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INTERFEROMETRY

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I. INTRODUCTION

For a telescope whose aperture is r, the angular resolution $\Delta\theta \sim \frac{\lambda}{r}$. Since $\lambda_{radio} >> \lambda_{light}$, the resolution of radio telescopes is comparatively poor.

<u>Techniques for Obtaining High Angular Resolution</u>

1. Lunar occultations

$\Delta\theta$ depends on source strength.

For strong sources $\Delta\theta \sim 0.5''$.

Gives considerable information, but only for small number of sources.

2. Interplanetary scintillations

Irregularities in the electron density of the solar corona (scale $\sim$ 100 km) cause a very small source to flicker or scintillate. A large source does not. Frosted glass effect. Study of variation in these scintillations with distance from the sun can give information about small-scale structure. Useful but rather qualitative.

3. Interferometry

Trick which uses two or more small antennas separated by a distance r, called the "physical baseline." This gives same angular resolution as a large telescope whose aperture covers the whole baseline.

Same angular resolution.
(Not same sensitivity.)

This is the most powerful method of achieving high resolutions and has made a major contribution to astronomy.

Types of Interferometers

1. Cable link interferometers

   Only practical for $r \lesssim 3$ km.

   [NRAO, Cambridge, CalTech, Malvern, Parkes]

2. Radio link interferometers

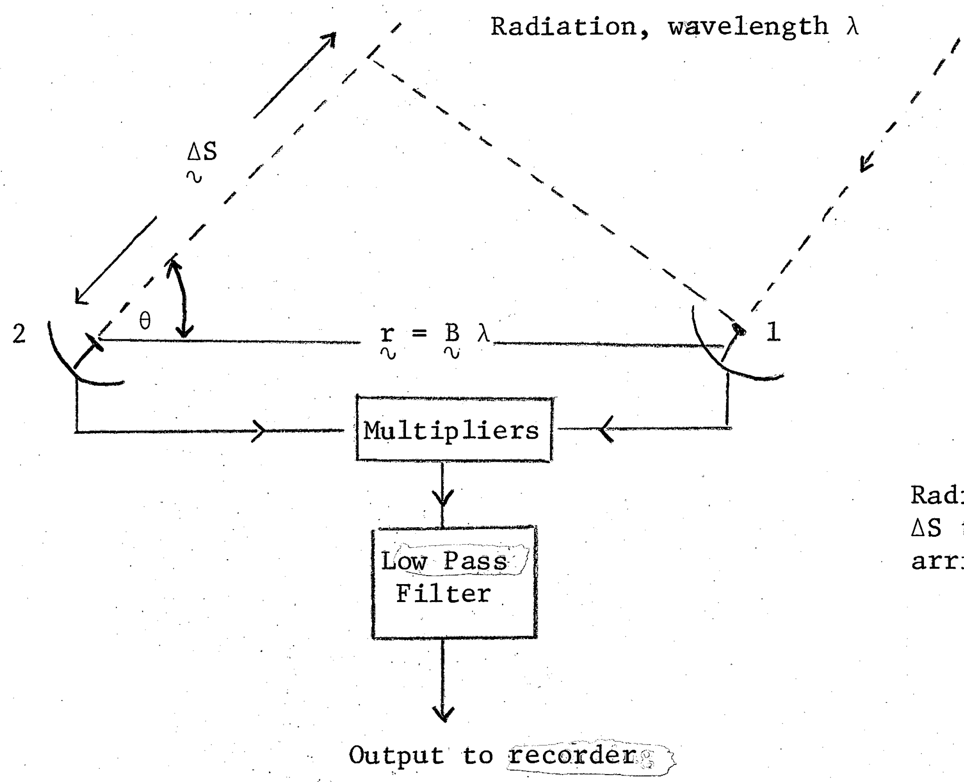
   $r \lesssim 200$ km. For larger r many repeater stations are needed. This means complicated and unreliable electronics.

   [Jodrell Bank, NRAO]

3. Interferometers without a physical link

   Signals from each antenna are recorded on tape and processed at a later date. Completely independent receivers at each antenna. Only recently possible due to development of accurate frequency standards. [NRAO, Canada] [CalTech, Jodrell Bank]

II. RESPONSE TO A POINT SOURCE



<u>Assumptions</u>

1. Point source at infinity.
2. Monochromatic radiation.
3. Unpolarized radiation.
4. Identical antennas.
5. Equal path lengths from antennas to multiplier.

