NATIONAL RADIO ASTRONOMY OBSERVATORY
COMPUTER DIVISION
INTERNAL REPORT

COMMENTS ON THE FREQUENCY TO VELOCITY CONVERSION IN THE 413 CHANNEL AUTOCORRELATION RECEIVER PROGRAMS

BY

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SUMMARY:

In the 413 channel autocorrelation receiver programs, frequencies are converted into velocities by using the equation for the relativistic Doppler effect. The high frequency resolution seems to justify this very accurate treatment of the data.

I doubt that anything is gained by such a refinement, and I think that only confusion is caused. I propose to compute radial velocities in the classical way, with moderate accuracy, but to calculate and print the redshift parameter z with high accuracy.

1. INTRODUCTION

In one of the computer programs* developed at the NRAO for the new 413 channel autocorrelation receiver, the apparent radial velocity, u, is calculated with the equation

$$u = c \frac{\nu_o^2 - \nu^2}{\nu_o^2 + \nu^2} \qquad (1)$$

Here, ν is the apparent frequency and ν_o is the rest frequency. Equation (1) is the rigorous solution of the equation

$$\nu = \nu_o \cdot \frac{1 - \frac{u}{c}}{\sqrt{1 - (\frac{u}{c})^2}} \qquad (2)$$

the relativistic representation of the doppler effect as seen by an observer who moves with velocity u along the line of sight.

I do not know who originally made the request to incorporate the relativistic doppler effect in standard data reduction programs. But, as a matter of fact, the existing programs carry out the frequency to velocity conversion in this manner, and at least some of the users know about it and seem to appreciate it.

To my knowledge, all radial velocities published until now were obtained from observed wave length or frequency shifts by using the classical doppler effe t equation. The reasons for this are quite clear: The precision of measurements was much too small to detect relativistic effects in most of the applications. An exception is extragalactic objects with large redshifts, but here it is the general practice to discuss the redshift parameter z (which follows in a direct way from the observations, without making any hypothesis) cosmologically, rather than radial velocity.

The new receiver represents a big step forward in terms of frequency resolution, and this is certainly the reason for the relativistic handling of the observed frequency: Why should we not include a well known effect in our data reductions if our measurements are precise enough to detect it?

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*AC4142; see Computer Division Internal Report No. 5, section VI A.

In principle, there is nothing wrong with such a decision. However, one must bare in his mind that the inclusion of a relativistic term in radial velocities is a break with tradition. Such breaks are by no means unusual in astronomy. For example, when enough evidence existed for the modern accuracy of star positions, second order terms had to be introduced in the normal procedure of reducing an apparent place to a mean place. However, this change was very carefully discussed and did become effective only after international agreement. The reasons for this carefulness are very much clear: One could foresee that a large amount of fundamental data would be collected in the future, to be used for all sorts of basic investigation, and all these data would be affected by such a decision.

I don't know how important it is to take standard computer programs at NRAO very seriously, in the sense of what I have just tried to describe. But there is no question that during the coming years many astronomers will use these programs with the feeling that very accurate observed data are reduced in a very accurate manner. This seems to be enough justification to ask two questions:

(a) Does the improvement of frequency resolution, as given by the NRAO 413 channel autocorrelation receiver, justify the inclusion of relativistic terms in the frequency to velocity conversion?

(b) What are the principle astronomical problems which may occur if one leaves the traditional way?

It is clear that if the answer to question (a) is "No", we can stop these considerations. We should then, of course, eliminate the relativistic features from our standard computer programs. At least we would gain some computer time and computer memory. Also, we would not betray ourselves in terms of accuracy.

If the answer is "Yes", however, we should be aware that we are going to break with an old tradition, that we are planning to make much more accurate observations, that we correspondingly treat these data in a way different from the previous one, and that a later interpretation of the data might lead to new results which never could have been obtained with the old equipment and with the old methods. In other words, if the answer to question (a) is "Yes", we should start thinking about question (b).

