Geometrical Aspects of Interferometry

C. M. Wade and G. W. Swenson, Jr.

December 1964

The present notes give a brief summary of the basic geometrical formulae needed in working with a two-element interferometer.

We use a set of notation and definitions which we feel to be unambiguous and practical. It is based on the systems of Read (Ap.J. 138, 1, 1963) and Rowson (MN 125, 177, 1963). There is an obvious need for an accepted terminology in this field, and we suggest that the NRAO Interferometer Group adopt the one used below as standard.

I. The Interferometer Triangle

The basic spherical triangle is shown as $\mathcal{S}$NB in Figure 1. $\mathcal{S}$ denotes the point of observation, N is the north celestial pole, and B is the more northerly baseline pole. The angular quantities are as follows:

δ = declination of $\mathcal{S}$

H = hour angle of $\mathcal{S}$

d = declination of B

h = hour angle of B

ϑ = distance between $\mathcal{S}$ and B

σ = position angle of B as seen from $\mathcal{S}$.

We also define

D = baseline length

λ = free-space wavelength corresponding to the local oscillator frequency.

D and λ are linear quantities which should be expressed in the same units.

The effective length of the interferometer baseline for observation in the $\mathcal{S}$-direction is $D \sin \vartheta$. The degree of resolution is specified by the fringe separation, which is $\frac{D}{\lambda} \sin \vartheta$ (in radians). The effective orientation of the baseline is determined by the position angle of the line normal to the fringes at $\mathcal{S}$. The fringes always cross the line $\mathcal{S}B$ at right angles; therefore σ is the position angle of the normal to the fringes.

It follows from the law of cosines that

$$\cos \vartheta = \sin d \sin \delta + \cos d \cos \delta \cos (H-h) \tag{1}$$

and from the four-parts formula that

$$\operatorname{ctn} \sigma = \tan d \cos \delta \csc (H-h) - \sin \delta \operatorname{ctn} (H-h) \tag{2}$$

Let us specify the east-west and north-south components of the effective baseline (in wavelengths) by u and v, respectively. Then

$$u = \frac{D}{\lambda} \sin \vartheta \sin \sigma,$$
$$v = \frac{D}{\lambda} \sin \vartheta \cos \sigma.$$

The law of sines gives

$$\sin \vartheta \sin \sigma = \cos d \sin (H-h). \tag{3}$$

Multiplying (2) by (3), we get

$$\sin \vartheta \cos \sigma = \sin d \cos \delta - \cos d \sin \delta \cos (H-h)$$

Therefore

$$u = \frac{D}{\lambda} \cos d \sin (H-h) \tag{4}$$

$$v = \frac{D}{\lambda} (\sin d \cos \delta - \cos d \sin \delta \cos(H-h)) \tag{5}$$

These relations are the parametric equations of an ellipse, (H-h) being the parameter. This ellipse has the following properties:

(a) Its major axis lies parallel with the u-axis.

(b) Its center is at $u = 0$, $v = \frac{D}{\lambda} \sin d \cos \delta$.

(c) Its major and minor semi-axes are $\frac{D}{\lambda} \cos d$ and $\frac{D}{\lambda} \cos d \sin \delta$, respectively.

(d) Its axial ratio is $\sin \delta$; i.e., its eccentricity is $\cos \delta$.

II. The Fringe Pattern

The crests of the fringe pattern occur in directions where the difference in path length from the two antennas is an integral number of wavelengths. Clearly, the fringe pattern is a family of small circles centered on the baseline poles. The interferometer response in directions between the fringe crests depends on the degree of cancellation between the contributions from the individual antennas. The system response as a function of hour angle and declination is

$$f(H,\delta) = \cos \left\{2\pi \frac{D}{\lambda} \cos \vartheta(H,\delta) + a \right\} \tag{6}$$

where $\underline{a}$ is a phase constant which depends on the difference in electrical path length between the local oscillator and the antennas.