Radiation travels extra distance ΔS to antenna 2. It therefore arrives later by time $\frac{\Delta S}{C}$.

Antennas give voltage out proportional to electric field at input. Therefore, voltages at multiplier inputs are

$$V_1 = V_{01} \cos 2\pi ft$$

$$V_2 = V_{02} \cos (2\pi ft + \phi) \quad \text{where } \phi = \frac{2\pi}{\lambda} (\Delta S)$$

Output of multiplier is

$$M = V_1 V_2$$

$$= V_{01} V_{02} \cos (2\pi ft) \cos (2\pi ft + \phi)$$

Using simple trigonometry

$$M = \frac{1}{2} V_{01}V_{02} \cos \phi + \frac{1}{2} V_{01}V_{02} \cos [2\pi(2f) + \phi] .$$

The low pass filter rejects the higher frequency term.

Output of correlator

$$R \propto V_{01}V_{02} \cos \phi .$$

Now

$$V_{01}V_{02} \propto |\text{electric field of radiation}|^2$$
$$\propto S$$

where S is the flux density of the radiation received from the source.

Therefore,

$$R \propto S \cos \phi$$

Now

$$\phi = \frac{2\pi}{\lambda} (\Delta S)$$
$$= 2\pi B \cos \theta$$

$$\boxed{R \propto S \cos [2\pi B \cos \theta]} \qquad (1)$$

This is the fundamental equation of interferometry. (In practical interferometers there is usually also a constant instrumental phase term which can be calibrated out.)

As the earth rotates the angle θ changes and the output of our interferometer is a set of "interference fringes" whose amplitude is proportional to the flux density of the source.

Depending on the position of the source in the sky, the output of an interferometer will be positive or negative. Thus the output of an interferometer can be visualized as a set of imaginary positive and negative lobes in the sky--the interferometer polar diagram. We can think of this comb-like structure as being fixed as shown in Fig. 1. The source passes through the lobes as the earth rotates. It is important to realize that the lobe structure of the interferometer is determined entirely by the physical baseline and is independent of the antennas. If the antennas are directional, their pointing will not effect the position of the lobes but will effectively "illuminate" different parts of the polar diagram.

One measure of the resolving power of an interferometer is the lobe separation or fringe width. This is the angular separation of two adjacent lobes in the fringe pattern. From equation 1 it can be seen that a maximum will occur for every value of θ that satisfies the relation

$$2\pi B \cos\theta = 2n\pi \quad n = 0,\ n = 0, \pm 1, \pm 2, \ldots$$

If θ_1 and θ_2 are values of θ which correspond to two adjacent minima

$$2\pi B (\cos\theta_1 - \cos\theta_2) = 2(n+1)\pi - 2n\pi$$

Now $$\theta_2 = \theta_1 + \Delta\theta$$

where $\Delta\theta$ is the fringe width

$$\cos\theta_1 - \cos\theta_2 = \cos\theta_1 - \cos(\theta_1 + \Delta\theta)$$

$$= \Delta\theta \sin\theta .$$

Therefore

$$\boxed{\Delta\theta = \frac{1}{B\sin\theta} = \frac{1}{B_p}} \qquad (2)$$

$B_p = B\sin\theta$ is the component of the physical baseline perpendicular to the direction of the source and it is called the "projected baseline" of the interferometer. If B is expressed in wavelengths then $\Delta\theta$ will be in radians.

From equation 2 it is clear that the projected baseline and the fringe width vary with θ, and the angular position of the radio source in the sky.

III. THE EFFECT OF FINITE BANDWIDTHS

So far the radiation has been assumed monochromatic. If the interferometer receives radiation over a band of frequencies, then the total response is the sum of the responses due to the various components. If the voltage gains in the two channels depend on frequency f, and are given by $G_1(f)$ and $G_2(f)$, respectively, then the response of the interferometer to an infinitesimal frequency band df is proportional to $G_1 G_2 \cos\phi \, df$.

Now

$$\phi = \frac{2\pi}{\lambda}(\Delta S)$$

$$= 2\, f(\Delta\tau)$$

where $\Delta\tau \frac{\Delta S}{C}$ is the overall differential time delay from the source to the two inputs of the multipliers.