2. RELATIVISTIC VELOCITY TERMS AND THE FREQUENCY RESOLUTION OF THE 413 CHANNEL RECEIVER

If the source of the ν_o - radiation moves along the line which connects source and observer, the rigorous relation between ν_o, ν and u is given by equation (2). Let us first see what the classical equivalent is. In optical astronomy, the so-called redshift parameter z is defined by:

$$z = \frac{\lambda - \lambda_o}{\lambda_o} = \frac{\nu_o - \nu}{\nu}^{*} \tag{3}$$

Radio astronomers seem to prefer a slightly different definition:

$$z_R = \frac{\nu_o - \nu}{\nu_o}^{*} \tag{4}$$

In a relativistic first order theory, the two definitions are completely equivalent. This is clearly seen if we compute z and z_R by using the relativistic equation (2):

$$z = \frac{\sqrt{1 - (\frac{u}{c})^2}}{1 - \frac{u}{c}} - 1 = \frac{u}{c} + \tfrac{1}{2}(\frac{u}{c})^2 + \tfrac{1}{2}(\frac{u}{c})^3 + \cdots\cdot \tag{5a}$$

$$z_R = 1 - \frac{1 - \frac{u}{c}}{\sqrt{1 - (\frac{u}{c})^2}} = \frac{u}{c} - \tfrac{1}{2}(\frac{u}{c})^2 + \tfrac{1}{2}(\frac{u}{c})^3 - \cdots\cdot \tag{5b}$$

By conversion of these power series we obtain:

$$u = cz_R + \tfrac{1}{2}cz_R^2 \cdots\cdot \tag{6a}$$

$$= cz - \tfrac{1}{2}cz^2 \cdots\cdot \tag{6b}$$

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*In classical physics, these two definitions correspond to the two cases: (a) Observer rests, source moves. (b) Source rests, observer moves. In relativity, this distinction becomes meaningless.

Which one of these two we are using is not at all important. Let us use the first one (6a). Then:

$$cz_R = \text{"classical term"}$$

$$\tfrac{1}{2}cz_R^2 \cdot \cdot = \text{"relativistic term"}$$

In the following considerations, we assume that the velocities involved are small enough to let

$$\boxed{u_{rel} = \tfrac{1}{2}\, cz_R^2} \qquad (7)$$

be a reasonable approximation of the relativistic term. This is the term which, essentially, does now exist in the computer programs. We have to compare it to the velocity resolution of the 413 channel receiver. The receiver gives the following

$$\text{"channel spacing"} \quad \delta f = \frac{\text{Bandwidth}}{\text{Number of channels}}$$

A table of all possible values of δf can be found in NRAO Electronics Division Report 75, p. 5. For the frequency resolution we may then write:

$$\text{"frequency resolution"} = \delta\nu = a \cdot \delta f \qquad (8)$$

where the numerical factor a is of the order of 1.

What the actual frequency resolution is in the case of an observation should be determined from the obtained line profile. Let us keep, therefore, the $\delta\nu$ itself as the basic argument.

Whereas $\delta\nu$ will be approximately constant over the entire receiver band, the velocity resolution (δu) will vary with the observed frequency (ν):

$$\boxed{\delta u = c\,\frac{\delta\nu}{\nu} \qquad \text{"velocity resolution"}} \qquad (9)$$

with equations (7) and (9), we can now calculate the ratio

$$Q = \frac{u_{rel}}{\delta u} = \frac{\nu_o}{2 \cdot \delta\nu} \cdot z_R^2\,(1 - z_R)$$

which compares relativistic term and velocity resolution as long as z_R is not too large; if z_R becomes too large, equation (7) is no longer a reasonable representation of the relativistic effect. This restriction allows us then, of course, to ignore the third order term in the last equation:

$$Q = \frac{\nu_o}{2 \cdot \delta\nu} \cdot \bar{z}_R^2 \qquad (10)$$

We are now prepared to answer question (a) in the introduction:

> The relativistic term becomes detectable if $Q \geq 1$. This, on the other hand, will occur when $|z_R|$ is larger than a certain limit, or if
>
> $$|u| \geq u_{min}$$
>
> where u_{min} is a characteristic function of ν_o and $\delta\nu$.

