes simply $\mathbf{N^2/3}$.