OPTIMUM RATE OF SAMPLING OF A SIGNAL MIXED WITH NOISE

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Abstract

A general method of analy zing the combined effects of the filtering and sampling processes on the signal to noise ratio is proposed. It is based on some of the properties of the Fourier series, the stationary random functions, and the linear filters, and it leads to the estimate of the resultant standard deviation on the amplitude of each of the Fourier series components which would represent the signal alone in any limited interval. The magnitude of these standard deviations is then used as a criterion of the efficiency of these processes used together.

One immediate application is the determination of the optimum rate of sampling discrete values of a signal mixed with noise. This optimum is a function of the frequency response of the noise filter, the spectrum bandwidth of the signal, and the relative increase of the standard deviation introduced by the sampling process which is considered acceptable. An example is given, from radio astronomy, in the case where a resistance capacity filter is used in the receiver output. It is shown that for any given acceptable relative increase of the standard deviations, optimum relations can be established between the values of the filter time constant, the sampling interval, and the spatial frequency bandwidth of the antenna used.

I. Introduction

In the reduction of the data from a record of signal and random noise, a problem is often met of limiting the number of points of measurement, to a minimum without losing any, or at least hardly any, of the information contained in the original record. This problem arises, for instance, when the degree of computation involved, or when the length of record to be put in a digital form, is extensive enough to make desirable a cutting of the quantity of data to be used.

In the ideal case where the signal s(t) is free of noise and contains no frequencies higher than B cps, the answer is given by the sampling theorem, [1], which states that s(t) is completely determined by the knowledge of an infinite series of discrete values equidistant by no more than $\frac{1}{2B}$:

$$s(t) \equiv \sum_{-\infty}^{+\infty} s\left(\frac{c}{2B}\right) \frac{\sin 2\pi B\,(t - \frac{c}{2B})}{2\pi B\,(t - \frac{c}{2B})}$$

When the signal is observed only over a finite interval T, the preceding theorem still applies, to the extent that s(t), instead of being completely determined in the interval T from the knowledge of a finite number of points, is now determined only to a very high degree of approximation everywhere within T, except at the edges (the smaller the amplitude of s(t) outside T, the better the approximation near the edges), [2 - 3].

In the case where the frequency spectrum of s(t) is not strictly limited to a bandwidth B, but contains only a relatively small amount of energy outside B, the sampling theorem still leads to a very good approximation of s(t), in the interval T, from the knowledge of $\frac{T}{1/2B} = 2BT$ equidistant values [2]. For these two last cases a necessary condition is to have $2BT \gg 1$.

One might consider extending the use of this theorem to the general case where a random noise x(t) is superimposed upon the signal s(t). To the extent that the power density spectrum of the noise can be considered as very small outside a certain bandwidth B^1, one can say that, by sampling any record of signal and noise at equal discrete intervals smaller or equal to the smallest of the two quantities $\frac{1}{2B}$ and $\frac{1}{2B^1}$, one disposes of almost all, if not all, the information necessary to restore the original record.

Actually, in the latter case, there are two objections against the use of the sampling theorem as a means of defining an optimum rate of sampling. One is that the definition of an equivalent bandwidth B^1 of the noise power spectrum, which is ultimately shaped by the frequency response of the receiver output filter, is somewhat arbitrary (unless, the ideal case of a rectangular filter is considered). Furthermore, due to the fact that such a filter is generally submitted to the requirement of distorting the signal as little as possible, its frequency gain may be still significant rather far away from the range of the signal bandwidth. Consequently, a conservative estimate of its bandwidth B^1 would in that case lead to a value of the sampling interval much smaller than it would have been for the signal alone.

This last point leads to the second objection which is that one is generally not interested in being able to restore both signal function and noise function, but only the signal by reducing the error due to the noise to a minimum. And indeed, as it will be shown, consideration of the problem directly in that form leads to the estimate of a sampling interval larger than it should have been if both signal and noise had to be restored.

