Holes in the u-v Diagram

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If we have an N by N/2 u-v diagram containing n holes (u_i, v_i), the extra sidelobe (x,y) is, in the principal solution

$$B = \frac{4}{N^2} \sum_{L=1}^{n} \cos (xu_i + yv_i) \quad (1)$$

The maximum possible sidelobe is thus obviously $\frac{2n}{N^2}$. We may also compute the rms sidelobe level

$$\overline{B^2} = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\pi}^{\pi} \left[\frac{2}{N^2} \sum_{L=1}^{n} \cos (xu_i + yv_i)\right]^2 dx\, dy \quad (2)$$

$$= \frac{1}{2\pi^2} \frac{n}{N^4} , \quad (3)$$

essentially Parsevall's theorem. For the diagrams we have been considering this is a very small number, as N is quite large. Typically, N = 200, n = 3000

$$\sqrt{\overline{B^2}} = .00031 = -35 \text{ dB}.$$

It is also of interest to know the largest sidelobe generated by the holes. The maximum possible is $\frac{2n}{N^2}$. This will be very nearly attained in one case of interest: where most of the holes are concentrated near the center of the u,v plane, they give rise to a comparatively broad pattern surrounding the array beam, whose height is $2n/N^2$. For the maximum sidelobes far from the array beam we may make a quick estimate by replacing the cosine wave of eq. (1) by a square unit of absolute value, which changes sign at odd multiples of $\pi/2$. Then, far from the origin, -1 and 1 are about equally likely and the probability distribution of B approaches the binomial distribution

$$\text{Prob}\left(\frac{N^2}{2} B = m\right) = 2^{-n} \frac{n!}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!} \tag{4}$$

$$\approx \sqrt{\frac{2}{\pi m}}\; e^{-\frac{m^2}{2n}} \tag{5}$$

The probability that the sidelobe is less than x is

$$\text{Prob}\,(|B| < x) = 2\,\text{Erf}\left(\frac{N^2 x}{2\sqrt{n}}\right). \tag{6}$$

The field of view contains about $\frac{N^2}{2}$ independent locations, so the strongest sidelobe in the field of view would have probability about $\frac{1}{N^2}$. So,

$$x_o = \frac{2\sqrt{n}}{N^2}\,\text{Erf}^{-1}\,\frac{1}{2}\left(1 - \frac{1}{N^2}\right) \tag{7}$$

Again with N = 200, n = 3000,

$x_o = .012 = -19$ dB.

Returning to the area near the array beam, if we say that

$$\cos\,(xu_i + yv_i) \approx 1 \text{ for } \sqrt{(u_i^2 + u_i^2)\,(x^2 + y^2)} < \frac{\pi}{2} \tag{8}$$

and is equally distributed between +1 and -1 otherwise, then the random sidelobe distribution (5) is superimposed on a hill

$$\frac{2}{N^2}\; n\left(\frac{\pi}{2\sqrt{x^2 + y^2}}\right) \tag{9}$$

where n(m) is the number of holes within radius m. The size of this hill is, along with n, a suitable measure of the convenience of using the array. This suggests new figures of merit, x_o, defined by eq. (7), the far-out sidelobe **maximum**, and

$$p = 2\int_0^{\pi/2} n\left(\frac{\pi}{2r}\right)\, 2\pi r dr \tag{10}$$

$$= \frac{\pi^3}{2} \sum_{i=1}^{n} \frac{1}{u_i^2 + v_i^2}\,, \tag{11}$$

The proportion of power in this error plateau compared to that in the main array beam. In eq. (10) and (11) it has been assumed that the u_i and v_i are measured in units of cell size.

There is a second question relating to holes: given data with holes, how does one process it? There are two obvious things to do with holes; they can be ignored or one may interpolate across them. Although at first sight the second may sound better than the first, this is not clear on further reflection.

Ignoring the holes is the best one can do in a sense of minimizing the square error. If one fits the synthesized beam by least squares to a desired beam shape over the entire field of view, since the cos $(u_i x + v_i y)$ are orthogonal on the field of view, the matrix of condition is diagonal and the coefficient assigned to any spacing is independent of the presence or absence of data from any other.

However, if one tries to minimize the absolute values of the largest sidelobes, or tires to minimize the sidelobes only over a portion of the field of view, one can do much more. However, the solutions are difficult, and are not just a simple interpolation. As an example of an interpolation, let us consider a one dimensional case, a uniformly spaced, uniformly weighted line. Let there be a hole at u. Then if this hole is ignored, there is an error term $-\cos 2\pi ux$, appearing in the synthesized beam. If we simply interpolate across the hole between u-1 and u+1, the error term becomes

$$\frac{1}{2}(\cos 2\pi(u-1)x + \cos 2\pi(u+1)x) - \cos 2\pi ux$$

$$= -2 \sin^2 \pi x \cos 2\pi ux$$

which over the field of view, u = -1, 1 has a larger maximum value (2 instead of 1) and rms value ($\sqrt{3/4}$ instead of $\sqrt{1/2}$). It does, however, avoid contributing to the central error plateau.

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