National Radio Astronomy Observatory
Summer Lecture Series - 1966

Theory of Interferometers and Aperture Synthesis

W. C. Tyler

1. Interferometer Response to a Point Source

Consider a simple interferometer of the type shown in Figure 1. The two antennas comprising the interferometer are separated by a baseline distance D/λ (in wavelengths). The antennas have effective areas A_1 and A_2, and associated with each antenna is electronic equipment having power gains G_1 and G_2 respectively. The common bandwidth of the receivers is ΔF.

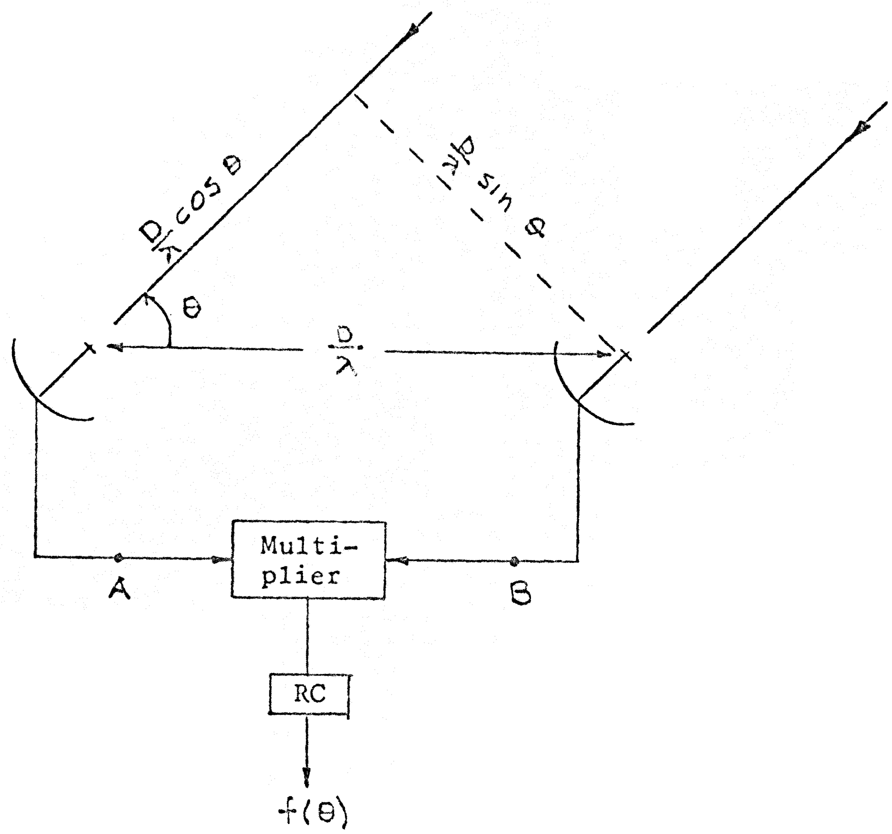

Figure 1.

We now wish to determine the response of the interferometer to a point source of flux density S. The line joining the phase centers of the antennas makes an angle θ with the direction to the source. The powers available at the (polarized) antenna feed terminals are, respectively,

$$\frac{1}{2} A_1 S\Delta f$$

and

$$\frac{1}{2} A_2 S\Delta f$$

The voltages at points A and B, neglecting impedance factors, are (approximately, for small Δf)

$$V_A = \frac{1}{\sqrt{2}} \sqrt{A_1 G_1 \Delta f S} \sin\left[\frac{2\pi}{\lambda}(ct - D\cos\theta)\right]$$

$$= \frac{1}{\sqrt{2}} \sqrt{A_1 G_1 \Delta f S} \sin\left(\omega t - \frac{2\pi D}{\lambda}\cos\theta\right)$$

$$V_B = \frac{1}{\sqrt{2}} \sqrt{A_2 G_2 \Delta f S} \sin(\omega t)$$

We have neglected here any instrumental phase differences up to points A and B.

