NATIONAL RADIO ASTRONOMY OBSERVATORY
Green Bank, West Virginia

December 7, 1964

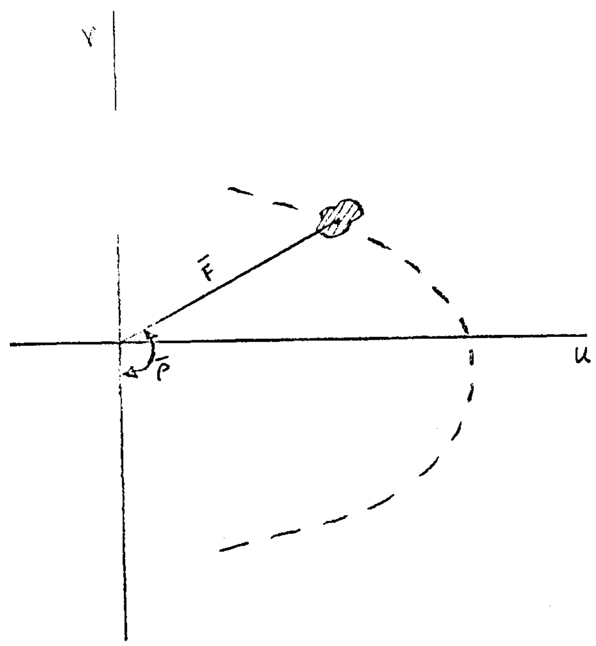
MEMORANDUM

To: Addressee

From: Nigel Keen

Subject: The Response of One or More Interferometers

The intention of this memorandum is to set up the form of the fundamental interferometer response equation, making no assumptions about the fringe shape. Even in the event of instantaneous sinuosidal response, it appears that such terms as "baseline", "fringe rate", "fringe period", etc., only lead to misinterpretations in the rigorous case. For simplicity, all phase and amplitude noise is considered stationary and gaussian. Since spatial frequency information is the result of the total aperture illumination, we concern ourselves with individual antenna apertures.



Since synthesis information is usually described in the fourier transform (spatial frequency) plane, one or more interferometers become spatial frequency filters in this plane. We will use polar coordinates F and ρ in this plane (rather than cartesian coordinates u and v) for conceptual simplicity. For interferometers with varying declinations of instrumental pole, little further simplification is possible by transforming the fourier plane coordinates.

We have to consider the information which will appear in this fourier plane. Ignoring system noise, we are concerned with the phase and amplitude of the absolute coefficient of cross-correlation of the noise voltages from each antenna: this cross-correlation coefficient is a vector defined by a phase and an amplitude: in the correlation interferometer it is the phase and amplitude of this coefficient which must appear in two conjugate spatial frequency planes. We will consider the form of the cross-correlation vector coefficient later in this memo.

The response of two point antennas to a point source with zero bandwidth in both i.f. amplifiers is the simplest form of interferometer problem. We are now concerned with finite individual antenna apertures which are not necessarily pointing in the direction of the source, as well as with finite source sizes and amplifier bandwidths. All source distributions may be considered instantaneously in the coordinates of hour-angle (H) and declination (δ). Since only instantaneous spatial frequency filter functions are obtainable (they are constantly changing) all time variations are contained in terms in H. Finite amplifier bandwidth requires definition in terms of wavelength (λ) or frequency: we use the former. To describe the individual antennas we require three coordinates, two of which are common to source distribution descriptions: we use H, δ and physical length D. Furthermore we must describe responses in the two conjugate F-ρ planes in terms of H, δ, D and λ.

