GBT Memo 270

A Mathematical Examination of 180 Degree and 90 Degree Hybrids in Differential Radiometers

September, 2010
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Abstract
Microwave hybrids are generally available in 180 degree or ‘Magic Tee’ versions and 90 degree or ‘Quadrature’ versions. Opinions solicited as to the implications of substituting 90 degree for 180 degree hybrids in differential radiometer architecture, specifically that of the GBT 26-40 GHz Receiver were anything but unanimous, thus those implications are examined here.

Introduction
Differential radiometers such as the GBT 26-40 GHz Receiver typically use 180 degree hybrids. The 180 degree hybrids combine the signals at the input ports to produce at respective output ports a signal proportional to the sum and difference of the signals at the input ports. Two of these hybrids along with some device that causes a 180 degree phase shift of the signal can be used in differential receiver architecture to reduce 1/f noise from the HEMT amplifiers when switched at a sufficiently rapid rate and to enable rapid beam switching. A typical architecture, a simplified version of the GBT 26-40 GHz Receiver that performs this function is shown in figure 1.

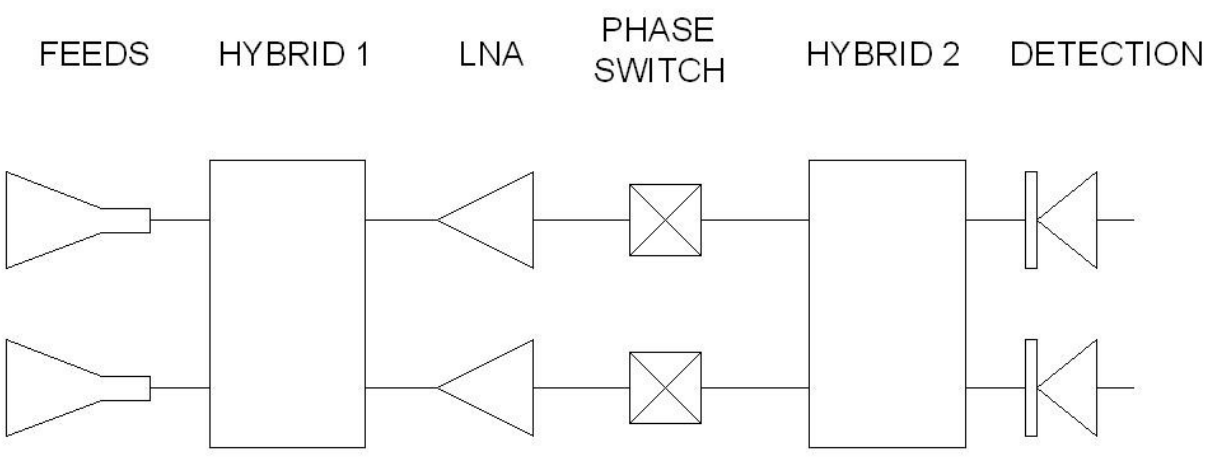

Figure 1, Differential Radiometer.

By changing the state of the phase switch between the hybrids the inputs to the first hybrid can be interchanged at the outputs of the second hybrid.

Using 180 Degree Hybrids

The 180 Degree, or 'Magic Tee' Hybrid
In waveguide the 180 degree hybrid appears as shown in figure 2 (1).

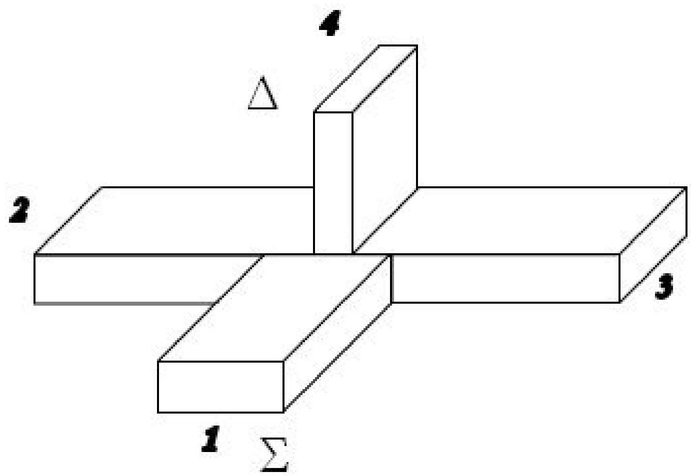

Figure 2, the 180 degree hybrid in waveguide.

