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HEAT FLOW IN A COOLED COAXIAL TRANSMISSION LINE

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ABSTRACT

Presented here are the results of a theoretical study of the heat flow and temperature gradients within a coaxial transmission line. The type of cable studied is frequently used in microwave research where cryogenic temperatures are required to maintain a minimal noise level in the circuits. Two refrigerating stations were taken into account, including their placement and efficiency. The solution to the linear differential model is presented in its entirety along with graphical displays of the effects of geometry and materials on temperatures, the equivalent conductivity of the inner conductor, and heat flow.

TABLE OF CONTENTS

LIST OF FIGURES

Introduction

The problem under consideration here is the flow of heat through a coaxial cable connecting an outside source to a microwave device cooled to 20° K. The geometry of the cable is taken to be cylindrically symmetric over its 5 cm length, with refrigerating stations in thermal contact with the outer conductor (See Figure 1). In setting up a differential model of heat flow I found it necessary to make the following assumptions:

1. The lengthwise flow of heat through the electrically insulating dielectric is negligible when compared to that in the two surrounding metal conductors.
2. The variation in thermal conductivity with temperature can be approximated by a simple average without adding unreasonable uncertainty to the calculations.
3. The refrigerators are in ideal thermal contact with the outer conductor.

Along with these assumptions the knowledge that temperature drop in a given material is proportional to heat flow allowed the construction of a linear model with a familiar solution.

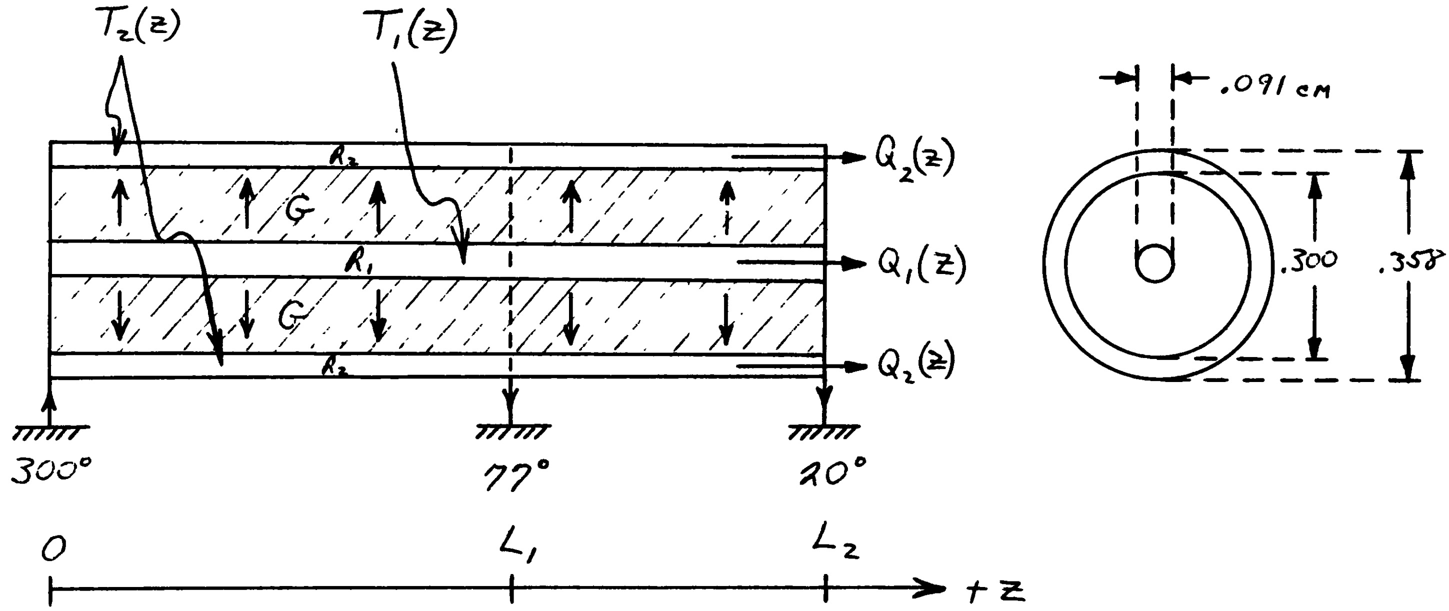


FIGURE 1: Sketch of the coaxial transmission line.