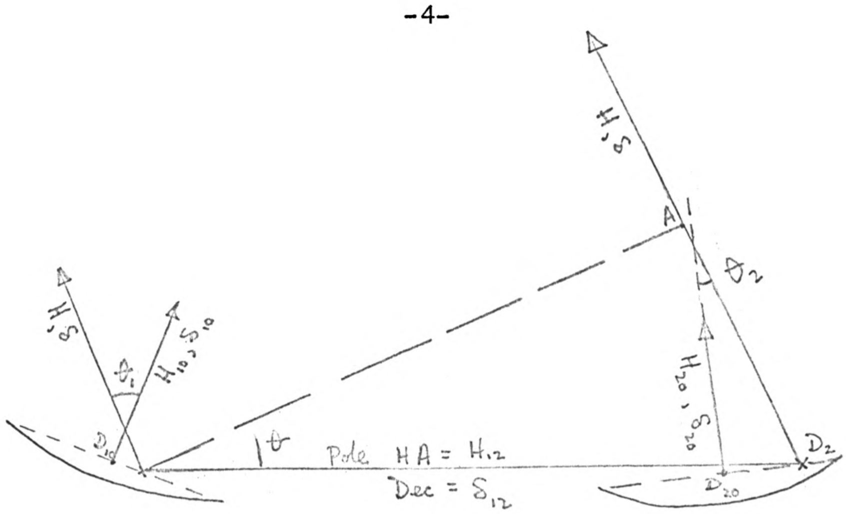
We now consider the meaning of an interferometer response. If the response is of the form

$$A(H,\delta,D,\lambda)\ e^{j\phi(H,\delta,D,\lambda)}$$

we commence by assuming a point source. To determine vector amplitudes, we have to determine the individual antenna-plus-amplifier voltage gains, which will be the sum of all individual gains for zero bandwidth and a point aperture: the first antenna we have

$$\int_\lambda \int_H \int_\delta \int_D G_1(\lambda)\ R_1\ (D_1 - D_{10}, \lambda, H, H_{10}, \delta, \delta_{10})\ dH dD d\lambda d\delta$$

where H_{10}, δ_{10} represent reference directions (directions of maximum R_1) and D_{10} represents a reference point such as the individual aperture center. G_1 is the amplifier gain. A similar expression applies to the second antenna. The terms G and R are complex quantities. Let us now consider the relationship between various points on two different antennas. The interferometer response is the integral of the products of the individual responses, whether a correlation or total power-plus-detection system is used. We assume parallel planes of polarization, and only consider points in the aperture plane.



When considering two antennas of an interferometer, we are concerned with differential paths such as AD_2: Let us call this distance ℓ, where

$$\ell = (D_2 - D_1)\ \sin\theta\ (H_{12}, H, \delta_{12}, \delta)$$

where $\sin\theta = \sin\delta_{12}\ \sin\delta + \cos\delta_{12}\ \cos\delta\ \cos(H_{12} - H)$

If we remember that θ for a single antenna is the angle between H_0, δ_0 and H, δ <u>projected onto the plane containing D and D_0</u>, then we can define

$$G_1 = G_{10}(\lambda)\ e^{j\phi_1(\lambda)}$$

$$G_2 = G_{20}(\lambda)\ e^{j\phi_2(\lambda)}$$

$$R_1 = R_{10}\left(\frac{D_1 - D_{10}}{\lambda}\right) \cdot e^{j\frac{D_1 - D_{10}}{\lambda}\sin\theta_1}$$

and

$$R_2 = R_{20}\left(\frac{D_2 - D_{20}}{\lambda}\right) \cdot e^{j\left[\frac{D_2 - D_{20}}{\lambda}\sin\theta_2 + \frac{D_2 - D_1}{\lambda}\sin\theta\right]}$$

Since R represents the vector voltage response of one antenna, where R_1 is the aperture illumination function (assumed circularly symmetrical, as is reasonable for scalar feeds), the interferometer response is

$$\int_\lambda\int_H\int_\delta\int_D G_1 G_2^* R_1 R_2^* \, e^{j\ell/\lambda} \, d\lambda . dH . d\delta , dD$$

where conjugate complex terms are employed since phase must be subtracted

$$\iiiint G_{10} G_{20} R_{10} R_{20} \, e^{j\frac{(D_1 - D_{10}) \sin \theta_1 - (D_2 - D_{20}) \sin \theta_2 + (D_1 - D_2) \sin \theta + L}{\lambda}}$$

$$e^{j(\phi_1 - \phi_2)} \, d\lambda dH d\delta dD \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where $\sin \theta_1 = \sin \delta_{101} \sin \delta_1 + \cos \delta_{101} \cos \delta_1 \cos (H_{101} - H_1)$ and

$$\sin \theta_2 = \sin \delta_{202} \sin \delta_2 + \cos \delta_{202} \cos \delta_2 \cos (H_{202} - H_2)$$

L represents the difference between the path lengths from the respective antenna apertures to the correlation point.