A signal applied to port 1, also known as the H-Plane arm or sum port, will divide equally between ports 2 and 3, also known as the Main Guide arms with no energy coupled to port 4. A signal applied to port 4, also known as the E-Plane arm or difference port will also divide equally between ports 2 and 3 with no energy coupled to port 1 but the signals at ports 2 and 3 will be 180 degrees out of phase (2).

If **A** is the signal applied to port 1 and **B** is the signal applied to port 2 the mathematical result is:

Port 2 = $1/\sqrt{2}\ (\mathbf{A} + \mathbf{B})$

Port 3 = $1/\sqrt{2}\ (\mathbf{A} - \mathbf{B})$

Because we are dealing with voltages, the outputs have a square-root-of-two factor.

Assigning **A'** to the signal at port 2 and **B'** to the signal at port 3 and in turn applying those signals to the input ports of the second hybrid in the architecture of figure 1 the result is:

Second port 2 = $1/\sqrt{2}\ (\mathbf{A'} + \mathbf{B'}) = 1/\sqrt{2}\ [(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) + (\mathbf{A}/\sqrt{2} - \mathbf{B}/\sqrt{2})]$

Second port 3 = $1/\sqrt{2}\ (\mathbf{A'} - \mathbf{B'}) = 1/\sqrt{2}\ [(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) - (\mathbf{A}/\sqrt{2} - \mathbf{B}/\sqrt{2})]$

Rearranging and Reducing,

Second port 2 = $1/\sqrt{2}\ (\mathbf{A'} + \mathbf{B'}) = 1/\sqrt{2}\ [\mathbf{A}/\sqrt{2} + \mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2} - \mathbf{B}/\sqrt{2}] = 1/\sqrt{2}\ [2\mathbf{A}/\sqrt{2}] = \mathbf{A}$

Second port 3 = $1/\sqrt{2}\ (\mathbf{A'} - \mathbf{B'}) = 1/\sqrt{2}\ [\mathbf{B}/\sqrt{2} - (\ -\mathbf{B}/\sqrt{2}) + \mathbf{A}/\sqrt{2} - \mathbf{A}/\sqrt{2}] = 1/\sqrt{2}\ [2\mathbf{B}/\sqrt{2}] = \mathbf{B}$

So no phase shift in either arm results in a 'straight through' response.

Phase Switching

Phase switching introduces a 180 degree shift to either or both terms after the first hybrid. If we phase switch the upper or **A'** leg and apply that to what we started with above, after the first hybrid and the phase switches we have

$$\text{Second port 2} = 1/\sqrt{2}\,(-\mathbf{A'} + \mathbf{B'}) = 1/\sqrt{2}\,[-(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) + (\mathbf{A}/\sqrt{2} - \mathbf{B}/\sqrt{2})]$$

$$\text{Second port 3} = 1/\sqrt{2}\,(-\mathbf{A'} - \mathbf{B'}) = 1/\sqrt{2}\,[-(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) - (\mathbf{A}/\sqrt{2} - \mathbf{B}/\sqrt{2})]$$

Rearranging and Reducing,

$$\text{Second port 2} = 1/\sqrt{2}\,[-\mathbf{B}/\sqrt{2} + (-\mathbf{B}/\sqrt{2}) + \mathbf{A}/\sqrt{2} - \mathbf{A}/\sqrt{2}] = 1/\sqrt{2}\,[-2\mathbf{B}/\sqrt{2}] = \mathbf{-B}$$

$$\text{Second port 3} = 1/\sqrt{2}\,[-\mathbf{A}/\sqrt{2} + (-\mathbf{A}/\sqrt{2}) + \mathbf{B}/\sqrt{2} - \mathbf{B}/\sqrt{2}] = 1/\sqrt{2}\,[-2\mathbf{A}/\sqrt{2}] = \mathbf{-A}$$

The outputs have interchanged relative to the non-switched example. Since the receiver is a total power instrument the sign of the output voltage is not important.