Substituting (1) into (6), we have

$$f(H,\delta) = \cos \left\{ 2\pi \frac{D}{\lambda}(\sin d \sin \delta + \cos d \cos \delta \cos (H-h)) + a\right\} \tag{7}$$

The rotation of the earth sweeps the fringe pattern across the sky. The rate at which the fringes cross a given source depends on its hour angle and declination as well as the baseline parameters. This rate is the "fringe frequency". It is simply the time (i.e., hour angle) derivative of the number of r.f. cycles in the differential path to the two antennas:

$$R(H,\delta) = \left| \left(\frac{\partial \frac{D}{\lambda} \cos \vartheta}{\partial H} \right)_\delta \right|$$

$$= \frac{D}{\lambda} \cos d \cos \delta \sin (H-h)$$

in cycles per radian of hour angle. Thus the fringe frequency in cycles per sidereal second is

$$R = \frac{D}{\lambda} \; \frac{\cos d \cos \delta \sin (H-h)}{13751} \tag{8}$$

III. An Alternative Notation

We can express (4), (5), (7), and (8) more compactly if we define

$$B_1 = \frac{D}{\lambda} \sin d,$$

$$B_2 = \frac{D}{\lambda} \cos d,$$

$$B_3 = a/2\pi.$$

Then we have

$$u = B_2 \sin (H-h) \tag{9}$$

$$v = B_1 \cos \delta - B_2 \sin \delta \cos (H-h) \tag{10}$$

$$f(H,\delta) = \cos \{2\pi\, (B_1 \sin \delta + B_2 \cos \delta\, (H-h) + B_3)\} \tag{11}$$

$$R = \frac{B_2 \cos \delta \sin (H-h)}{13751} \tag{12}$$

IV. The Relation Between the Baseline Parameters and the Survey Data

The survey data specify the station latitude φ, the baseline length D, and the azimuth A and elevation ζ of the north baseline pole. Figure 2 shows how A and ζ are related to h and d.

The law of cosines and the four-parts formula give respectively

$$\sin d = \sin \zeta \sin \varphi + \cos \zeta \cos \varphi \cos A \tag{13}$$

$$\operatorname{ctn} h = \sin \varphi \operatorname{ctn} A - \tan \zeta \cos \varphi \csc A. \tag{14}$$