Therefore,

$$dR(f,\Delta\tau) \propto G_1(f) \cos (2\pi f \Delta\tau) \, df.$$

The total response of the interferometer is then

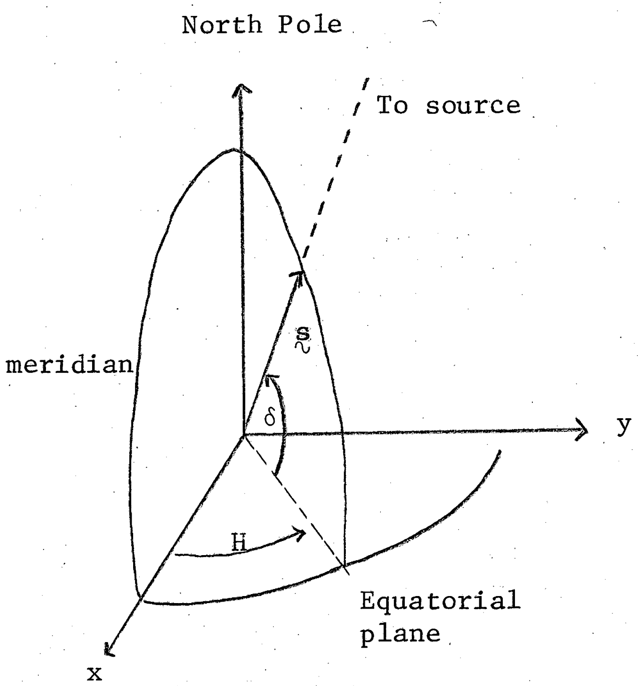
$$R(\Delta\tau) \propto \int_{-\infty}^{\infty} G_1(f) \, G_2(f) \cos (2\pi f \Delta\tau) \, df. \tag{3}$$

If the receivers have rectangular passbands of width Δf_o, equation (3) reduces to

$$\boxed{R(\Delta\tau) \propto \frac{\sin (\pi \Delta f_o \Delta\tau)}{\pi \Delta f_o \Delta\tau}} \tag{4}$$

The situation is analogous to the case of white light fringes in optics. R is a maximum when $\Delta\tau = 0$ and all the various frequency components are in phase. To maintain this condition, a compensating delay must be inserted in one arm of the interferometer, and this must be continuously adjusted as the source moves through the sky. From equation (4) it is clear that the larger the bandwidth Δf_o the more accurately the delay must be adjusted to minimize $\Delta\tau$ and make R a maximum.

III. GEOMETRY OF THE BASELINE - UV ELLIPSES



The direction of a radio source can be specified in astronomical coordinates by two angles:

declination δ
hour angle H

Alternatively it can be described by a unit vector $\underset{\sim}{s}$ in the direction of the source.

We can resolve $\underset{\sim}{s}$ into components

$$s_x = \cos\delta \cos H$$

$$s_y = \cos\delta \sin H$$

$$s_z = \sin\delta$$

Now the physical baseline $\underset{\sim}{B}$ is also a vector and can be resolved into B_x, B_y, B_z. The angle between the direction of the source and the direction of the baseline is given by

$$\cos\theta = \frac{\underset{\sim}{B} \cdot \underset{\sim}{s}}{|\underset{\sim}{B}|}$$

$$\boxed{\cos\theta = \frac{1}{B}\left(B_x \cos\delta \cos H + B_y \cos\delta \sin H + B_z \sin\delta\right)} \qquad (5)$$

Of more interest, however, is the projected baseline B_p since this determines the angular resolution of our interferometer. B_p is the baseline as seen from the source and is in the plane perpendicular to $\underset{\sim}{s}$. Although B_p changes with hour angle, it can at any instant be resolved into a north-south component "v" and an east-west component "u" (note that B_p is only 2-dimensional and has no component in the direction of the source). The relation of u and v to B_x, B_y, and B_z is shown in Fig. 2.

By inspection it follows that

$$\boxed{\begin{aligned} u &= B_y \cos H - B_x \sin H \\ v &= B_z \cos\delta - B_x \sin\delta \cos H - B_y \sin\delta \sin H \end{aligned}} \qquad (6)$$