From equation (10), we obtain easily:

$$u_{min} = 424 \sqrt{\frac{\delta\nu}{\nu_o}} \text{ km/sec} \qquad (11)$$

$\delta\nu$ = actual frequency resolution in KHz,
ν_o = rest frequency in GHz.

To illustrate this relation, u_{min} is plotted in the diagram of Figure 1. The three curves (actually, straight lines in a log-log plot) belong to ν_o = 5000 MHz, 2600 MHz and 1420 MHz. Abscissa is our actual frequency resolution, ordinate is u_{min} (km/sec). The interpretation of the diagram is simply that, for a given frequency resolution, the velocities below the curves cause a doppler effect so small that the relativistic term could not be detected by the receiver. The relativistic term is large enough for velocities above the curve. The scale on the right edge of the diagram shows the km/sec equivalent of the relativistic term. The scale along the curves gives some even values of the main term of the total doppler effect in MHz; they are, of course, on different positions along the various curves and are connected by the smaller straight lines.

An example may explain the use of this figure. We assume that we observe at ν_o = 1420 MHz (21 cm). The receiver configuration may be such that 192 channels are distributed over an 156 KHz band. The table 1 in the Electronics Division Report 75 shows that the channel spacing is δf = 0.81 KHz. If our actual

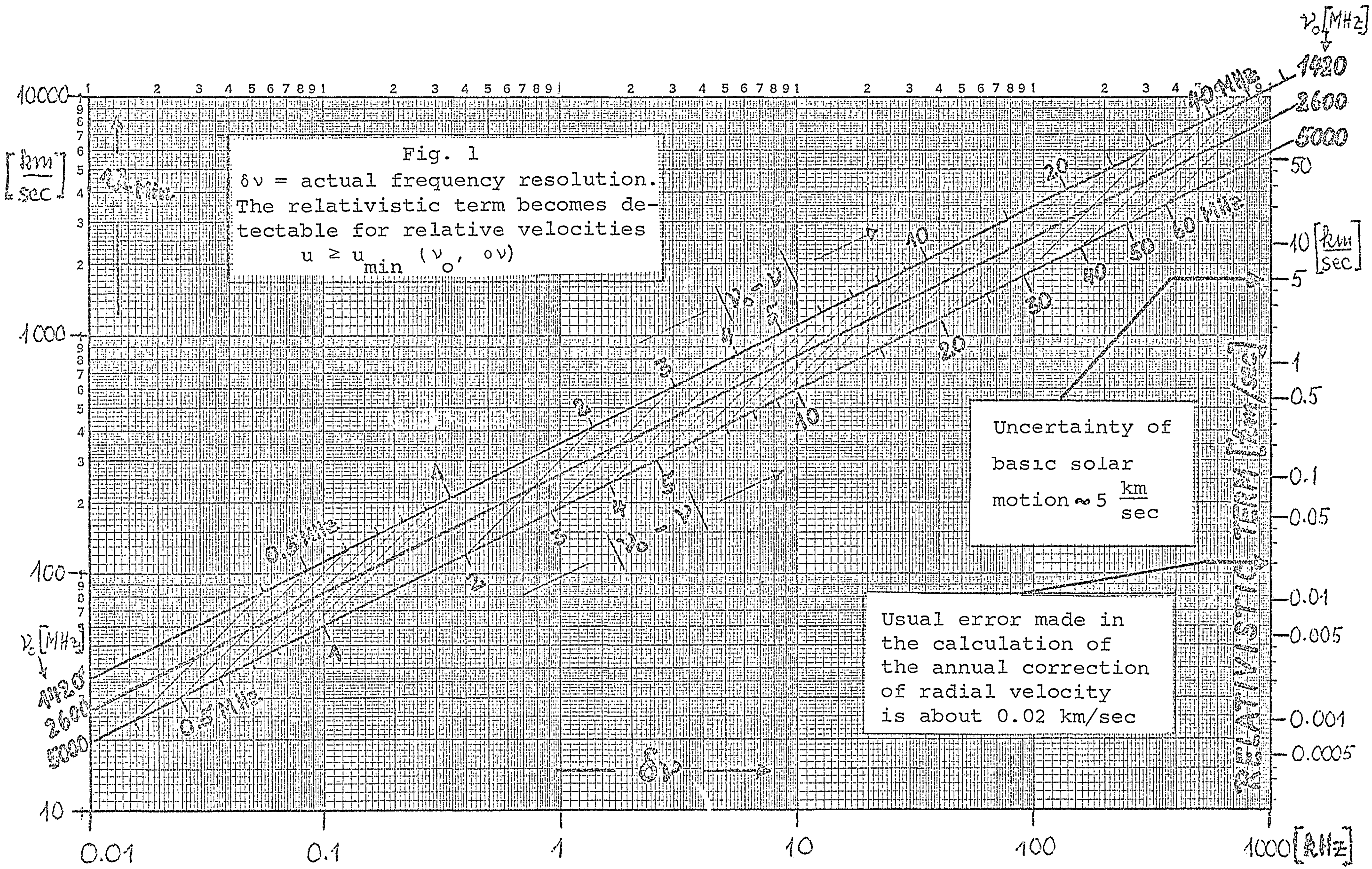

Fig. 1
$\delta\nu$ = actual frequency resolution. The relativistic term becomes detectable for relative velocities $u \geq u_{min}(\nu_o, \delta\nu)$

frequency resolution is the same, namely $\delta\nu = 0.81$ KHz, we obtain from the diagram: $u_{min} = 340$ km/sec. This means that for relative velocities smaller than that we need not include the relativistic term which here is ~ 0.1 km/sec as shown at the right scale. Only for velocities larger than 340 km/sec, or for doppler effects larger than 1 MHz (see the scale along the curve), becomes the relativistic effect significant enough. In our example, the conclusion would be that in all types of galactic work the relativistic effect is too small to be detected.

For extragalactic work where we deal with much larger velocities, the relativistic term is significant in practically all possible receiver configurations, provided that the actual frequency resolution is of the same order as the channel spacing.

In general, it seems so as if the answer to question (a) indeed should be "Yes", at least for certain types of applications.