Switching the Other Leg

This time we invert the signal in the lower or **B'** leg,

$$\text{Second port 2} = 1/\sqrt{2}\,[\mathbf{A'} + (-\mathbf{B'})] = 1/\sqrt{2}\,[(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) + (-\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2})]$$

$$\text{Second port 3} = 1/\sqrt{2}\,[\mathbf{A'} - (-\mathbf{B'})] = 1/\sqrt{2}\,[(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) - (-\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2})]$$

Rearranging and Reducing,

$$\text{Second port 2} = 1/\sqrt{2}\,[\mathbf{B}/\sqrt{2} + (\mathbf{B}/\sqrt{2}) + \mathbf{A}/\sqrt{2} - \mathbf{A}/\sqrt{2}] = 1/\sqrt{2}\,[2\mathbf{B}/\sqrt{2}] = \mathbf{B}$$

$$\text{Second port 3} = 1/\sqrt{2}\,[\mathbf{A}/\sqrt{2} + (\mathbf{A}/\sqrt{2}) + \mathbf{B}/\sqrt{2} - \mathbf{B}/\sqrt{2}] = 1/\sqrt{2}\,[2\mathbf{A}/\sqrt{2}] = \mathbf{A}$$

Again, the outputs have interchanged relative to the non-switched example.

Switching Both Legs

$$\text{Second port 2} = 1/\sqrt{2}[-(\mathbf{A'} + \mathbf{B'})] = 1/\sqrt{2}\,[(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) + (\mathbf{A}/\sqrt{2} - \mathbf{B}/\sqrt{2})]$$

$$\text{Second port 3} = 1/\sqrt{2}[-(\mathbf{A'} - \mathbf{B'})] = 1/\sqrt{2}\,[(\mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2}) - (\mathbf{A}/\sqrt{2} - \mathbf{B}/\sqrt{2})]$$

Reducing,

$$\text{Second port 2} = 1/\sqrt{2}\,(\mathbf{A'} + \mathbf{B'}) = 1/\sqrt{2}\,[\mathbf{A}/\sqrt{2} + \mathbf{A}/\sqrt{2} + \mathbf{B}/\sqrt{2} - \mathbf{B}/\sqrt{2}] = 1/\sqrt{2}\,[2\mathbf{A}/\sqrt{2}] = \mathbf{A}$$

$$\text{Second port 3} = 1/\sqrt{2}\,(\mathbf{A'} - \mathbf{B'}) = 1/\sqrt{2}\,[\mathbf{B}/\sqrt{2} - (-\mathbf{B}/\sqrt{2}) + \mathbf{A}/\sqrt{2} - \mathbf{A}/\sqrt{2}] = 1/\sqrt{2}\,[2\mathbf{B}/\sqrt{2}] = \mathbf{B}$$

The same result as if neither leg was switched.

Using 90 Degree Hybrids

I will look at a branchline style quadrature waveguide coupler, and use the short-hand notation that is often used when referring to hybrids known as phasor notation.

Figure 3 is an example where the branchline coupler is used as a combiner as taken from a microwave text (3). The input signals are vectors of magnitude A and B resulting in the outputs as shown.

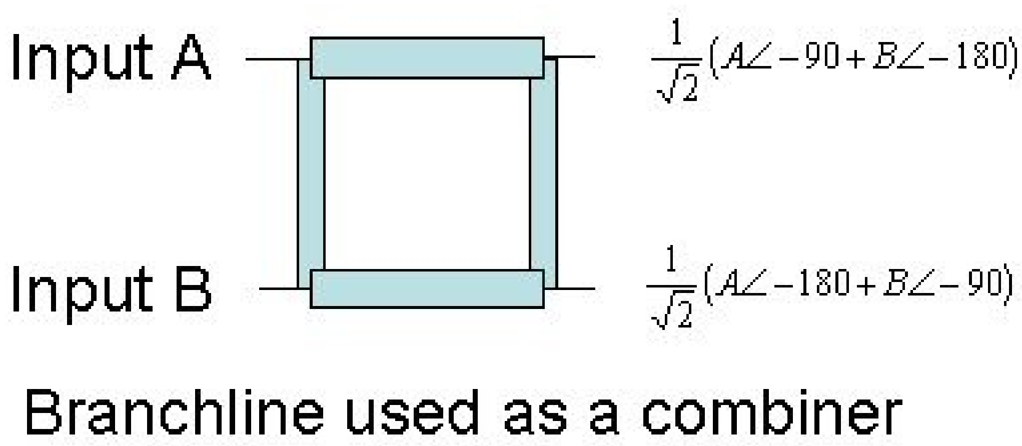

Figure 3, a Branchline Quadrature Coupler

Now I'll apply a second branchline coupler to the outputs of the first as would be done in the architecture shown in figure 1.