It can easily be verified that

$$\frac{u^2}{a^2} + \frac{(v - v_o)^2}{b^2} = 1$$

where

$$a = \sqrt{B_x^2 + B_y^2}$$

$$b = \sin\delta \sqrt{B_x^2 + B_y^2}$$

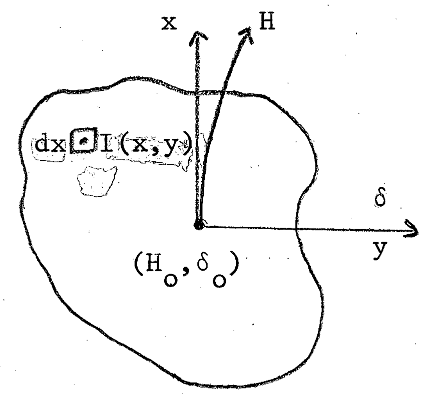
$$v_o = B_z \cos\delta$$

This is the equation for an ellipse where a is the semi-major axis, b is the semi-minor axis. The ellipse is centered at the point $(0, v_o)$ and has eccentricity $\cos\delta$.

Thus as the hour angle of the source changes the tip of the projected baseline vector describes an ellipse. An observer at the source looking at our interferometer would see one telescope appear to describe an ellipse about the other due to the rotation of the earth. Note that this ellipse depends on both the declination of the source and on the parameters of the baseline. For sources at the equator ($\delta = 0$) the ellipse degenerates to a straight line and for sources at the pole ($\delta = \pi/2$) the ellipse becomes a circle. The "resolution ellipse" or the "u-v ellipse" is very important in deriving the structure of extended radio sources from observations with a tracking interferometer.

V. RESPONSE TO AN EXTENDED SOURCE

We are now fully equipped to examine the response of our interferometer to an extended source of radiation. We still assume that this source is quasi-monochromatic, unpolarized, and that it is much smaller than the primary beam pattern of the antenna. Such an extended source can be imagined to consist of a very large number of infinitesimal point sources, each having an hour angle H and a declination D. Let the brightness centroid of the source have an hour angle H_o and a declination D_o. Since hour angle and declination are not orthogonal coordinates, it is convenient to define a Cartesian coordinate system, x, y whose origin is at the brightness centroid by



$$x = H \cos \delta - H_o \cos \delta_o = (H-H_o) \cos \delta$$

and

$$y = \delta - \delta_o .$$

x and y are now orthogonal and this will greatly simplify our problem.

Suppose that an infinitesimal element of area dxdy at the point (x,y) on the source contributes a brightness of intensity I(x,y)dxdy. Treating it as a point source and using equation (1), we can write down the response of our interferometer to this element as

$$dR \propto I(x,y) \cos (2\pi B \cos \theta) \, dxdy. \qquad (7)$$

We can use a Taylor series to expand cos θ about the centroid.

$$\cos \theta = \cos \theta_o + \frac{\partial}{\partial x} (\cos \theta) \, x + \frac{\partial}{\partial y} (\cos \theta) \, y$$

$$= \cos \theta_o + \frac{1}{\cos \delta} \left[\frac{\partial}{\partial H} (\cos \theta) \right] x + \left[\frac{\partial}{\partial \delta} (\cos \theta) \right] y.$$

Using equations (5) and (6) (the expressions we derived in celestial coordinates for cos θ, and the effective baseline components u and v), this immediately simplifies to

$$\cos \theta = \cos \theta_o + (u/B)x + (v/B) \, y. \qquad (8)$$

Combining equations (7) and (8), we have

$$dR(u,v) \propto I(x,y) \cos [2\pi B \cos \theta_o + 2\pi(ux+vy)]dxdy.$$

If we expand this expression and put $\phi_o = 2\pi B \cos \theta_o$ (the phase term due to a point source at the brightness centroid), we obtain

$$dR(u,v) \propto I(x,y) \{\cos [2\pi(ux+vy)] \cos \phi_o - \sin [2\pi(ux+vy)] \sin \phi_o\} \ dxdy.$$

By integrating we can immediately find the response to an extended source as

$$R \propto \cos \phi_o \iint_{-\infty}^{\infty} I(x,y) \cos [2\pi(ux+vy)] \, dxdy$$

$$- \sin \phi_o \iint_{-\infty}^{\infty} I(x,y) \sin [2\pi(ux+vy)] \, dxdy.$$

This can be written as

$$R \propto [C(u,v) \cos \phi_o - S(u,v) \sin \phi_o] \tag{9}$$

where

$$S = \iint_{-\infty}^{\infty} I(x,y)dxdy \tag{10}$$ is the flux density of the source.

$$C(u,v) = \frac{1}{S}\iint_{-\infty}^{\infty} I(x,y) \cos [2\pi(ux+vy)] \, dxdy \tag{11}$$

and

$$S(u,v) = \frac{1}{S}\iint_{-\infty}^{\infty} I(x,y)\ \sin\ [2\pi(ux+vy)]\ dxdy \tag{12}$$