Some additional comments should be made which do not have anything to do with the receiver. First, all observed velocities are reduced to the sun, taking into account the annual motion of the earth. In all normal computer programs, this effect is computed with an error of about 0.01-0.02 km/sec, because the periodic perturbations are not included. They could be included, of course, and the error would become much smaller, but for a standard data reduction program this would be much too elaborate. A relativistic term of 0.02 km/sec would occur for radial velocities of 100 km/sec. It is very much questionable, therefore, whether it is meaningful to calculate a relativistic term <0.02 km/sec if, in another part of the data reduction, an error of 0.02 km/sec is introduced. I have heard people saying that the standard reductions made at different observatories differ by even larger amounts, up to 0.3 km/sec. This would indicate that a relativistic term becomes meaningful only for velocities $\gtrsim 400$ km/sec, i.e., definitely only for extragalactic work.

Furthermore, in extragalactic work, as far as one wants to reduce the data to the LSR, the basic solar motion has to be computed. The motion of the sun relative to the LSR is certainly not more precise than about 5 km/sec, which means that to include relativistic terms for velocities below 2000 km/sec is meaningless.

Ignoring these difficulties which might be solved eventually in one way or the other, we have seen that the new receiver is capable, under certain circumstances, to detect the relativistic term of the doppler effect. This conclusion brings me then to section 3 where I will try to discuss some of the problems which are involved in the relativistic doppler effect, and which are usually overlooked if one does not really try to understand the definitions of special relativity.

3. RELATIVISTIC DOPPLER EFFECT AND ABERRATION, AND THE USUAL ASTRONOMICAL DEFINITION OF RADIAL VELOCITIES

Since section 2 of this report has shown that relativistic effects start becoming significant, we will discuss these effects a little bit more generally. We are talking, of course, about little effects in most practical cases; but, as I said in the introduction: Either the effects are so little that we don't like to talk about them, in which case we simply should not include them in our reductions; or, we feel they have to be included in the reductions although they are so little. Then, whether we like that or not, we must at least understand what they mean. In order to do so, we assume now the position that we are capable of measuring everything with "unlimited accuracy".