The output of the multiplier and low pass filter is then seen to be

$$\boxed{f(\theta) = \frac{1}{4} A\cdot G\cdot \Delta f\cdot S\cdot \cos\left[\frac{2\pi D}{\lambda}\cos\theta\right]}$$

where $A = \sqrt{A_1 A_2}$, $G = \sqrt{G_1 G_2}$ and θ is a slowly varying function of time (due to motion of the source). The term $\cos\theta$ will now be developed in more useful terms.

2. Source - Baseline Geometry, Fringe Frequency

In Figure 2, we depict that portion of the celestial sphere above the local horizon. The source position will be given in terms of hour angle H and declination δ. The projection along the baseline to the celestial sphere establishes a point which we designate as the instrumental pole I. This point has a fixed hour angle h and declination d.

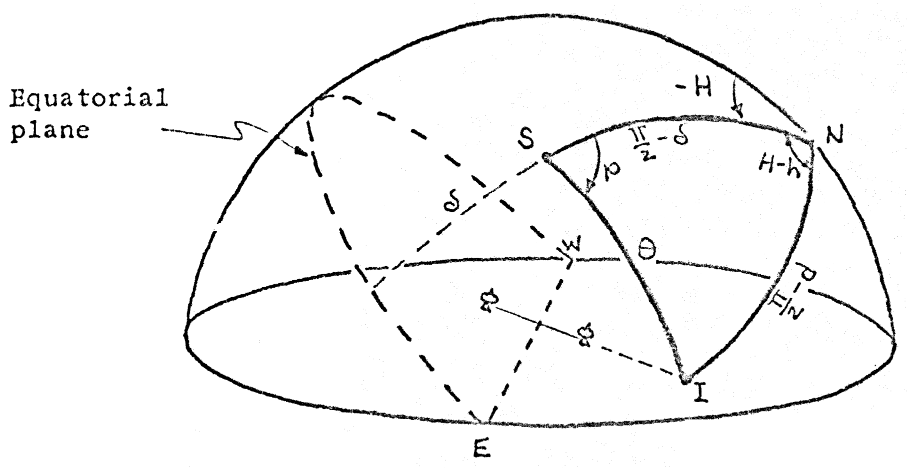

Figure 2.

The basic spherical triangle formed by the source (S), the north celestial pole (N) and the north instrumental pole (I) allows us to express cos θ in terms of the more fundamental source (H, δ) and baseline (h, d) coordinates. From the law of cosines we have

$$\cos\theta = \sin d \sin\delta + \cos d \cos\delta \cos(H-h)$$

Thus the interferometer response to a point source of flux density S at (H,δ) is

$$f(H,\delta) = \frac{1}{4} A\cdot G\cdot \Delta f\cdot S\cdot \cos\left[\frac{2\pi D}{\lambda}\{\sin d \sin \delta + \cos d \cos \delta \cos (H-h)\}\right]$$

This equation determines the output fringe frequency and phase as a function of source position (H,δ), baseline orientation (h,d) and baseline length (D/λ). For a given source, the only time dependent term is H. Therefore the fringe frequency R is obtained from

$$R(H,\delta) = \left|\frac{\partial}{\partial H}\left(\frac{D}{\lambda}\cos\theta\right)\right| \cdot \frac{dH}{dt}$$

$$R(H,\delta) = \left(\frac{2\pi}{86,400}\right) \frac{D}{\lambda} \cos d \cos \delta \sin (H-h) \text{ cycles/sider. sec.}$$

3. Projected Baseline - Length and Orientation

From Figures 1 and 2 it is seen that the projection of the baseline in direction of the source is $\frac{D}{\lambda}\sin\theta$ and lies at a position angle p. The East-West and North-South components of the projected baseline are then