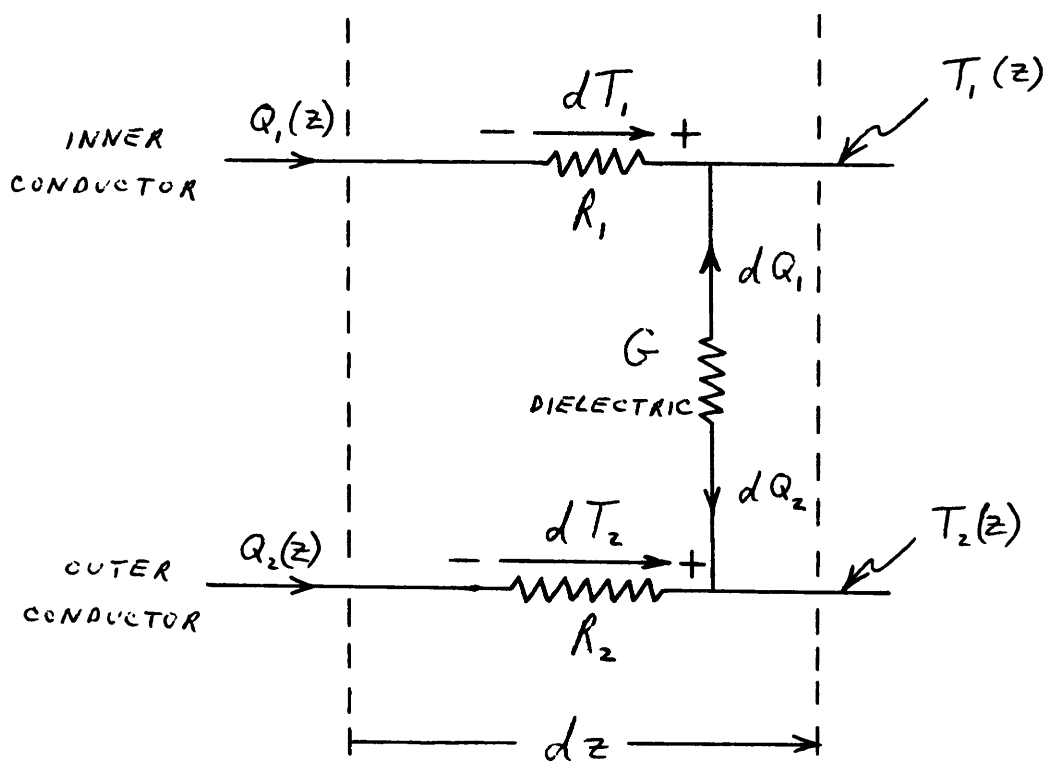


FIGURE 2: The model of a differential element dz.

The Differential Model and its Solution

The model I have sketched in Figure 2 represents the heat currents and temperatures of interest in the coaxial line. Longitudinal flow toward the cold end is taken as the +z direction and radial flow is pictured as the perpendicular path between the inner and outer conductors. The variables have the following connotations:

R_1, R_2 - thermal resistivity of the inner and outer conductors respectively (units $\frac{\text{degrees}}{\text{watt cm}_z}$)

G - thermal conductivity of the teflon cylinder around the inner conductor ($\frac{\text{watts}}{\text{degree cm}_z}$)

Q(z) - steady state heat flow (watts)

T(z) - steady state temperature (degrees K)

The thermal constants of the materials themselves are related to R_1, R_2 and G by geometrical factors of the line.

In the infinitesimal length dz there is an equal influx and outflux of heat with some temperature drop in the direction of flow. The four z-dependent variables we are seeking are T_1, T_2, Q_1 and Q_2. From the model, the following four equations evidently characterize the process:

$$(1)\quad dT_1 = -(R_1\, dz)\, Q_1$$

$$(2)\quad dT_2 = -(R_2\, dz)\, Q_2$$

$$(3)\quad dQ_1 = (G\, dz)(T_2 - T_1)$$

$$(4)\quad dQ_2 = (G\, dz)(T_1 - T_2)$$

Each differential change in the four variables is in balance with the heat flow or temperature gradient existing in the length dz.

The coupled set of linear differential equations shown above can be solved by proposing a solution of the form $e^{\lambda z}$ such that the operator $\frac{d}{dz}$ can be replaced by the constant λ. By dividing equations 1 to 4 by dz and making this substitution, we arrive at a set of linear algebraic equations in the four unknowns. In matrix form these are written:

$$\begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{bmatrix} \lambda & 0 & R_1 & 0 \\ 0 & \lambda & 0 & R_2 \\ G & G & \lambda & 0 \\ -G & -G & 0 & \lambda \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ Q_1 \\ Q_2 \end{bmatrix} \begin{array}{c} = \\ = \\ = \\ = \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Knowing that the determinant of the matrix must now be identically zero yields the characteristic equation in eigenvalue λ:

$$\lambda^4 - (GR_1 + GR_2)\lambda^2 = 0$$

$$\text{Roots:} \quad \lambda = 0,\ 0,\ \pm\sqrt{GR_1 + GR_2}$$

The presence of two zero-roots allows a constant and linear term to enter into the solution which then becomes:

$$(5) \quad T_1(z) = A + Bz + Ce^{\lambda z} + De^{-\lambda z}$$

By separately applying 3 of the 4 equations, we can determine the form of the other variables.

$$(6) \quad \text{Apply (1):} \quad Q_1(z) = \frac{-B}{R_1} - \frac{C}{R_1}\lambda e^{\lambda z} + \frac{D}{R_1}\lambda e^{-\lambda z}$$

$$(7) \quad \text{Apply (3):} \quad T_2(z) = A + Bz - \frac{R_2}{R_1}Ce^{\lambda z} - \frac{R_2}{R_1}De^{-\lambda z}$$

$$(8) \quad \text{Apply (2):} \quad Q_2(z) = \frac{-B}{R_2} + \frac{C}{R_1}\lambda e^{\lambda z} - \frac{D}{R_1}\lambda e^{-\lambda z}$$

Finally, with the thermal properties of each material known, the only remaining task is to apply 4 appropriate boundary conditions from the physical make-up of the problem in order to determine the constants A, B, C and D.