All terms now contain D/λ and angles. In the spatial frequency domain

$$D/\lambda \equiv F_{\text{absolute}}$$

and θ and ρ have the analytical relationships

$$\cos \theta = \sin d \sin \delta + \cos d \cos \delta \cos (H - \hbar)$$

$$\cot \rho = \tan d \cos \delta \operatorname{cosec} (H - \hbar) - \sin \delta \operatorname{cotan} (H - \hbar)$$

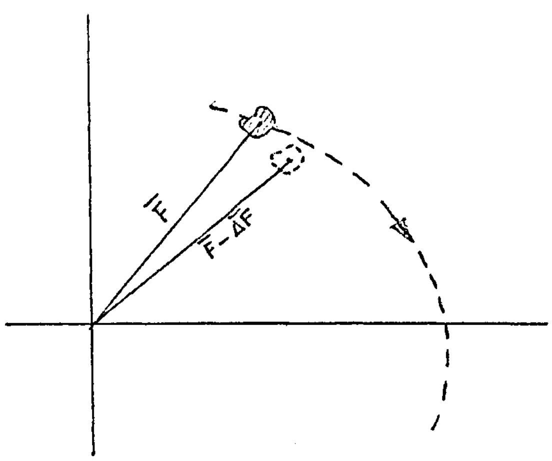
The expression reduces to the familiar form for interferometer response when

$$\left.\begin{array}{l}\delta_{10} = \delta_{20} \\ H_{10} = H_{20}\end{array}\right\} \text{(i.e. } \theta_1 = \theta_2\text{)}$$

$$R_{10} = R_{20}$$

$$G_{10} = G_{20}$$

A final point remains to be discussed: the relative movement of the dish aperture centers due to their not being the center of rotation of the dish. Assuming that the form of this movement is known, and that we are only concerned with the relative movement of one aperture center relative to the other: this is equivalent to a variation of $(D_1 - D_2)$, H_{12} and δ_{12} with $H_{10}, \delta_{10}, H_{20}$ and δ_{20}. Even the simplest form of such a variation introduces a considerable complication in the phase term. The simplest case, where $(D_{10} - D_{20})$ changes linearly with time while H_{12} and δ_{12} remain constant, produces a movement of the spatial frequency filter response in the spatial frequency plane. Hence, a "baseline change" occurs although it is more rigorous to say that the spatial frequency filter response has moved from its ideal elliptical track. If H_{12} and δ_{12} also change, the integrated effect will produce a further change in $\bar{F}$ ($\overline{\Delta F'}$, say) and a change in $\bar{\rho}$ ($\overline{\Delta\rho}$). Some changes in the form of the amplitude and phase characteristics would also occur. Even without these



changes, the assumption of sine wave response is not correct. If the effect of changes in $(D_{10} - D_{20})$, H_{12} and δ_{12} is to be determined, the sine wave approximation must be discarded: otherwise all such second order effects will be completely masked.

Conclusions

The complex nature of the general equation of the spatial frequency filter response demands some approximations, but the application of such approximations should be cautious. Above all, the response and its time rate of change should be checked experimentally in a quantitative way.

Since source spatial frequency information is filtered by the interferometer response, the interferometer response to a distributed (non-point) source is

$$\iiiint G_{10} G_{20} R_{10} R_{20}\, e^{j\alpha} \,.\, S(H,\delta)\, e^{j\beta(H,\delta)}\, d\lambda, dH.d\delta.dD$$

where α is the total interferometer phase term and $Se^{j\beta}$ is the (complex) fourier transformation of the source brightness distribution (assumed invariant over the observing bandwidth).