Taking the outputs of the above example and setting them equal to **A'** and **B'** respectively,

A' = 1/√2 (**A**∠-90 + **B**∠-180)

B' = 1/√2 (**A**∠-180 + **B**∠-90)

Putting this through the second coupler and setting that equal to **A''** and **B''** respectively,

A'' = 1/√2 (**A'**∠-90 + **B'**∠-180)

B'' = 1/√2 (**A'**∠-180 + **B'**∠-90)

Substituting the original expressions for **A'** and **B'**,

A'' = 1/√2 (**[**1/√2 (**A**∠-90 + **B**∠-180)**]**∠-90 + **[**1/√2 (**A**∠-180 + **B**∠-90)**]**∠-180)

B'' = 1/√2 (**[**1/√2 (**A**∠-90 + **B**∠-180)**]**∠-180 +**[**1/√2 (**A**∠-180 + **B**∠-90)**]**∠-90)

Factoring out the 1/√2's,

A'' = 1/2 (**[** (**A**∠-90 + **B**∠-180)**]**∠-90 + **[** (**A**∠-180 + **B**∠-90)**]**∠-180)

B'' = 1/2 (**[** (**A**∠-90 + **B**∠-180)**]**∠-180 +**[** (**A**∠-180 + **B**∠-90)**]**∠-90)

Applying the phase shifts from outside the brackets,

$$\mathbf{A''} = 1/2\ (\ [\ (\mathbf{A}\angle{-180} + \mathbf{B}\angle{-270})] + [\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})]\ )$$

$$\mathbf{B''} = 1/2\ (\ [\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})] + [\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-180})]\ )$$

Rearranging terms,

$$\mathbf{A''} = 1/2\ (\ [\ (\mathbf{A}\angle{-180} + \mathbf{A}\angle{-360})] + [\ (\mathbf{B}\angle{-270} + \mathbf{B}\angle{-270})]\ )$$

$$\mathbf{B''} = 1/2\ (\ [\ (\mathbf{A}\angle{-270} + \mathbf{A}\angle{-270})] + [\ (\mathbf{B}\angle{-360} + \mathbf{B}\angle{-180})]\ )$$

-360 is 0 in degrees, -180 and 0 (or -360) cancel, so we have

$$\mathbf{A''} = 1/2\ [\ \mathbf{B}\angle{-270} + \mathbf{B}\angle{-270}\ ] = \mathbf{B}\angle{-270}$$

$$\mathbf{B''} = 1/2\ [\ \mathbf{A}\angle{-270} + \mathbf{A}\angle{-270}\ ] = \mathbf{A}\angle{-270}$$

So the outputs with 90 degree hybrids have interchanged compared to 180 deg. hybrids and have a phase lag of 270 degrees, but there is no relative phase difference introduced between **A** and **B**.

Phase Switching

If we phase switch the upper or **A'** leg and apply that to what we started with above, after the first hybrid and the phase switches we have

$$\mathbf{A'} = 1/\sqrt{2}\ [(\mathbf{A}\angle{-90} + \mathbf{B}\angle{-180})]\angle{-180} = 1/\sqrt{2}\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})$$

$$\mathbf{B'} = 1/\sqrt{2}\ (\mathbf{A}\angle{-180} + \mathbf{B}\angle{-90}) \qquad \text{(unaltered)}$$