Boundary Conditions

Given just one refrigerator, we can set up 4 boundary conditions from the problem by inspection. The temperature at the hot end for both conductors is room temperature (300°). The temperature at the single refrigerator is known (20°) and we'll insulate the inner conductor at the cold end by keeping it unconnected and in a vacuum ($Q_1(z)$ goes to 0 at the cold end). With this configuration the temperature at the cold end of the inner conductor is:

$$T_1(L) \cong T_2(L) + (1 + R_2/R_1)\,\frac{T_2(0) - T_2(L)}{\lambda L + R_2/R_1}$$

where L = distance from hot end to the refrigerator

$T_2(L)$ = temperature of the refrigerator

$T_2(0)$ = room temperature at the hot end -- (the formula valid for $\lambda L > 3$ so that $\sinh \lambda L \cong \cosh \lambda L$)

The general case has more than a single refrigerator and must be treated as separate sections of cable (perhaps each with different materials) joined in a "continuous" way. Each section has its own λ and constants A, B, C, D which describe it. Thus, each requires 4 boundary conditions. The two temperatures at the hot end and temperature and heat flow at the cold end are the same given any number of sections. For each section after the first, 4 more B.C.'s are obtained as follows: The outer conductor temperature for both sections is the refrigerator's temperature. At the inner conductor the heat flow out of one section is the same as the flow into the next, and there can be no temperature jump there. Simply stated, 2 continuities plus 1 refrigerator give the 4 boundary conditions. The constants for all the sections can then be solved simultaneously with one large matrix inversion that will be demonstrated later.

Equivalent Thermal Resistance of the Inner Conductivity

Up to now the end of the inner conductor has been taken as completely insulated from heat flow, and we have seen how to find the "open circuit" temperature at that point. When the cable is connected to a device however, heat will flow into that device based on its temperature and the equivalent thermal resistance of the coax and the device. Because of the linear coupling of heat flow to temperature and constant coefficients in the differential equations, the solution is indeed a linear one such that each boundary condition has a linear effect on each of the 4 variables of interest. For example, this means that the temperature at the cold end of the inner conductor varies linearly with the boundary condition for the amount of heat draining from it:

$$T_1(L) = -R_{equiv}Q_1(L) + T_o$$

The quantity T_o is the "open circuit" temperature (which exists when $Q_1(L) = 0$), and the constant of proportionality is the equivalent resistance of the inner conductor. Before we can determine R_{equiv} we must solve the matrix equation for the constants A, B, C, D.

The General Solution for the Constants

The method of generating 4 boundary conditions for each section of the coax was described above. Here we will examine the practical case of two refrigerators at distances L_1 and L_2 along the line and show how the constants can be found.

The functional forms of T(z) and Q(z) are known from equations (5) - (8) except for the 4 constants for each section of the coax. Let these be treated as variables and z be fixed for each particular boundary condition. Then 8 equations are generated as follows:

$T_1(0) = 300$ room temperature		$T'_2(L_2) = 20$	second refrigerator
$T_2(0) = 300$ room temperature		$T_1(L_1) - T'_1(L_1) = 0$ continuity	
$T_2(L_1) = 77$ first refrigerator		$Q_1(L_1) - Q'_1(L_1) = 0$ continuity	
$T'_2(L_1) = 77$ first refrigerator		$I'_1(L_2) = 0$	insulated end

The primed variables denote the second section of line. In matrix form these are written:

$[M]\ C] = B]$ and thus $C] = [M]^{-1}\ B]$

where [M] is the matrix of 1, z, $e^{\lambda z}$, $e^{-\lambda z}$ terms evaluated at fixed z

C] is the vector of coefficients (A, B, C, D, A', B', C', D')

B] is the vector of boundary conditions (300, 300, 77, 77, 20, 0, 0, 0)

From the matrix form of the solution it is again evident that each boundary condition has a linear effect on the coefficients. By setting all $B_i = 0$ except $I'_1(L_2)$ which we will denote by Q_o into the device, the ratio of $T'_1(L_2)$ to Q_o can be determined. It is in fact:

$$\frac{T'_1(L_2)}{Q_o} = -R_{equiv} = m_{5,8} + m_{6,8}L_2 + m_{7,8}e^{\lambda' L_2} + m_{8,8}e^{-\lambda' L_2}$$

where $[m] = [M]^{-1}$

Along with the open circuit temperature, this equivalent resistance (in degrees/watt) completes the characterization of the heat flow through the inner conductor.

Calculations

The physical dimensions of the coax are shown in Figure 1. The cross-sectional areas of the inner and outer conductors and dielectric are .0065, .0300, and .0642 cm^2, respectively. Taking averages of the thermal conductivities between the refrigerator temperatures, we obtain the following numbers:*

Conductivities in Watts/cm degree

	Beryllium Copper (inner conductor)	Stainless Steel (outer conductor)	Teflon (dielectric)
300° - 77°	.63	.11	.0033
77° - 20°	.26	.05	.0021

The values for R_1 and R_2 used in the model are related to the above by $R_1 = 1/(\text{area}_1 g_1)$ and $R_2 = 1/(\text{area}_2 g_2)$ where g_1, g_2 are the conductivities. To find the geometrical factor for G we integrate its conductivity radially over the hollow cylinder of dielectric and find $G = g_{\text{tefl}} 2\pi/\ln(\frac{r_2}{r_1})$, where r_i denotes the radius. These values are shown below:

	R_1	R_2	G
300° - 77°	244	303	.0174
77° - 20°	592	667	.0111

The distance to the 77° refrigerator is taken as $L_1 = 2.7$ cm, and the distance to the 20° refrigerator is $L_2 = 5.0$ cm. This is the "standard" configuration referred to in the graphs.

The first assumption I made can now be checked. The smallest ratio of conductivities (for longitudinal heat flow) between conductor and dielectric

* It is also possible to propose a change of variables at this point to account for some non-constant function g(T). One integrates g(T) over the appropriate temperature range, and thus obtains a new variable in which the differential equations are again linear. The restriction, however, is that all 3 conductivities must have the identical T-dependence for the new variable to be useful. No practical use was found for the particular materials here, but with some compromises an application for this change of variable could be found.

is 11 and occurs between the stainless steel outer conductor and the teflon at the cold end. Thus, there is at least 11 times as much longitudinal heat flow anywhere in either of the conductors as in the dielectric. Having two conductors then makes the first assumption valid to within a 5% correction.