EW

$$U = \frac{D}{\lambda}\sin\theta\cdot\sin p$$

NS

$$V = \frac{D}{\lambda}\sin\theta\cos p$$

Using the law of sines and the previous relation for cos θ we obtain

$$U = \frac{D}{\lambda}\cos d \sin (H-h)$$

$$V = \frac{D}{\lambda}[\sin d \cos \delta - \cos d \sin \delta \cos (H-h)]$$

Note that these relations are given directly by

$$\dot{U} = \frac{1}{\cos\delta}\frac{\partial}{\partial H}\left(\frac{D}{\lambda}\cos\theta\right)$$

$$V = \frac{\partial}{\partial\delta}\left(\frac{D}{\lambda}\cos\theta\right)$$

It is seen from the previous expression for the fringe frequency R(H,δ) and the above expression for U that

$$R(H,\delta) = \frac{U\cos\delta}{13{,}751} \text{ cycles/sidereal second}$$

and is independent of V, the North-South component of the projected baseline.

4. Response to an Extended Source

We have developed the expression for the response of the interferometer to a point source at (H,δ) having flux density S:

$$f(H,\delta) = \frac{1}{4}\,GA\Delta fS\cos\left(\frac{2\pi D}{\lambda}\cos\theta\right)$$

where

$$\cos\theta = \sin d\sin\delta + \cos d\cos\delta\cos(H-h)$$

Now consider an extended source as shown in Figure 3. The source dimensions are generally considered to be small compared to the beam width of the individual antennas comprising the interferometer. The source-centered coordinate system (x, y) is defined in terms of the astronomical co-

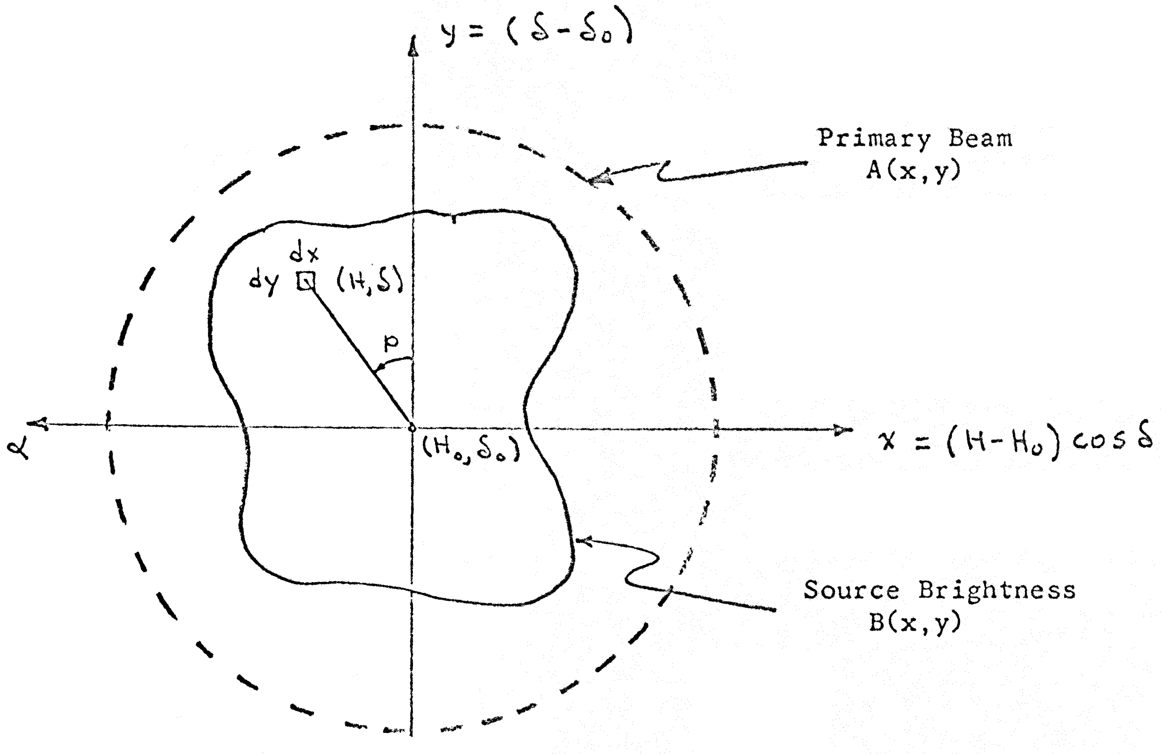

Figure 3.