Putting this through the second coupler, setting that equal to **A''** and **B''** respectively and substituting the original expressions for **A'** and **B'**,

$$\mathbf{A''} = 1/\sqrt{2}\ (\ [1/\sqrt{2}\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})]\angle{-90} + [1/\sqrt{2}\ (\mathbf{A}\angle{-180} + \mathbf{B}\angle{-90})]\angle{-180})$$

$$\mathbf{B''} = 1/\sqrt{2}\ (\ [1/\sqrt{2}\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})]\angle{-180} + [1/\sqrt{2}\ (\mathbf{A}\angle{-180} + \mathbf{B}\angle{-90})]\angle{-90})$$

Factoring out the 1/√2's,

$$\mathbf{A''} = 1/2\ (\ [\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})]\angle{-90} + [\ (\mathbf{A}\angle{-180} + \mathbf{B}\angle{-90})]\angle{-180})$$

$$\mathbf{B''} = 1/2\ (\ [\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})]\angle{-180} + [\ (\mathbf{A}\angle{-180} + \mathbf{B}\angle{-90})]\angle{-90})$$

Applying the phase shifts from outside the brackets,

$$\mathbf{A''} = 1/2\ (\ [\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-90})] + [\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})]\ )$$

$$\mathbf{B''} = 1/2\ (\ [\ (\mathbf{A}\angle{-450} + \mathbf{B}\angle{-180})] + [\ (\mathbf{A}\angle{-270} + \mathbf{B}\angle{-180})]\ ) \qquad \text{Note: } -450 = -90$$

Rearranging terms,

$$\mathbf{A''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-360} + \mathbf{A}\angle{-360})\mathbf{]} +\ \mathbf{[}\ (\mathbf{B}\angle{-90} + \mathbf{B}\angle{-270})\mathbf{]}\ )$$

$$\mathbf{B''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-90} + \mathbf{A}\angle{-270})\mathbf{]} +\ \mathbf{[}\ (\mathbf{B}\angle{-180} + \mathbf{B}\angle{-180})\mathbf{]}\ )$$

-90 and -270 cancel, so we have

$$\mathbf{A''} = 1/2\ \mathbf{[}\ \mathbf{A}\angle{-360} + \mathbf{A}\angle{-360}\ \mathbf{] = A}$$

$$\mathbf{B''} = 1/2\ \mathbf{[}\ \mathbf{B}\angle{-180} + \mathbf{B}\angle{-180}\ \mathbf{] = B}\angle{-180}$$

So the outputs have interchanged compared to the unswitched state and now the phase of **B’’** lags **B** by 180 degrees, or is opposite electrical polarity relative to **B**.

Switching the Other Leg

After the first hybrid and the phase switches we have

$$\mathbf{A'} = 1/\sqrt{2}\ (\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-180}) \qquad \text{(unaltered)}$$

$$\mathbf{B'} = 1/\sqrt{2}\ \mathbf{[}(\ \mathbf{A}\angle{-180} + \mathbf{B}\angle{-90})\mathbf{]}\angle{-180}\ =\ 1/\sqrt{2}\ (\ \mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})$$

Putting this through the second coupler, setting that equal to **A’’** and **B’’** respectively and substituting the original expressions for **A’** and **B’**,

$$\mathbf{A''} = 1/\sqrt{2}\ (\ \mathbf{[}1/\sqrt{2}\ (\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-180})\mathbf{]}\angle{-90} +\ \mathbf{[}1/\sqrt{2}\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})\mathbf{]}\angle{-180})$$

$$\mathbf{B''} = 1/\sqrt{2}\ (\ \mathbf{[}1/\sqrt{2}\ (\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-180})\mathbf{]}\angle{-180} +\mathbf{[}1/\sqrt{2}\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})\mathbf{]}\angle{-90})$$

Factoring out the 1/√2’s,

$$\mathbf{A''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-180})\mathbf{]}\angle{-90} +\ \mathbf{[}\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})\mathbf{]}\angle{-180})$$

$$\mathbf{B''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-180})\mathbf{]}\angle{-180} +\mathbf{[}\ (\mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})\mathbf{]}\angle{-90})$$

Applying the phase shifts from outside the brackets,

$$\mathbf{A''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-180} + \mathbf{B}\angle{-270})\mathbf{]} +\ \mathbf{[}\ (\ \mathbf{A}\angle{-180} + \mathbf{B}\angle{-450})\mathbf{]}\ ) \qquad \text{Note: } -450 = -90$$

$$\mathbf{B''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})\mathbf{]} +\ \mathbf{[}\ (\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-360})\mathbf{]}\ )$$