Actually, the problem we are considering is the following one: given a signal $s(t)$, of which nothing is known except that it has no frequency components over a certain bandwidth B, and which is mixed with a random stationary noise $x(t)$ whose power density spectrum is known; to what extent will the fact of knowing only discrete values of $s(t) + x(t)$, at periodic intervals of given width Θ affect the final quality of the restoration of $s(t)$?

These conditions are encountered in radio astronomy observations: so far as the signal is concerned we know [4] that whatever the spatial frequency distribution of the source is, the spectrum of the image given by an antenna is ~~strictly~~ practically limited to a cut-off whose value is proportional to the ratio of the diameter of the antenna to the wavelength of observation. Also the power density spectrum of the receiver output noise is known from the measurement of the frequency response of the selective filter. A last point is that, for an observation being made over an interval of time T, what is desired is to restore, in this interval only, the signal $s(t)$ to its best.

In the following, we will consider the combined effects of the sampling rate and the frequency response of the noise filter on the final quality of the restored signal. This will be done by considering the value of the standard deviation which affects the amplitude of each of the components of the Fourier series which would represent the signal alone over the interval T of observation. Based on some of the properties of the Fourier series, the sampling theorem, and the linear filters, this approach seems to have the twofold advantage, of leading to an exact and rather simple estimate of the respective contributions of the filtering and of the sampling and of allowing this estimate to be obtained as a function of the signal frequency component considered. To begin with, a resume of the basic formulae used is given.

II. Basic Concepts and Formulae

A. A few properties of the Fourier series.

One knows that a regular function s(t), whether it is periodical or not, can be identically represented by a Fourier series in any interval T, but in this interval only (the components of the Fourier series being themselves limited to the interval T):

$$s(t) \equiv b_o + \sum_{m=1}^{\infty} b_m \cos 2\pi \frac{m}{T} t + \sum_{m=1}^{\infty} a_m \sin 2\pi \frac{m}{T} t \tag{1}$$

for: $0 \leq t \leq T$

with:

$$\begin{aligned} a_m &= \frac{2}{T} \int_o^T s(t) \sin 2\pi \frac{m}{T} t \, dt \\ b_m &= \frac{2}{T} \int_o^T s(t) \cos 2\pi \frac{m}{T} t \, dt \\ b_o &= \frac{1}{T} \int s(t) \, dt \end{aligned} \tag{2}$$

In the case where the number of components is limited to a certain value m_o, the

$(2m_o + 1)$ unknown coefficients a_m and b_m can be calculated from the following linear system:

$$s(t_1) = b_o + \sum_{m=1}^{\infty} b_m \cos 2\pi \frac{m}{T} t_1 + \sum_{m=1}^{\infty} a_m \sin 2\pi \frac{m}{T} t_1$$

$$s(t_p) = b_o + \sum_{m=1}^{\infty} b_m \cos 2\pi \frac{m}{T} t_p + \sum_{m=1}^{\infty} a_m \sin 2\pi \frac{m}{T} t_p \qquad (3)$$

$$s(t_{2m_o+1}) = b_o + \sum_{m=1}^{\infty} b_m \cos 2\pi \frac{m}{T} t_{2m_o+1} + \sum_{m=1}^{\infty} a_m \sin 2\pi \frac{m}{T} t_{2m_o+1}$$

of $(2m_o + 1)$ equations, knowing $(2m_o + 1)$ values (whatever they are) of $s(t)$ in the interval T. If the $(2m_o + 1)$ values of $s(t)$ are chosen equidistant, the interval between consecutive points being:

$$\Theta = \frac{T}{2m_o}$$

the solution of (3) lead to:

$$a_m = \frac{2}{2m_o + 1} \sum_{p=0}^{2m_o} s(p\Theta) \sin 2\pi \frac{m}{T} p\Theta$$

$$b_m = \frac{2}{2m_o + 1} \sum_{p=0}^{2m_o} s(p\Theta) \cos 2\pi \frac{m}{T} p\Theta$$

$$b_o = \frac{1}{2m_o + 1} \sum_{p=0}^{2m_o} s(p\Theta) \qquad (4)$$

When the sampling interval is smaller than $\frac{T}{2m_o}$, the coefficients a_m and b_m can be estimated by still using relations (4), in which $(2m_o + 1)$ is replaced by n, the number of values of s(t) sampled. It is interesting to note also that, should the number of sampled values be less than the number of coefficients to be calculated, relations (4) would give the best approximation of these coefficients (considering the criterion of the minimum mean square error: $\frac{1}{n}\sum_{p=0}^{n-1}[s(t_p) - S(t_p)]^2$, where S(t) is the Fourier series calculated in that manner). [5].