ordinates (H,δ) by

$$\left.\begin{aligned} x &= (H-H_o)\cos\delta \\ y &= (\delta-\delta_o) \end{aligned}\right\} \text{(radians)}$$

We now consider the elemental solid angle $d\Omega = dxdy$ located at (H,δ) or (x,y), having brightness B(x,y). This solid angle contributes a flux density $dS = B(x,y)dxdy$ from the source. If the effective area of the (assumed identical) primary beams is A(x,y) then the response of the interferometer to this small portion of the source is

$$d^2f(H,\delta) = \frac{1}{4} G\Delta f\ B(x,y)\cdot A(x,y)\ \cos\left(\frac{2\pi D}{\lambda}\cos\theta\right) dxdy$$

(Note: If the effective areas are not identical, A(x,y) must be replaced by $\sqrt{A_1(x,y)\cdot A_2(x,y)}$).

We now expand cos θ about the origin of source coordinates (H_o, δ_o), using a Taylor series expansion and retaining only the first order terms.

$$\cos\theta \cong \cos\theta_o + \frac{1}{\cos\delta}\frac{\partial}{\partial H}(\cos\theta)\cdot x + \frac{\partial}{\partial\delta}(\cos\theta)\cdot y$$

From the previous expressions for U and V we may write this as

$$\frac{D}{\lambda}\cos\theta \cong \frac{D}{\lambda}\cos\theta_o + (Ux + Vy)$$

Therefore

$$\cos\left(\frac{2\pi D}{\lambda}\cos\theta\right) = \cos\left[\frac{2\pi D}{\lambda}(\cos\theta_o) + 2\pi(Ux + Vy)\right]$$

Expanding this and substituting it in the equation for $d^2f(H,\delta)$, we obtain

$$d^2f(H,\delta) = \frac{1}{4}G\Delta f\ \{\cos\left(\frac{2\pi D}{\lambda}\cos\theta_o\right) B(x,y)\ A(x,y)\ \cos\left[2\pi(Ux + Vy)\right]$$

$$- \sin\left(\frac{2\pi D}{\lambda}\cos\theta_o\right)\cdot B(x,y)\cdot A(x,y)\cdot\sin\left[2\pi(Ux + Vy)\right]\}dxdy$$

We now define the normalized antenna response as

$$a(x,y) = \frac{A(x,y)}{A}$$

where $A \doteq A(0,0)$. We also use the Rayleigh-Jeans approximation to relate brightness $B(x,y)$ to brightness temperature $T_B(x,y)$

$$B(x,y) = \frac{2k}{\lambda^2}\ T_B(x,y)$$

where k is Boltzmann's constant. Therefore

$$B(x,y)\cdot A(x,y) = \frac{2kA}{\lambda^2}\cdot a(x,y)\ T_B(x,y)$$

Using this and integrating over the brightness distribution we obtain

$$f(H,\delta) = \frac{1}{4} GAS\cdot\Delta f[C(U,V) \cos (\frac{2\pi D}{\lambda} \cos \theta_o) - S(U,V) \sin (\frac{2\pi D}{\lambda} \cos \theta_o)]$$

where

$$C(U,V) = \frac{\int\int a(x,y)\; T_B(x,y) \cos [2\pi(Ux + Vy)]\; dxdy}{\int\int T_B(x,y)\; dxdy}$$

$$S(U,V) = \frac{\int\int a(x,y)\; T_B(x,y) \sin [2\pi(Ux + Vy)]\; dxdy}{\int\int T_B(x,y)\; dxdy}$$

and where we have used the expression for flux density S

$$S = \frac{2k}{\lambda^2} \int\int T_B(x,y)\; dxdy$$

We may express $f(H,\delta)$ in yet another form which may be more illustrative

$$f(H,\delta) = \frac{1}{4}\; G\cdot A\cdot S\cdot \Delta f\cdot\; A(U,V) \cos [\frac{2\pi D}{\lambda} \cos \theta_o + \Phi(U,V)]$$