Rearranging terms,

$$\mathbf{A''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-180} + \mathbf{A}\angle{-180})\mathbf{]} +\ \mathbf{[}\ (\mathbf{B}\angle{-270} + \mathbf{B}\angle{-90})\mathbf{]}\ )$$

$$\mathbf{B''} = 1/2\ (\ \mathbf{[}\ (\ \mathbf{A}\angle{-270} + \mathbf{A}\angle{-90})\mathbf{]} +\ \mathbf{[}\ (\mathbf{B}\angle{-360} + \mathbf{B}\angle{-360})\mathbf{]}\ )$$

Adding or canceling angles,

$$\mathbf{A''} = 1/2\ \mathbf{[}\ \mathbf{A}\angle{-180} + \mathbf{A}\angle{-180})\mathbf{] = A}\angle{-180}$$

$$\mathbf{B''} = 1/2\ \mathbf{[}\ \mathbf{B}\angle{-360} + \mathbf{B}\angle{-360})\mathbf{] = B}$$

A similar result to switching the **A'** leg.

Switching Both Legs

After the first hybrid and the phase switches we have

$$\mathbf{A'}\angle{-180} = 1/\sqrt{2}\ \mathbf{[}(\ \mathbf{A}\angle{-90} + \mathbf{B}\angle{-180})\mathbf{]}\ \angle{-180} = 1/\sqrt{2}\ (\ \mathbf{A}\angle{-270} + \mathbf{B}\angle{-360})$$

$$\mathbf{B'}\angle{-180} = 1/\sqrt{2}\ \mathbf{[}(\ \mathbf{A}\angle{-180} + \mathbf{B}\angle{-90})\mathbf{]}\ \angle{-180} = 1/\sqrt{2}\ (\ \mathbf{A}\angle{-360} + \mathbf{B}\angle{-270})$$

After the second coupler and setting that equal to **A''** and **B''** respectively,

$$\mathbf{A''} = 1/2\ (\mathbf{[A}\angle{-270} + \mathbf{B}\angle{-360}\mathbf{]}\angle{-90} + \mathbf{[A}\angle{-360} + \mathbf{B}\angle{-270}\mathbf{]}\angle{-180})$$

$$\mathbf{B''} = 1/2\ (\mathbf{[A}\angle{-270} + \mathbf{B}\angle{-360}\mathbf{]}\angle{-180} + \mathbf{[A}\angle{-360} + \mathbf{B}\angle{-270}\mathbf{]}\angle{-90})$$

Applying the phase shifts from outside the brackets,

$$\mathbf{A''} = 1/2\ (\mathbf{[A}\angle{-360} + \mathbf{B}\angle{-90}\mathbf{]} + \mathbf{[A}\angle{-180} + \mathbf{B}\angle{-450}\mathbf{]})$$

$$\mathbf{B''} = 1/2\ (\mathbf{[A}\angle{-450} + \mathbf{B}\angle{-180}\mathbf{]} + \mathbf{[A}\angle{-450} + \mathbf{B}\angle{-360}\mathbf{]})$$

Which reduces to

$$\mathbf{A''} = \mathbf{B}\angle{-90}$$

$$\mathbf{B''} = \mathbf{A}\angle{-90}$$

Conclusions

Two 90 degree hybrids produce the same result as two 180 degree hybrids with the exception of the phase switch state for a given result. For 90 degree hybrids it would seem that in the non-zero phase difference for continuum observing is moot, continuum observing detects power which is the square of the voltage and thus not phase sensitive. GBT Spectrometry is also not phase sensitive, so it appears two 90 degree branch line hybrids would be sufficient for replacing 180 degree hybrids in a differential radiometer such as the GBT Ka Receiver.

References

(1) http://www.microwaves101.com/encyclopedia/magictee.cfm

(2) 'Introductory Microwave Techniques,' Richard W. Tinnel, 1965 Holt, Rinehart and Winston, Inc.

(3) http://www.microwaves101.com/encyclopedia/Branchline_couplers.cfm