TWO BIT CORRELATOR

by B. G. Clark

In one bit correlation one records only the sign -- + or -. In two bit correlation, we have a sign and a magnitude bit, giving four possible states. Let us denote the digitization of a variable A by [A] and denote the four possible states by

$$[A] = \begin{cases} +L & \text{if } \quad A > H \\ +S & \text{if } \quad H > A > 0 \\ -S & \text{if } \quad 0 > A > -H \\ -L & \text{if } \quad -H > A \end{cases}$$

where H is the second clipping level. We now have the freedom to define a multiplication table [A] * [B], which, because of the symmetries of the problem may, without loss of generality, be taken to be

[A] * [B] =

[B]\[A]	-L	-S	+S	+L
-L	n	m	-m	-n
-S	m	1	-1	-m
+S	-m	-1	1	m
+L	-n	-m	m	n

if A and B are equal independent gaussion noises from the receivers, and C is the noise from the source,

$$\text{Prob}\ \{\ X < A < X + dX\ \} = \text{Prob}\ \{\ X < B < X + dX\ \} = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{x^2}{2\sigma^2}}\ dx$$

$$\text{Prob}\ \{\ X < C < X + dX\ \} = \frac{1}{\sqrt{2\pi}\ \sigma_1}\ e^{-\frac{x^2}{2\sigma_1^{\ 2}}}\ dX$$

The correlator output is $< [A+C] * [B+C] >$.

Employing all the symmetries of the problem,

$$< [A+C] * [B+C] > \; = 2n\,[\,\text{Prob}\,\{A + C > H \cap B + C > H\}$$
$$- \text{Prob}\,\{A + C > H \cap B + C < -H\}\,]$$
$$+ 4m\,[\,\text{Prob}\,\{A + C > H \cap H > B + C > 0\}$$
$$- \text{Prob}\,\{A + C > H \cap 0 > B + C > -H\}\,]$$
$$+ 2\,[\,\text{Prob}\,\{H > A + C > 0 \cap H > B + C > 0\}$$
$$- \text{Prob}\,\{H > A + C > 0 \cap 0 > B + C > -H\}\,]$$

$$= 2 \int_{x=-\infty}^{\infty} \text{Prob}\,\{X < C < X + dX\}\,[\,n\,\text{Prob}\,\{A > H - X\}$$
$$(\,\text{Prob}\,\{B > H - X\} - \text{Prob}\,\{B > H + X\}\,)$$
$$+ 2m\,\text{Prob}\,\{A > H - X\}\;(\,\text{Prob}\,\{H - X > B > -X\}$$
$$- \text{Prob}\,\{H + X > B > X\}\,) + \text{Prob}\,\{H - X > A > -X\}$$
$$(\,\text{Prob}\,\{H - X > B > -X\} - \text{Prob}\,\{H + X > B > X\}\,)\,]$$

$$= 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_1}\, e^{-\frac{x^2}{2\sigma_1^2}}\,[\,n\,(\tfrac{1}{2} - \text{Erf}\,(\tfrac{H-X}{\sigma}))$$
$$(\,\text{Erf}\,(\tfrac{H+X}{\sigma}) - \text{Erf}\,(\tfrac{H-X}{\sigma}))$$
$$+ 2m\,(\tfrac{1}{2} - \text{Erf}\,(\tfrac{H-X}{\sigma}))\,(2\,\text{Erf}\,(\tfrac{X}{\sigma}) + \text{Erf}\,(\tfrac{H-X}{\sigma})$$
$$- \text{Erf}\,(\tfrac{H+X}{\sigma}) + (\text{Erf}\,(\tfrac{H-X}{\sigma}) + \text{Erf}\,(\tfrac{X}{\sigma}))\,(2\,\text{Erf}\,(\tfrac{X}{\sigma})$$
$$+ \text{Erf}\,(\tfrac{H-X}{\sigma}) - \text{Erf}\,(\tfrac{H+X}{\sigma}))\,]\;dX$$

where

$$\text{Erf}\,(y) = \int_0^y \frac{1}{\sqrt{2\pi y}}\, e^{-\frac{t^2}{2}}\, dt$$

If we approximate a low signal-to-noise ratio, $\sigma_1 << \sigma$, then for all X of interest $X << \sigma$ and we can expand the error functions