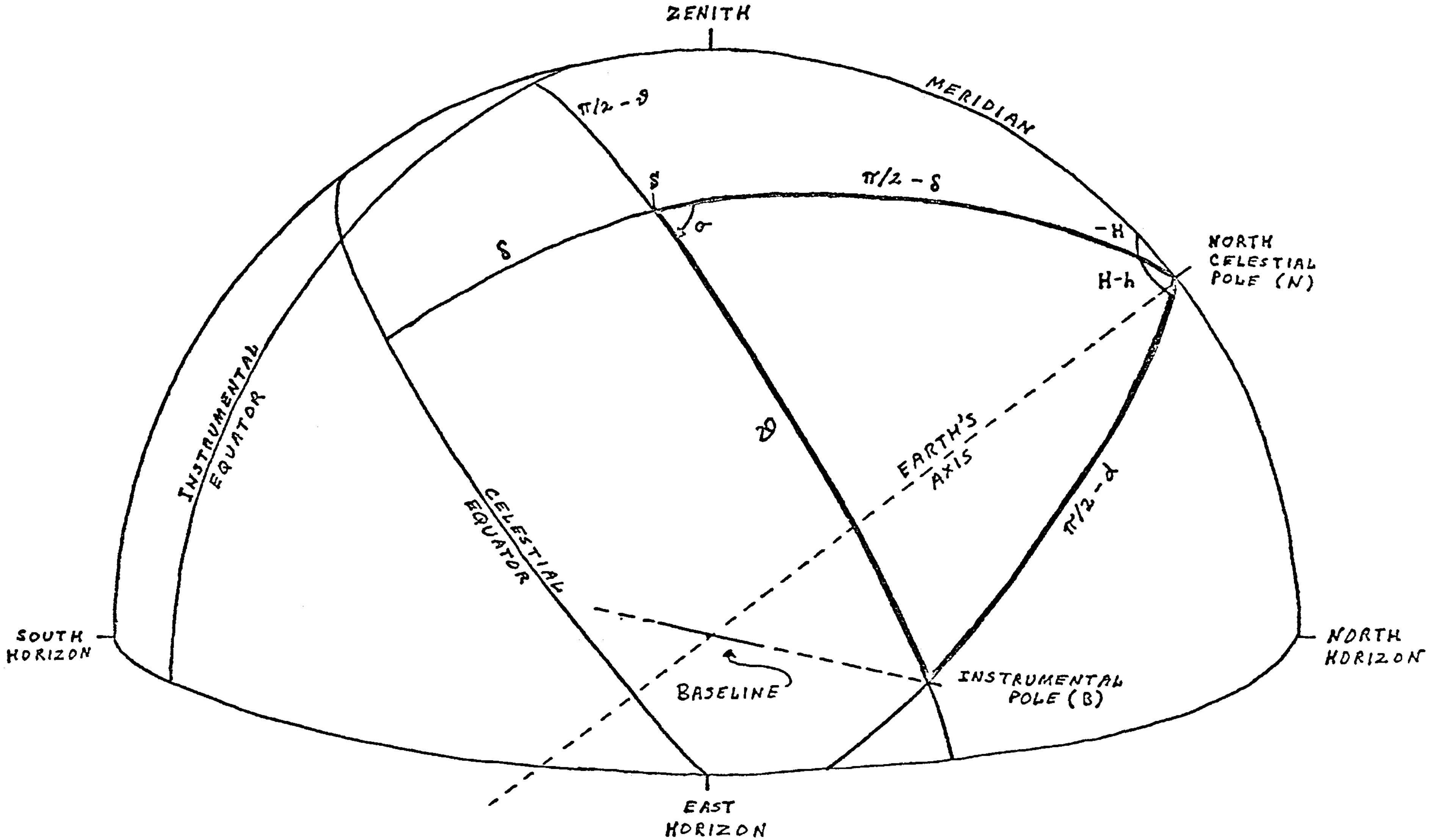
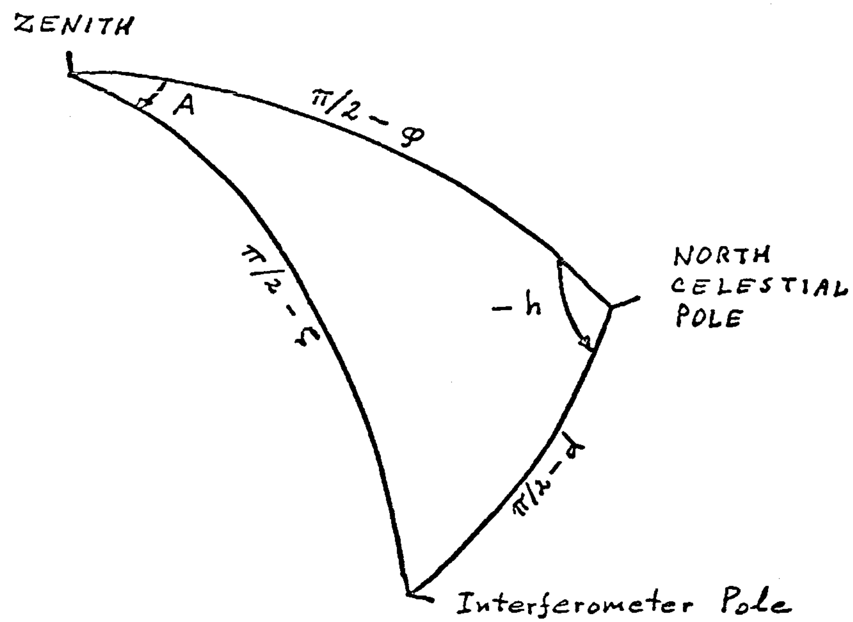

ZENITH
MERIDIAN
$\pi/2-\vartheta$
S
$\pi/2-\delta$
σ
δ
$-H$
NORTH CELESTIAL POLE (N)
$H-h$
INSTRUMENTAL EQUATOR
ϑ
EARTH'S AXIS
$\pi/2-d$
CELESTIAL EQUATOR
SOUTH HORIZON
NORTH HORIZON
BASELINE
INSTRUMENTAL POLE (B)
EAST HORIZON

Fig. 1



ZENITH
A
π/2 − φ
NORTH
CELESTIAL
POLE
− h
π/2 − ζ
π/2 − d
Interferometer Pole

Fig. 2