C(u,v) and S(u,v) are the normalized cosine and sine Fourier transforms of the source brightness distribution. We can write equation (9) in an alternative form

$$\boxed{R(u,v) \propto \gamma(u,v)\ \cos\ \Phi(u,v)} \tag{13}$$

where

$$\gamma(u,v) = [C^2(u,v) + S^2(u,v)]^{1/2} \tag{14}$$

and

$$\Phi(u,v) = 2\pi B\ \cos\ \theta_o + \tan^{-1}\left[\frac{S(u,v)}{C(u,v)}\right] \tag{15}$$

$$= \phi + \phi(u,v)\ .$$

Now $\gamma(u,v)$ and $\phi(u,v)$ are the amplitude and phase of the "complex visibility function." This is defined by

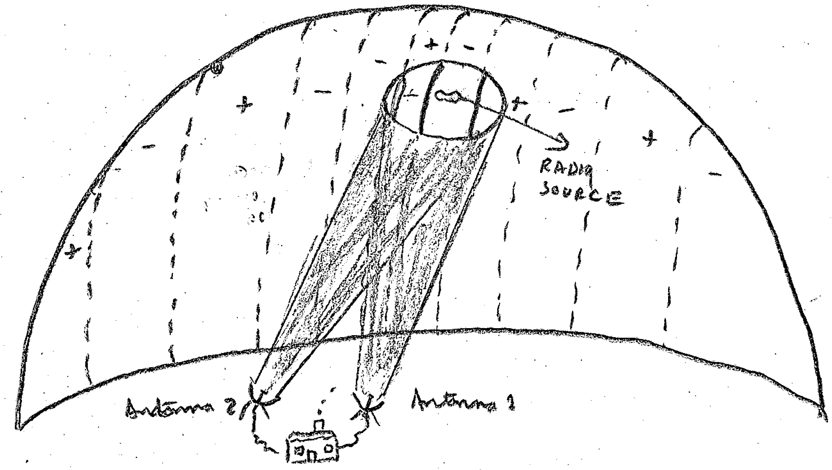
$$\Gamma(u,v) = \frac{1}{S}\tilde{I}(x,y) = \gamma(u,v)e^{i\phi(u,v)} \tag{16}$$

where $\tilde{I}(x,y)$ is the complex Fourier transform of the source brightness distribution I(x,y).

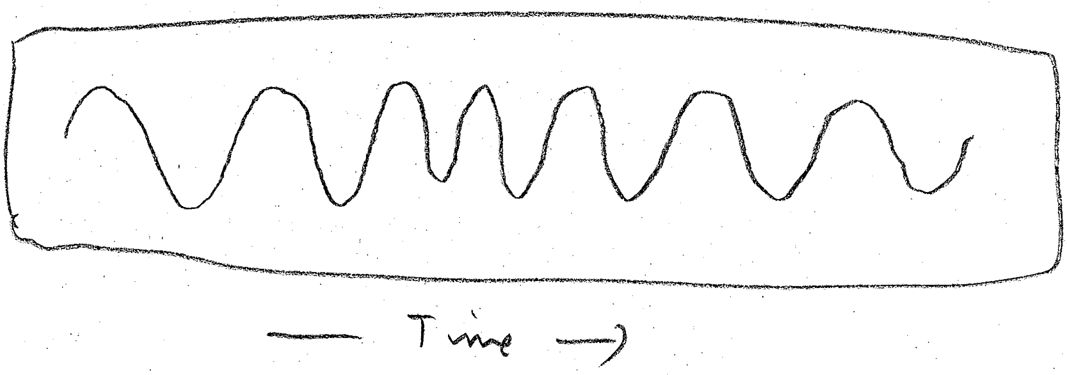
Therefore we can write the output of our interferometer as the

$$\boxed{R(u,v) \propto \mathrm{Re}[\Gamma(u,v)e^{i\phi_0(u,v)}]} \tag{15}$$

This is a set of sinusoidal fringes which are similar to those that would be observed from a point source located at the brightness centroid but modulated in amplitude and phase by the visibility function of the source. If both γ and ϕ are determined for all values of u and v, inverse Fourier transformation enables the brightness distribution of the source to be determined uniquely (synthesis). However, we have seen that for a fixed physical baseline the rotation of the earth gives rise to only a limited range of effective baselines. In the u-v plane these correspond to the resolution ellipse of the source. For long baselines the absolute measurement of phase is difficult and there is usually no information about $\phi(u,v)$. Under these circumstances it is usual to propose a model for the source structure. The assumed $I(x,y)$ is then used to derive γ's for the projected baselines that are observed. These predicted γ's are then compared with the observed γ's.