$$\mathrm{Erf}\,(C + y) \approx \mathrm{Erf}\,(C) + y \frac{1}{\sqrt{2\pi}} e^{-\frac{C^2}{2}}$$

then

$$< [A + C] * [B + C] > = 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{x^2}{2\sigma_1^{\,2}}} \Big[n \frac{x^2}{\pi\sigma^2} e^{\frac{H^2}{\sigma^2}}$$

$$+ m \frac{2X^2}{\pi\sigma^2} e^{-\frac{H^2}{2\sigma^2}} \left(1 - e^{-\frac{H^2}{2\sigma^2}} \right)$$

$$+ \frac{X^2}{\pi\sigma^2} \left(1 - e^{-\frac{H^2}{2\sigma^2}}\right)^2 \Big] dX$$

$$= \frac{2}{\pi} \frac{\sigma_1^{\,2}}{\sigma^2} \Big[(n + 1 - 2m) e^{-\frac{H}{\sigma^2}}$$

$$+ (2m - 2) e^{-\frac{H^2}{2\sigma^2}} + 1 \Big]$$

The noise on the correlator output is

$$N^2 = < ([A] * [B])^2 > = n^2 \ \mathrm{Prob} \{ |A| > H \cap |B| > H$$

$$+ 2m^2 \ \mathrm{Prob} \{ |A| > H \cap |B| < H \}$$

$$+ \mathrm{Prob} \{ |A| < H \cap |B| < H \}$$

$$= n^2 \left(1 - 2\,\mathrm{Erf}\left(\frac{H}{\sigma}\right)\right)^2 + 4m^2 \left(1 - 2\,\mathrm{Erf}\left(\frac{H}{\sigma}\right)\right) \mathrm{Erf}\left(\frac{H}{\sigma}\right))$$

$$+ 4 \left(\mathrm{Erf}\left(\frac{H}{\sigma}\right)\right)^2$$

For continuous multiplication, the signal-to-noise ratio is simply $\frac{\sigma_1^2}{\sigma^2}$, so the degradation produced by the quantization is

$$D = (S/N) / (S_c/N_c) = \frac{2}{\pi} \left[(n + 1 - 2m) e^{-\frac{H^2}{\sigma^2}} + (2m - 2) e^{-\frac{H^2}{2\sigma^2}} + 1 \right]$$

$$\left[(4n^2 + 4 - 8m^2) \left(Erf\left(\frac{H}{\sigma}\right)\right)^2 + (4m^2 - 4n^2)\, Erf\left(\frac{H}{\sigma}\right) + n^2 \right]^{-\frac{1}{2}}$$

David Sun has programmed this equation and solves for the optimum $(H/_\sigma)$ given m and n. Below are two tables giving, respectively, D and $(H/_\sigma)$.

D

m \ n	2	3	4	6	8	12	16	20	24
1	.787	.838	.855	.858	.851	.839	.832	.828	.825
2	.745	.800	.842	.875	.878	.865	.853	.845	.839
3		.769	.802	.855	.869	.880	.870	.860	.852
4			.771	.821	.857	.881	.880	.872	.864
5				.790	.830	.872	.881	.879	.873
6				.765	.803	.856	.877	.881	.878
7					.779	.837	.867	.878	.880
8					.759	.818	.854	.872	.878

H/σ

m \ n	2	3	4	6	8	12	16	20	24
1	.701	.782	.821	.828	.796	.740	.707	.687	.674
2	1.371	1.049	.977	.967	.948	.864	.800	.759	.733
3		1.410	1.199	1.036	1.006	.953	.885	.830	.792
4			1.385	1.122	1.036	.989	.942	.891	.847
5				1.223	1.077	1.002	.970	.932	.892
6				1.305	1.135	1.012	.981	.955	.924
7					1.197	1.028	.987	.967	.944
8					1.251	1.053	.993	.972	.955

The optimum signal-to-noise ratio occurs at

$m = 4, \quad n = 14, \quad H/\sigma = .967, \quad D = .882.$

However, the convenience of having n and m be powers of two would recommend

$$m = 4, \quad n = 16, \quad H/_{\sigma} = .952, \quad D = .880$$

at a negligible loss in signal-to-noise ratio.

The signal-to-noise ratio of the two bit correlator is thus about .88 that of a continuous multiplier, as contrasted with .64 for a one bit correlator.

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