where $A(U,V) = \sqrt{C^2(U,V) + S^2(U,V)}$

$$\Phi(U,V) = -\tan^{-1}[\frac{S(U,V)}{C(U,V)}]$$

A(U,V) and $\Phi(U,V)$ define the complex visibility of the source brightness temperature distribution

$$\underset{\sim}{V}(U,V) = A(U,V)\; e^{-j\Phi(U,V)}$$

Thus it is seen that the output of the interferometer $f(H,\delta)$ is a sinusoid whose amplitude and phase determine the complex visibility function.

It is easily shown that $\underset{\sim}{V}(U,V)$ is the Fourier transform of the source brightness temperature distribution, weighted by the antenna pattern and normalized by the integral over the temperature distribution:

$$\underset{\sim}{V}(U,V) = \frac{\int\int a(x,y)\ T_B(x,y)\ e^{-j2\pi(Ux + Vy)}\ dxdy}{\int\int T_B(x,y)\ dxdy}$$

5. Aperture Synthesis

Given the value of $\underset{\sim}{V}(U,V)$ for all values of U,V, one could then obtain the brightness temperature distribution $T_B(x,y)$ by a Fourier inversion:

$$\frac{2ka(x,y)\ T_B(x,y)}{\lambda^2 S} = \int\int \underset{\sim}{V}(U,V)\ e^{+j2\pi(Ux + Vy)} dxdy$$

The practical limitation is, of course, that the value of $\underset{\sim}{V}(U,V)$ is not available for all values of U and V. The maximum values of U and V are limited by the maximum separation D/λ of the elements of the interferometer.

It is useful to see which values of U and V are available if a source at declination δ is tracked over the full hour angle range of the telescopes. We have from section 3:

$$U = \frac{D}{\lambda} \cos d \sin (H-h)$$

$$V = \frac{D}{\lambda} [\sin d \cos \delta - \cos d \sin \delta \cos (H-h)]$$

These equations are in the form

$$U = a \sin (H-h)$$

$$V = V_o + b \cos (H-h)$$

which are the parametric equations of an ellipse in the U,V plane:

$$\frac{U^2}{a^2} + \frac{(V-V_o)^2}{b^2} = 1$$

where

semi-major axis $a = D/\lambda \cos d$

semi-minor axis $b = D/\lambda \cos d \sin \delta$

center of ellipse $(0, V_o) = (0, -\frac{D}{\lambda} \sin d \cos \delta)$

eccentricity $\varepsilon = \cos \delta$

Thus, the interferometer, in tracking a source through the sky, samples the Fourier transform of the source brightness temperature distribution at points along an ellipse in the U,V plane. By changing the separation D/λ, a different elliptical track is sampled. This procedure can then be repeated until a large number of samples at different values of U and V are obtained. The question is, how many are needed and what must the distribution of points be?

Several considerations simplify the problem. First we note that since $T_B(x,y)$ is real then the complex visibility function must be Hermitian, i.e.

$$\underset{\sim}{V}(-U,-V) = \underset{\sim}{V}^*(U,V)$$

Thus, half the information in the U,V plane is redundant. Next the source dimensions are finite (in any case we are limited by the antenna beamwidth). Then the sampling theorem states that we only have to sample points which are separated by no more than

$$\Delta U,\ \Delta V \leq \frac{1}{2X},\ \frac{1}{2Y}$$

where X is the largest dimension of the source. Thus only discrete points need be sampled, the spacing being dependent on the source size.

In addition, practical considerations limit the resolution Δx and Δy which must be attained in the source reconstructions. This allows us to sample points only out to a maximum U and V given by

$$U_{max} \leq \frac{1}{2\Delta x}$$

$$V_{max} \leq \frac{1}{2\Delta y}$$

Thus we need sample only a number of points N given by

$$N \cong \frac{X \cdot Y}{\Delta x \cdot \Delta y}$$