From the standard configuration of refrigerators and for these materials we obtained the value of 37° for the open circuit temperature, and an equivalent thermal resistance of .31 degrees/mWatt or conductance of 3.2 mWatt/degree.

Conclusions

On the following pages I present a large amount of information in graphical form that was extracted from the model. The following conclusions can be drawn regarding the coax line, the equivalent conductivity, and refrigeration:

1. The temperature gradient is essentially linear along the line for the first few centimeters, but is highly dependent on the placement of refrigerators.
2. The temperature of each refrigerator has a very strong effect on the cold-end temperature of the inner conductor, and thus it is crucial to optimize the efficiency of the refrigerators.
3. Only the cold-end conductivities of the 3 materials have a significant effect on the final temperature, with the order of significance being inner conductor, dielectric, and outer conductor.
4. The value used for room temperature has little effect on the final temperature of the line.
5. For a given set of materials the equivalent conductivity at the end of the inner conductor is almost entirely a function of the distance between refrigerators.

6. The optimum placement of the 77° refrigerator, if it has 3 times the capacity of the 20° station, is 7/10 of the way to the 20° refrigerator, independent of the length of the line.

All of these investigations were done by the HP 9845 computer in the electronics division. The program remains on file for further study and to provide data for comparison with experimental measurements.

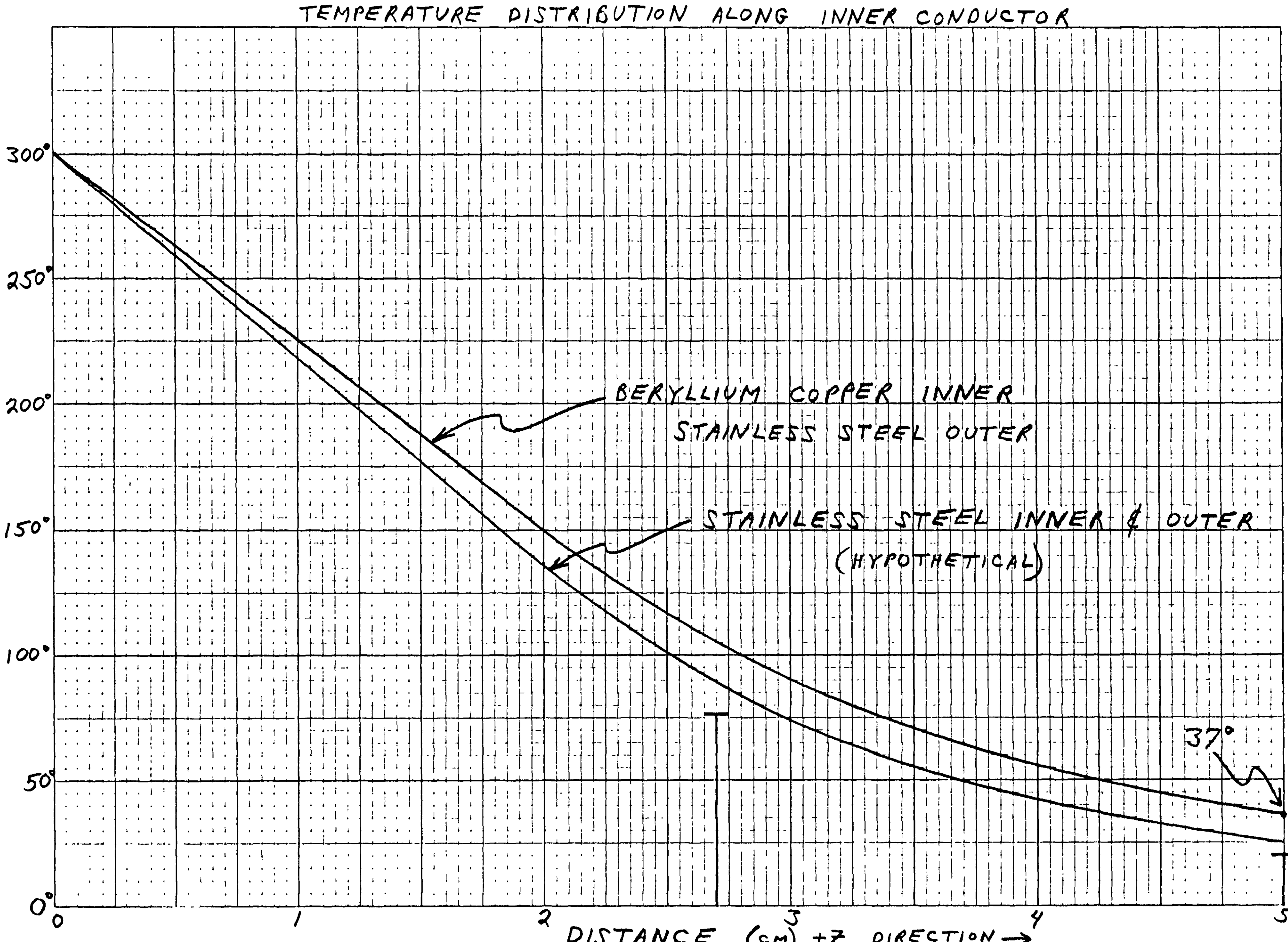

TEMPERATURE DISTRIBUTION ALONG INNER CONDUCTOR
300°
250°
200°
150°
100°
50°
0°
0
1
2
3
4
5
DISTANCE (CM) +Z DIRECTION →
BERYLLIUM COPPER INNER
STAINLESS STEEL OUTER
STAINLESS STEEL INNER & OUTER
(HYPOTHETICAL)
37°