Figure 2 shows what happens. Since the observer, T , is very far away from the source, S, the electromagnetic wave can be approximated by a plane wave when it arrives at T. If the observer at T would be in rest relative to S, he would find $\vec{s}_o$ to be the propagation vector of the wave, and would describe the electric vector of the field by

$$\vec{E} = \vec{A}\cdot\exp\left[2\pi i\nu_o\left(\frac{X_o\cos A_o + Y_o\cos B_o + Z_o\cos C_o}{c} - t_o\right)\right]$$

If T, however, moves along the x-axis with velocity u, the observer would find $\vec{s}$ to be the propagation vector, and would, correspondingly, define the field by

$$\vec{E} = \vec{A}\cdot\exp\left[2\pi i\nu\left(\frac{X\cos A + Y\cos B + Z\cos C}{c} - t\right)\right].$$

We have chosen an arbitrary relative motion, u, and then have defined the coordinate systems such that:

$$x_o \,\|\, x \,\|\, \vec{u}\ ; \qquad y_o \,\|\, y\ , \qquad z_o \,\|\, z$$

This is no limitation to generality.

The relation between ν, ν_o, A, A_o, B, B_o, C, C_o is found by introducing the Lorentz transformation into the first of the two equations:

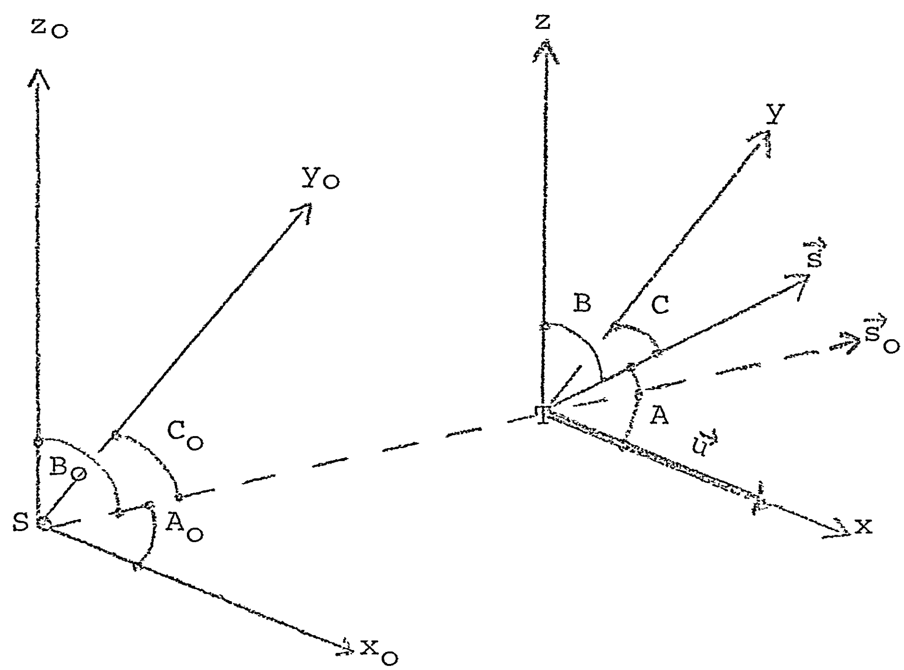

Fig. 2

S = Source, resting in system x_o, y_o, z_o

T = Observer, resting in system x, y, z

$\vec{s}_o$ = Propagation vector at T as seen by an observer resting in x_o, y_o, z_o

$\vec{s}$ = Propagation vector at T as seen by an observer resting in x, y, z

$\vec{u}$ = Motion of x, y, z relative to x_o, y_o, z_o

The angles 180-A, 180-B, 180-C describe the "apparent" position of the source. The angles $180-A_o$, $180-B_o$, $180-C_o$ describe the "true"position of the source. The difference between the two positions is the relativistic aberration.

$$x_o = \frac{x + ut}{\sqrt{1 - \beta^2}} ; \; y_o = y; \quad z_o = z; \quad t_o = \frac{t + \beta \frac{x}{c}}{\sqrt{1 - \beta^2}} ; \tag{12}$$

where $\beta = \frac{u}{c}$ (13)

