Interferometer Calibration and Source Position Measurement Using Long-Run Phase Behavior

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I. Introduction

The true visibility phase of an unresolved radio source is always equal to zero. But the phases calculated by the fringe reduction program [1] from observations of such sources are seldom near zero; furthermore, they frequently vary as a function of hour angle. This behavior is attributable to errors in the source positions and baseline parameters assumed in the fringe reduction program. The phase trend during the course of a long run on a point source can be analyzed to give corrections to the baseline parameters if the source position is known precisely, or conversely to give corrections to the position when the baseline parameters are well determined. A computer program has been written which performs both of these tasks. The purpose of the present report is to describe the method of solution used in the program.

Let the fringe pattern observed for an unresolved source be [2]

$$f'(T) = A \cos \{2\pi[B_1' \sin \delta' + B_2' \cos \delta' \cos (T-\alpha'-h') + B_3']\} \quad (1)$$

where T is the local sidereal time. The primed quantities on the right are the correct values of the source coordinates and the baseline parameters. The fringe reduction program determines the phase by reference to the fringe pattern expected for a point source at an assumed position (α, δ) with an assumed set of baseline parameters (h, B_1, B_2, B_3):

$$f(T) = A \cos \{2\pi\, [B_1 \sin \delta + B_2 \cos \delta \cos (T-\alpha-h) + B_3]\} \qquad (2)$$

Suppose that the assumed position and baseline parameters are slightly in error. Then f(T) and f'(T) will be out of step by an amount which depends on T; this will be revealed as a variation in the computed phase. In this report, we undertake the following:

a) To show how errors in the assumed constants affect the calculated phase as a function of time (Section II).

b) To show how the baseline parameters can be improved, using the phase behavior in observations of sources with very accurately known positions (Section III).

c) To show how accurate positions can be established for other point sources once the errors in the baseline parameters are known (Section IV).

II. The Basic Equation

Referring to equations (1) and (2), we have

$$\begin{aligned}
B_1' &= B_1 - \Delta B_1 \\
B_2' &= B_2 - \Delta B_2 \\
B_3' &= B_3 - \Delta B_3 \\
h' &= h - \Delta h \\
\alpha' &= \alpha - \Delta\alpha \\
\delta' &= \delta - \Delta\delta
\end{aligned}$$

where ΔB_1, ΔB_2, etc. are the errors in the assumed values of the corresponding quantities.

The phase given by the fringe reduction program is, in radians,

$$\Phi(T) = 2\pi \{B_2' \cos \delta' \cos(T-\alpha'-h') - B_2 \cos \delta \cos (T-\alpha-h) + K\} \qquad (3)$$

where

$$K = B_1' \sin \delta' + B_3' - B_1 \sin \delta - B_3$$

is constant for an observation of any given source. It is clear that errors in B_1 and B_3 have an effect on the numerical values of the calculated phases, but this effect is independent of time. The only errors that can result in a time-varying phase error are those in B_2, h, α, and δ.

Expanding the primed terms in (3), we get

$$\frac{1}{2\pi}\Phi(T) = (B_2\Delta\delta \sin \delta - \Delta B_2 \cos \delta) \cos \tau + B_2 \Delta\tau \cos \delta \sin \tau + K \qquad (4)$$

This assumes that the errors are small; i.e., that the terms involving $\Delta B_2 \Delta\delta$, $\Delta B_2 \Delta\tau$, and $\Delta\delta\Delta\tau$ are negligibly small in comparison with the remaining terms. In (4) we have set

$$\tau = T - \alpha - h$$

$$\Delta\tau = -(\Delta\alpha + \Delta h).$$

Equation (4) relates errors in source position and the baseline parameters to the behavior of the calculated phase as a function of time. The next two sections deal with its application to calibration and position measurement.