FIGURE 3

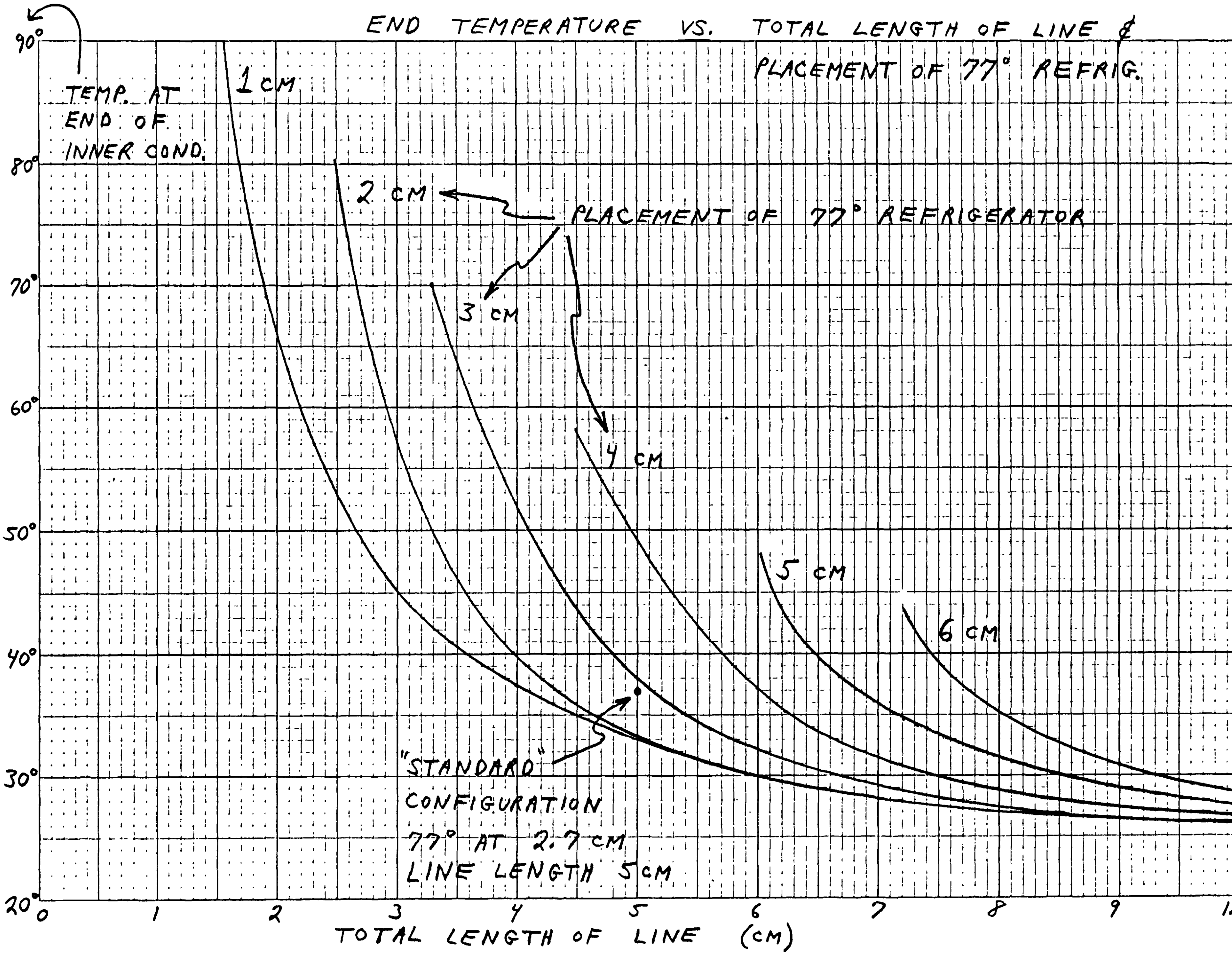
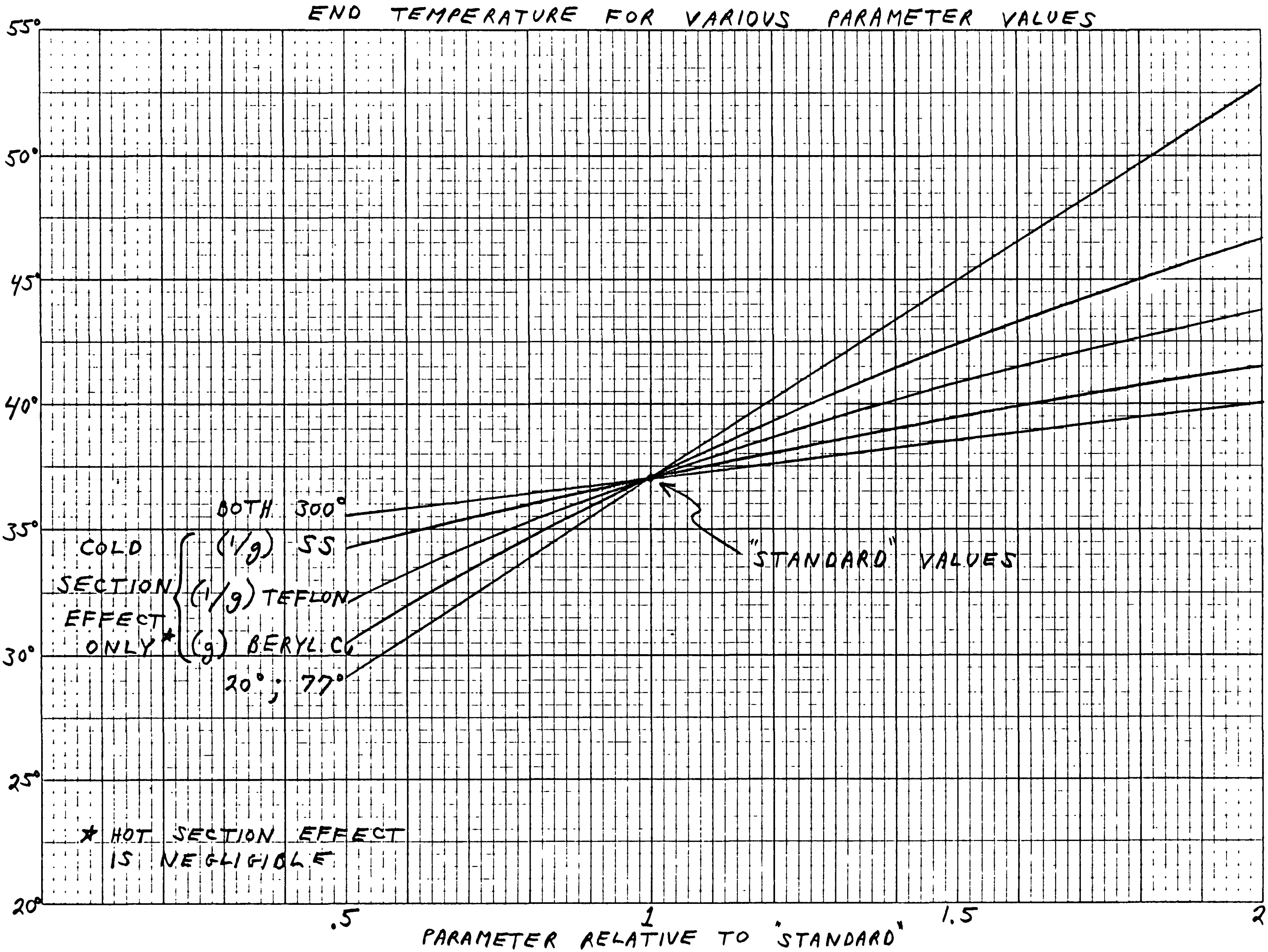

