MEMORANDUM August 28, 2002

TO: Mark Clark, GBT Commissioning Group Members
FROM: Fred Schwab
SUBJECT: Correlator Sensitivity Degradation due to Mis-balancing

Mark,

I've just worked out the sensitivity degradation implications of the spectrometer mis-balancing. They aren't awfully serious. In the three-level case the achieved efficiency would have been ~80.15%, versus the optimal efficiency of 80.98%. In the nine-level case the degradation would have been somewhat greater: an achieved efficiency of ~94.99%, vs. the optimal 96.93%. Here are the details of the calculation:

The efficiency of an n-level digital autocorrelator, in the weak-correlation limit $\rho \to 0$, when sampling at the Nyquist rate, is given (for the case of odd n and zero-mean stationary Gaussian input signals) by

$$\eta = \frac{2\left(\sum_{k=1}^{(n-1)/2} e^{-(2k-1)^2 v_1^2/2}\right)^2}{\pi \sum_{k=0}^{(n-3)/2} (2k+1)\,\mathrm{erfc}\left(\frac{(2k+1)v_1}{\sqrt{2}}\right)}, \tag{1}$$

where $\mathrm{erfc}\, z \equiv 1 - \mathrm{erf}\, z$ is the complementary error function; v_1 is the first positive input threshold, expressed in units of the r.m.s. input voltage level σ; and it is assumed that both the quantizer input threshold levels and output levels are equi-spaced. In the three-level case, this equation becomes simply

$$\eta = \frac{2e^{-v_1^2}}{\pi\,\mathrm{erfc}\left(\frac{v_1}{\sqrt{2}}\right)}. \tag{2}$$

This expression is maximized when $v_1 = 0.612003181$, in which case $\eta = \eta_{\mathrm{opt}} \equiv 80.9826\%$. For the nine-level case,

$$\eta = \frac{2\left(e^{-v_1^2/2} + e^{-9v_1^2/2} + e^{-25v_1^2/2} + e^{-49v_1^2/2}\right)^2}{\pi\left(\mathrm{erfc}\left(\frac{v_1}{\sqrt{2}}\right) + 3\,\mathrm{erfc}\left(\frac{3v_1}{\sqrt{2}}\right) + 5\,\mathrm{erfc}\left(\frac{5v_1}{\sqrt{2}}\right) + 7\,\mathrm{erfc}\left(\frac{7v_1}{\sqrt{2}}\right)\right)}; \tag{3}$$

this expression is maximized when $v_1 = 0.266911104$, and here $\eta_{\mathrm{opt}} = 96.9304\%$.

As I understand it, the so-called "balancing" of correlator input levels is achieved by examining the input signal population statistics during something like a 20-msec interval just prior to the start of an integration. In three-level mode, one looks at a "duty cycle" R which is defined as the ratio of the number of samples in the central (0) quantization zone to the sum of the number of samples in the lower (-1) zone and the upper ($+1$) zone. The target ratio should be $R = 0.850017$ in the three-level case, corresponding to occupancy rates of 27.0268%, 45.9464%, and 27.0268%, in the -1, 0, and $+1$ quantization zones, respectively.

In the nine-level case, the duty cycle R is defined as the ratio of the number of samples in the middle three quantization zones to the sum of the number in the lowermost three and the uppermost three zones. In this case, the target ratio should be $R = 1.36247$, corresponding to occupancy rates of 21.1643%, 57.6714%, and 21.1643%.

In my first message commenting on your memo about balancing, I noted that your assumed R values (of 0.84813 and 1.37578) were slightly inaccurate. As a matter of fact, the minima of the efficiency curves are so broad that the first case leads to an efficiency loss of less that 0.0001%, and the second to a loss of only 0.0004%. So I really needn't have commented. But I guess you subsequently discovered that the on-line correlator balancing code was aiming for a target ratio equal to the reciprocal of R, rather than R itself.

In Appendix D of my memorandum *Van Vleck Correction for the GBT Correlator* I give a Mathematica code to derive three-level or nine-level quantizer thresholds (v_1 values) from the quantizer population counts. In the three-level case, for a target $R = 1/0.84813 = 1.18789$, the corresponding quantization zone occupancy rates are 22.9456%, 54.1087%, and 22.9456%. The threshold value one would infer from these counts is $v_1 = 0.740639$, which implies a correlator efficiency of 80.153%.

In the nine-level case, for a target $R = 1/1.37578 = 0.726860$, the occupancy rates are 28.9543%, 42.0914%, and 28.9543%. This implies a threshold value $v_1 = 0.184907$ and a correlator efficiency of 94.9902%.

So, in conclusion, there would have been somewhat significant efficiency losses (of $\sim$0.8% and $\sim$1.9% in the respective two cases) as a result of the spectrometer mis-balancing—but not severe losses.