III. Application to Calibration

For calibration, we use sources whose positions are known to very high accuracy. Then we have

$$\Delta\delta = 0$$

$$\Delta\tau = -\Delta h$$

and equation (b) becomes

$$\frac{1}{2\pi}\,\Phi(T) = -\cos\,\delta(\Delta B_2 \cos\tau + B_2\Delta h \sin\tau) + K \tag{5}$$

The fringe reduction program gives a value of Φ for every minute of observing time. A telescope program aimed at system calibration will be designed in such a way that the reference source is observed over a wide range of hour angles in the course of a single day's run, and it will also ensure that a large number of observing minutes are provided. The best way to deduce ΔB_2 and Δh from such a mass of data is to make a least squares solution based on (5) as the type equation of condition.

Let us define

$$x = \Delta B_2$$

$$y = B_2\Delta h$$

$$z = -K/\cos\,\delta$$

$$a = -\Phi/360\,\cos\,\delta$$

where now Φ is assumed to be given in degrees instead of radians. Then the equation of condition is

$$x\cos\tau + y\sin\tau + z = a. \tag{6}$$

The procedure used in the computer program to accomplish the least squares solution is summarized in the Appendix. The solution gives the most probable values of x, y, and z, and also the mean errors in these quantities. Obviously

$$\Delta B_2 = x; \quad \text{m.e.}(\Delta B_2) = \text{m.e.}(x);$$

$$\Delta h = 13751\,y/B_2; \ \text{m.e.}(\Delta h) = 13751\,\text{m.e.}(y)/B_2;$$

$$K = z\cos\,\delta; \ \text{m.e.}(K) = \text{m.e.}(z)\cos\,\delta.$$

The factor 13751 converts Δh from radians to seconds of time.

IV. Application to Position Measurement

Now assume that ΔB_2 and Δh are known, and that we wish to find $\Delta\alpha$ and $\Delta\delta$. In this case,

$$\Delta\tau = -(\Delta\alpha + \Delta h).$$

With all the known quantities transferred to the right-hand side, equation (4) becomes

$$B_2\Delta\delta \sin\delta \cos\tau + B_2\Delta\tau \cos\delta \sin\tau + K = \frac{1}{2\pi}\Phi + \Delta B_2 \cos\delta \cos\tau \quad (7)$$

Again expressing Φ in degrees and defining

$$x = B_2\Delta\delta \sin\tau$$

$$y = B_2\Delta\tau \cos\tau$$

$$z = K$$

$$a = \Phi/360 + \Delta B_2 \cos\delta \cos\tau,$$

we can express (7) in the form (6), and make the least squares solution for $\Delta\alpha$ and $\Delta\delta$ in exactly the same way as we did for ΔB and Δh in the calibration case. This time we have

$$\Delta\delta = \frac{206265\, x}{B_2 \sin\delta}; \quad \text{m.e.}(\Delta\delta) = \frac{206265\ \text{m.e.}(x)}{B_2 \sin\delta}$$

$$\Delta\alpha = \frac{-13751\, y}{B_2 \cos\delta} - \Delta h; \quad \text{m.e.}(\Delta\alpha) = \frac{13751\ \text{m.e.}(y)}{B_2 \cos\delta}$$

$$K = z\ ; \quad \text{m.e.}(K) = \text{m.e.}(z)$$

The factors 206265 and 13751 convert the results to seconds of arc and time, respectively. Note that it is assumed that Δh is in seconds of time.

References

1. C. M. Wade, "A Method for Finding the Phase and Amplitude of Interferometer Fringe Patterns," NRAO report, November 1964.
2. C. M. Wade and G. W. Swenson, "Geometrical Aspects of Interferometry", NRAO report, December 1964.

Appendix

Although the solution of linear equations by least squares is common knowledge, it is desirable for the sake of definiteness to set down the actual procedure used in the computer program.