END TEMPERATURE VS. TOTAL LENGTH OF LINE &
PLACEMENT OF 77° REFRIG.
TEMP. AT
END OF
INNER COND.
90°
80°
70°
60°
50°
40°
30°
20°
1 CM
2 CM
3 CM
4 CM
5 CM
6 CM
PLACEMENT OF 77° REFRIGERATOR
"STANDARD"
CONFIGURATION
77° AT 2.7 CM
LINE LENGTH 5 CM
0
1
2
3
4
5
6
7
8
9
10
TOTAL LENGTH OF LINE (CM)

FIGURE 4


END TEMPERATURE FOR VARIOUS PARAMETER VALUES
55°
50°
45°
40°
35°
30°
25°
20°
BOTH 300°
COLD SECTION EFFECT ONLY*
(1/g) SS
(1/g) TEFLON
(g) BERYL. Co
20°; 77°
"STANDARD" VALUES
* HOT SECTION EFFECT IS NEGLIGIBLE
.5
1
1.5
2
PARAMETER RELATIVE TO "STANDARD"
FIGURE 5

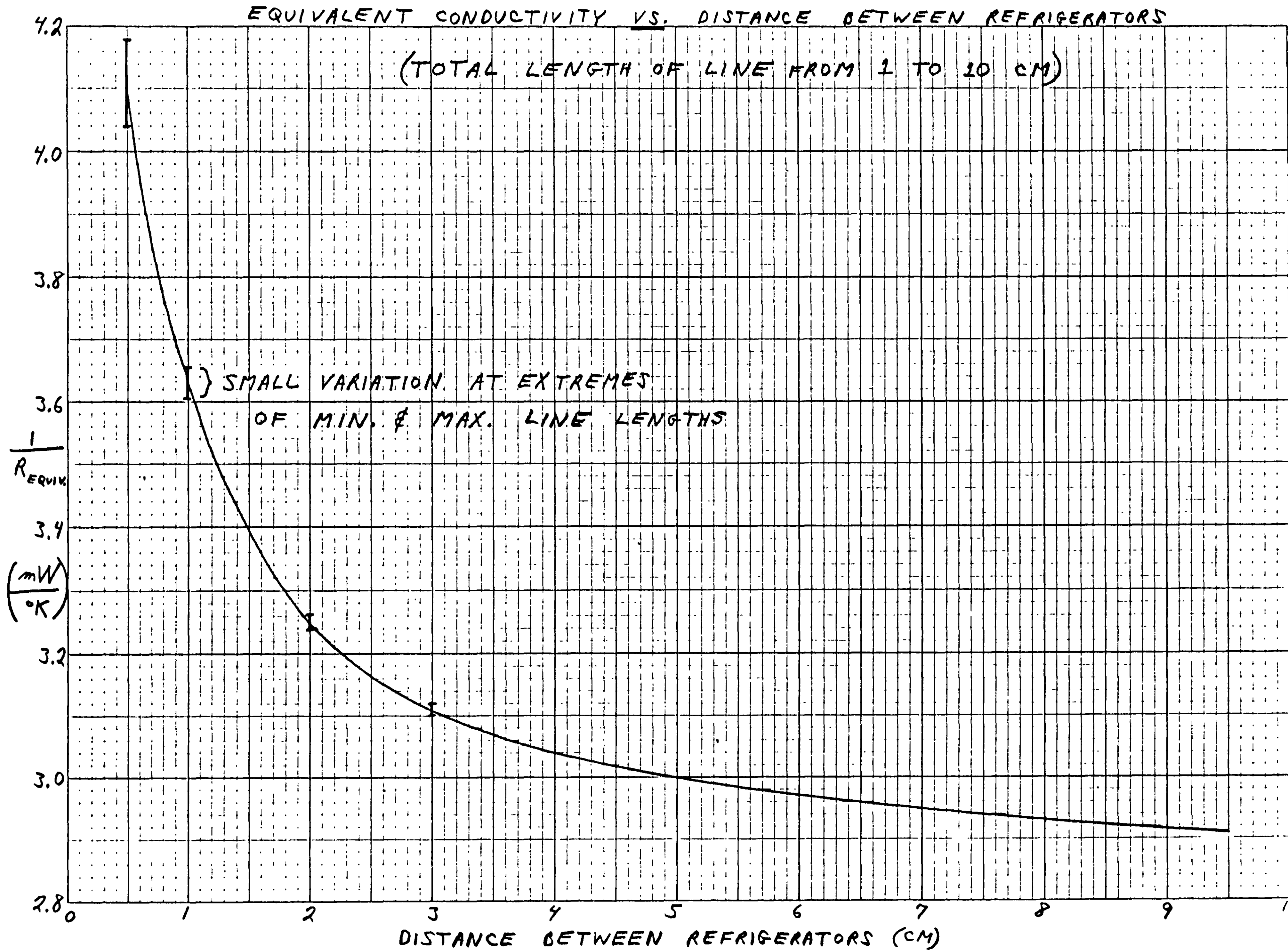

EQUIVALENT CONDUCTIVITY VS. DISTANCE BETWEEN REFRIGERATORS
(TOTAL LENGTH OF LINE FROM 1 TO 10 CM)
SMALL VARIATION AT EXTREMES
OF MIN. & MAX. LINE LENGTHS
$\frac{1}{R_{EQUIV.}}$
$\left(\frac{mW}{°K}\right)$
4.2
4.0
3.8
3.6
3.4
3.2
3.0
2.8
0
1
2
3
4
5
6
7
8
9
10
DISTANCE BETWEEN REFRIGERATORS (CM)