The interferometer 'phase polar diagram'.



The output fringes

Fig 1.

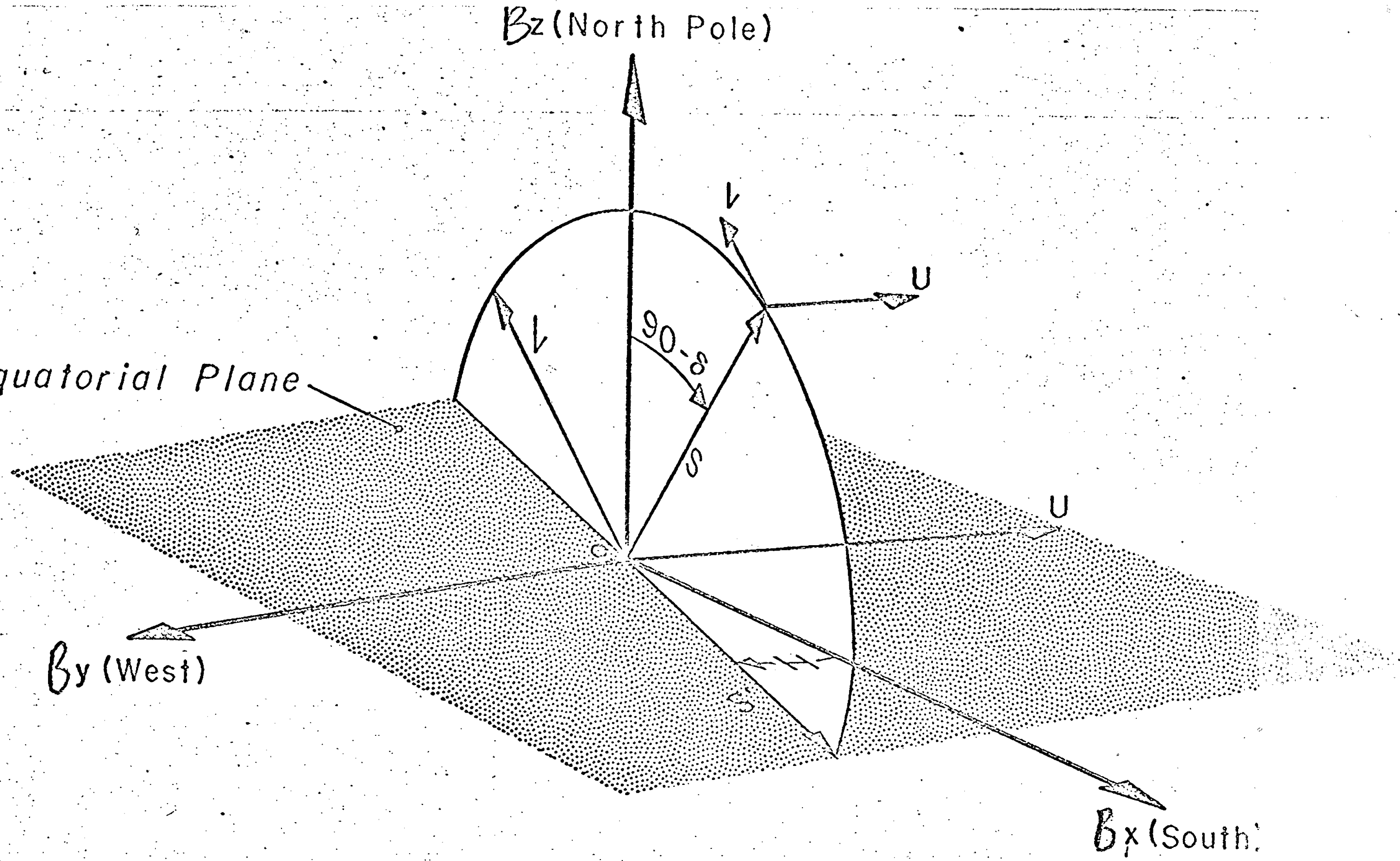


Fig 2