Our equation of condition is

$$x \cos \tau + y \sin \tau + z = a.$$

The corresponding normal equations are

$$P_1 x + P_2 y + P_3 z = Q_1$$
$$P_2 x + P_4 y + P_5 z = Q_2$$
$$P_3 x + P_5 y + P_6 z = Q_3$$

where

$$P_1 = \sum \cos^2 \tau$$
$$P_2 = \sum \sin \tau \cos \tau$$
$$P_3 = \sum \cos \tau$$
$$P_4 = \sum \sin^2 \tau$$
$$P_5 = \sum \sin \tau$$
$$P_6 = \text{no. of equations of condition}$$
$$Q_1 = \sum a \cos \tau$$
$$Q_2 = \sum a \sin \tau$$
$$Q_3 = \sum a$$
$$Q_4 = \sum a^2.$$

The determinant of the set of normal equations is

$$\Delta = \begin{vmatrix} P_1 & P_2 & P_3 \\ P_2 & P_4 & P_5 \\ P_3 & P_5 & P_6 \end{vmatrix} = P_1 P_4 P_6 + 2P_2 P_3 P_5 - P_1 P_5^2 - P_2^2 P_6 - P_3^2 P_4$$

The solutions for the unknowns are:

$$x = \frac{1}{\Delta} \begin{vmatrix} Q_1 & P_2 & P_3 \\ Q_2 & P_4 & P_5 \\ Q_3 & P_5 & P_6 \end{vmatrix} = \frac{Q_1(P_4 P_6 - P_5^2) + Q_2(P_3 P_5 - P_2 P_6) + Q_3(P_2 P_5 - P_3 P_4)}{\Delta}$$

$$y = \frac{1}{\Delta} \begin{vmatrix} P_1 & Q_1 & P_3 \\ P_2 & Q_2 & P_5 \\ P_3 & Q_3 & P_6 \end{vmatrix} = \frac{Q_1(P_3 P_5 - P_2 P_6) + Q_2(P_1 P_6 - P_3^2) + Q_3(P_2 P_3 - P_1 P_5)}{\Delta}$$

$$z = \frac{1}{\Delta} \begin{vmatrix} P_1 & P_2 & Q_1 \\ P_2 & P_4 & Q_2 \\ P_3 & P_5 & Q_3 \end{vmatrix} = \frac{Q_1(P_2 P_5 - P_3 P_4) + Q_2(P_2 P_3 - P_1 P_5) + Q_3(P_1 P_4 - P_2^2)}{\Delta}$$

To determine the mean errors of the solutions, we need the sum of the squares of the residuals:

$$\begin{aligned} \lambda &= \sum (x \cos \tau + y \sin \tau + z - a)^2 \\ &= P_1 x^2 + 2P_2 xy + 2P_3 xz + P_4 y^2 + 2P_5 yz + P_6 z^2 \\ &\quad -2Q_1 x - 2Q_2 y - 2Q_3 z + Q_4 \end{aligned}$$

The effective mean square residual is

$$\mu^2 = \frac{\lambda}{P_6 - 3}$$

The reciprocals of the weights of the unknown are:

$$w_x^{-1} = \frac{1}{\Delta} \begin{vmatrix} 1 & 0 & 0 \\ 0 & P_4 & P_5 \\ 0 & P_5 & P_6 \end{vmatrix} = \frac{P_4 P_6 - P_5^2}{\Delta}$$

$$w_y^{-1} = \frac{1}{\Delta} \begin{vmatrix} P_1 & 0 & P_3 \\ 0 & 1 & 0 \\ P_3 & 0 & P_6 \end{vmatrix} = \frac{P_1 P_6 - P_3{}^2}{\Delta}$$

$$w_z^{-1} = \frac{1}{\Delta} \begin{vmatrix} P_1 & P_2 & 0 \\ P_2 & P_4 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{P_1 P_4 - P_2{}^2}{\Delta}$$

The mean errors of the solutions are then:

$$\text{m.e.}(x) = \sqrt{\mu^2 \; w_x^{-1}}$$

$$\text{m.e.}(y) = \sqrt{\mu^2 \; w_y^{-1}}$$

$$\text{m.e.}(z) = \sqrt{\mu^2 \; w_z^{-1}} \;.$$