FIGURE 6

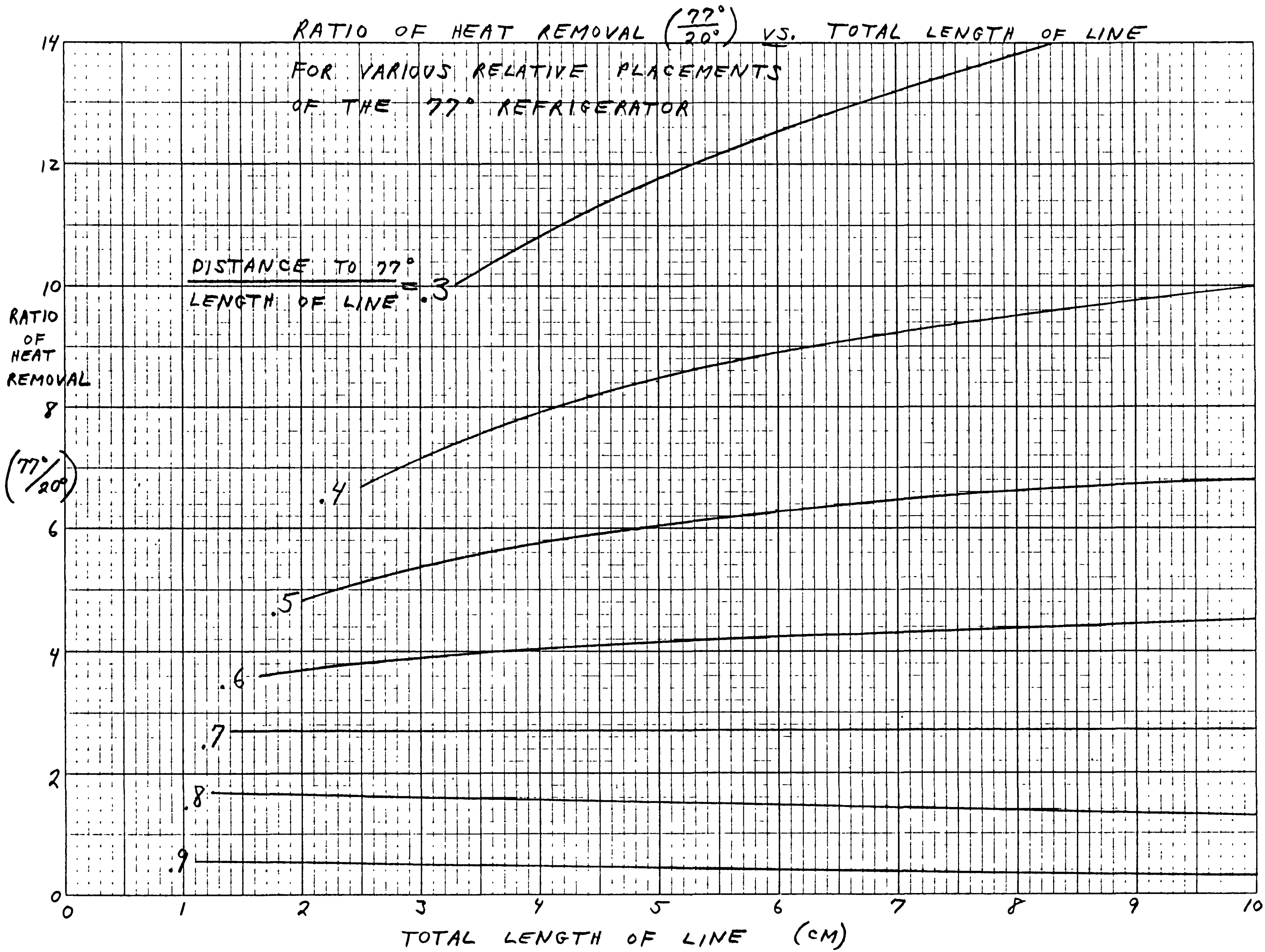

RATIO OF HEAT REMOVAL (77°/20°) VS. TOTAL LENGTH OF LINE
FOR VARIOUS RELATIVE PLACEMENTS
OF THE 77° REFRIGERATOR
DISTANCE TO 77° / LENGTH OF LINE = .3
.4
.5
.6
.7
.8
.9
RATIO OF HEAT REMOVAL (77°/20°)
0
2
4
6
8
10
12
14
0
1
2
3
4