This gives us the relativistic description of doppler effect and aberration:

$$\nu = \nu_o \frac{1 - \beta \cos A_o}{\sqrt{1 - \beta^2}} \tag{14}$$

$$\cos A = \frac{\cos A_o - \beta}{1 - \beta \cos A_o} \tag{15}$$

$$\cos B = \frac{\cos B_o \sqrt{1 - \beta^2}}{1 - \beta \cos A_o} \tag{16}$$

$$\cos C = \frac{\cos C_o \sqrt{1 - \beta^2}}{1 - \beta \cos A_o} \tag{17}$$

One should note that the angle which enters (14) is not the "observed" direction but the direction in the system of rest. From equation (15) we can derive:

$$\cos A_o = \frac{\cos A + \beta}{1 + \beta \cos A} \tag{15a}$$

If we substitute this in equation (14), we obtain the equivalent formula:

$$\nu = \nu_o \frac{\sqrt{1 - \beta^2}}{1 + \beta \cos A} \tag{14a}$$

From (14) and (14a) we get

$$z_R = \frac{\nu_o - \nu}{\nu_o} = 1 - \frac{1 - \beta \cos A_o}{\sqrt{1 - \beta^2}} \tag{18}$$

$$= 1 - \frac{\sqrt{1 - \beta^2}}{1 + \beta \cos A} \tag{18a}$$

For further discussions it will be helpful to develop some of these expressions in power series:

$$\cos A = \cos A_o - \beta \sin^2 A_o - \beta^2 \cos A_o \sin^2 A_o - \beta^3 \cos^2 A_o \sin^2 A_o \cdots$$

$$\cos A_o = \cos A + \beta \sin^2 A - \beta^2 \cos A \sin^2 A + \beta^3 \cos^2 A \sin^2 A \cdots \quad (19)$$

$$z_R = \beta \cos A_o - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\cos A_o \cdot \beta^3 \cdots \quad (20)$$

$$z_R = \beta \cos A - \tfrac{1}{2}\beta^2 (2\cos^2 A - 1) + \tfrac{1}{2}\beta^3 \cos A \, (2\cos^2 A - 1) \cdots \quad (20a)$$

The next step, in our consideration, must be to find a relation between the term "Radial Velocity" (as used until now in astronomy) and the relativistic doppler effect. In order to gain a clear picture of this relation, we should ignore the various standard corrections which normally are applied to observations such as diurnal correction, annual correction and correction due to basic solar motion. In other words we assume that the observer is identical with the local standard of rest. The mean position 1950.0, for instance, describes the direction to the source (in the 1950 coordinate system). Such a"mean position" is, actually, an "apparent" position because all our position catalogues contain still the individual aberration of the sources (caused by the motion of the source relative to the local standard of rest). Even if we wanted, we could not correct for this individual aberration because we do not know the direction of the spatial motion. The term "Radial Velocity" is defined as the projection of space velocity on the line of sight. In our notation, this means that

$$\text{Radial Velocity } V = u \cdot \cos A. \quad (21)$$

With the new abbreviation

$$\gamma = \frac{V}{c} \quad (22)$$

it follows that

$$\beta = \gamma \sec A. \quad (23)$$

From equation (18a) we obtain then the rigorous relation between frequency shift and radial velocity:

$$z_R = 1 - \frac{\sqrt{1 - \gamma^2 \sec^2 A}}{1 + \gamma} \tag{24}$$

Or, from equation (20a):

$$z_R = \gamma - \tfrac{1}{2}\gamma^2(2 - \sec^2 A) + \tfrac{1}{2}\gamma^3(2 - \sec^2 A) \cdots\cdots \tag{25}$$

Inversion of this series gives

$$\gamma = z_R + \tfrac{1}{2}z_R^2\ (2 - \sec^2 A) \cdots\cdots \tag{26}$$

or:

$$\boxed{\begin{aligned} &\text{Radial Velocity}\quad V = V_c + \Delta V_{rel} \\ &V_c = c\cdot z_R \ \text{(classical term)} \\ &\Delta V_{rel} = \tfrac{1}{2}c(2 - \sec^2 A)\cdot z_R^2 \ \text{(relativistic term)} \end{aligned}} \tag{27}$$