5
6
7
8
9
10
TOTAL LENGTH OF LINE (CM)

FIGURE 7

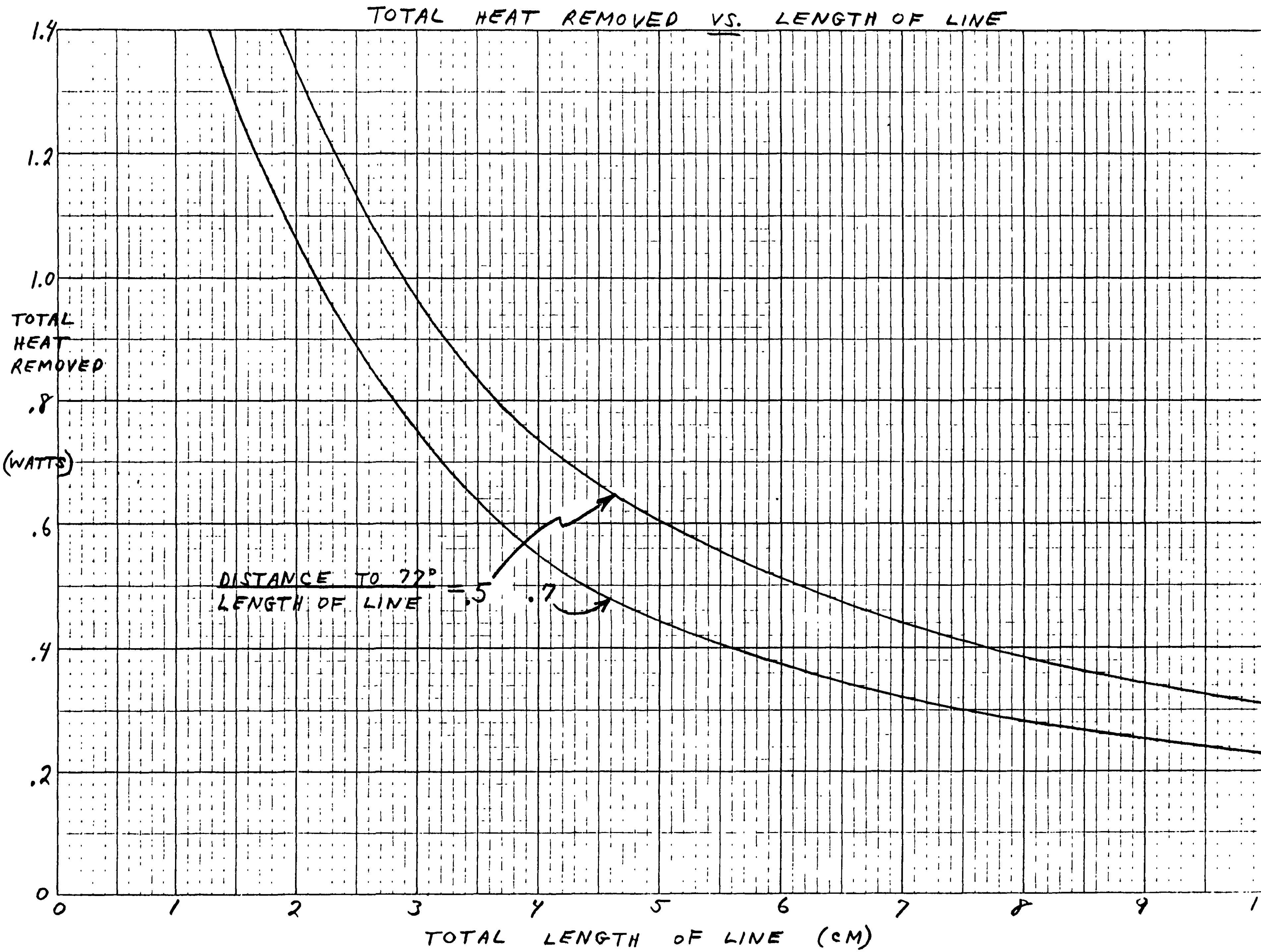

TOTAL HEAT REMOVED VS. LENGTH OF LINE
TOTAL HEAT REMOVED (WATTS)
1.4
1.2
1.0
.8
.6
.4
.2
0
0
1
2
3
4
5
6
7
8
9
10
TOTAL LENGTH OF LINE (CM)
DISTANCE TO 72° / LENGTH OF LINE = .5 .7

FIGURE 8