In section 2 we saw that in the existing computer programs, the relativistic term instead was calculated by

$$u_{rel} = \tfrac{1}{2}\, c\, z_R^2$$

i.e., in the naive application of special relativity, the classical radial velocity is corrected by a term which involves the assumption that the source has no tangential motion. The difference between the naive and the correct relativistic terms is determined by the function $2 - \sec^2 A$ which is given in the next table for some values of A:

A	$2 - \sec^2 A$
0°; 180°	+1.00
10 ; 170	0.97
20 ; 160	0.87
30 ; 150	0.67
40 ; 140	+0.30
50 ; 130	-0.42
60 ; 120	-2.0
70 ; 110	-6.6
80 ; 100	-31.2
(90)	(-∞)

If the space velocity is inclined by $\sim 45°$: $2 - \sec^2 A = 0$, i.e., in this case disappears the relativistic second order term. These considerations fail, of course, if A is in the neighborhood of $90°$. If the true spatial motion would be perpendicular to the "true" line of sight, i.e., if $A_o = 90°$, then we obtain from equation (18):

$$z_R = 1 - \frac{1}{1 - \sqrt{1 - \beta^2}} = -\tfrac{1}{2}\beta^2 - 3/8\ \beta^4 \cdots \qquad (28)$$

which is the "transversal doppler effect" (blue shift). Equation (15) shows that under this condition

$$\cos A = -\beta, \qquad (29)$$

i.e., the individual aberration of the source reaches its maximum value. For small space velocities, cosA would still be small and, therefore, $\sec^2 A$ would be large. Convergence of the series in equation (26) exists only if both γ and secA are not too large, i.e., if γ small and $|\cos A| \approx 1$. The latter condition, however, means that $|\cos A_o| \approx 1$, too, and this excludes the special case $A_o = 90°$ which was just considered.

These considerations show that it is not possible to apply a relativistic correction without knowing the space direction of the relative motion, or without measuring the "individual aberration". Application of equation (1), or inclusion of the term $\frac{1}{2}cz_R^2$, implies the assumption $A = 0°$ or $180°$. Perhaps this is a good assumption in a statistical sense for a large sample of sources, but for a statistics we can ignore the small relativistic correction anyway.

Some remarks should be made about a connected problem although this has certainly no practical meaning at the present. Let us assume that our z_R's were very accurate, even in the low velocity range ($u \leq 100$ km/sec). Let us further assume that we would know the direction of u. In this case, we could apply the relativistic correction properly. However, we would then of course also have to pay attention to the fact that for the annual velocity correction and for the basic solar motion correction the Einstein theorem of velocity addition must be applied rather than the classical vector addition. Otherwise we would again introduce an error in the relativistic terms.

Let us hope that relativistic corrections in the low velocity range never become important! For high velocities, of course, they are important, but this is a different thing. Cosmologists discuss z or z_R rather than radial velocities. Therefore, it might be a good idea if the standard data reduction programs would print accurate values of z and/or z_R - quantities which can be derived without hypotheses directly from observations and which will directly reflect the accuracy of the observations. The accuracy of z_R is given by

$$\left| \Delta z_R \right| = \frac{\delta \nu}{\nu_o} \cdot 10^{-6}$$

where $\delta \nu$ = actual frequency resolution in KKz, and ν_o = rest frequency in GHz. The highest possible accuracy for z_R, in the case of the 413 channel receiver, would be obtained with

$$\delta \nu = 0.1 \text{ KHz}, \ \nu_o = 1.42 \text{ GHz} : \ \left| \Delta z_R \right| = 7.10^{-8}$$

If the data reduction programs would compute classical velocities with moderate accuracy, but in addition give values of z and/or z_R with 7 decimal places, they would present useful and clearly defined data for further discussions.

The parameter z is preferable because this is the one which has been in use since Hubble's time. The only difference between z and z_R is that if β varies from -1 to +1, z goes from -1 to $+\infty$ whereas z_R goes from $-\infty$ to +1.