A last and useful property of the Fourier series is the following one: whatever is the spectrum of a function s(t) considered in an interval T, the Fourier series expansion limited to its first m_o terms is still the best trigonometric series of m_o terms to represent s(t), (with the same minimum mean square criterion).

B. Sampling theorem and Fourier series

Just as the Fourier series are either an identical representation of a function s(t) in a limited interval T, or a good approximation of it, also the sampling theorem in the time domain gives, for a strictly band limited function, either an exact representation of this function (from $-\infty$ to $+\infty$) or a high degree of approximation of it in a limited interval T (if BT >> 1, where B is the spectrum bandwidth):

$$s(t) \equiv \sum_{c=-\infty}^{+\infty} s\left(\frac{c}{2B}\right)\frac{\sin 2\pi B\left(t - \frac{c}{2B}\right)}{2\pi B\left(t - \frac{c}{2B}\right)} \quad \text{for} \quad -\infty < t < +\infty \tag{5}$$

$$s(t) \doteqdot \sum_{c=-BT}^{+BT} s\left(\frac{c}{2B}\right)\frac{\sin 2\pi B\left(t - \frac{c}{2B}\right)}{2\pi B\left(t - \frac{c}{2B}\right)} \quad \text{for} \quad -\frac{T}{2} \leq t \leq +\frac{T}{2} \tag{6}$$

The fact that (6) is a very good approximation of (5) when s(t) is only known in a finite interval T, is due to the rapid attenuation of the quantity $\left(\frac{\sin 2\pi B\left(t - \frac{c}{2B}\right)}{2\pi B\left(t - \frac{c}{2B}\right)}\right)$ when $\left(t - \frac{c}{2B}\right)$ diverges from zero. The effects of any term in (5) are therefore of consequence only within a relatively small number of intervals in the neighborhood of the corresponding sampling point [2]. The approximation given by (6) in the interval T is then good anywhere within T except at the edges.

Using this property of the sampling theorem, one can show, at least qualitatively, that a function s(t) which contains no frequencies higher than B can be represented with a good approximation in any interval T (with the condition that BT >> 1) by a Fourier series limited to m_o terms, with:

$$m_o = \text{integer part of BT}$$

We have been unable to find elsewhere a direct demonstration of this rather intuitive property, and propose the following approximate one.

Let us consider a regular function s(t) in the interval $-\frac{T}{2}$, $+\frac{T}{2}$. In this interval it is identically represented by equations (1) and (2) which can be written in complex terms:

$$s(t) \equiv \sum_{m=-\infty}^{+\infty} C_m \, e^{2\pi j \frac{m}{T} t} \qquad (7)$$

with

$$C_m = \frac{b_m - j\,a_m}{2} = \frac{1}{T} \int_{-T/2}^{+T/2} s(t)\, e^{-2\pi j \frac{m}{T} t} \qquad (8)$$

Replacing in (8), s(t) by its value given by (6), one has

$$C_m = \frac{1}{T} \int_{-T/2}^{+T/2} \sum_{c=-BT}^{+BT} s\left(\frac{c}{2B}\right) \frac{\sin 2\pi B\,(t - \frac{c}{2B})}{2\pi B\,(t - \frac{c}{2B})}\, e^{-2\pi j m \frac{t}{T}}\, dt$$

In consequence of the uniform convergence of the series one can reverse summation and integration:

$$C_m = \frac{1}{T} \sum_{c=-BT}^{+BT} s\left(\frac{c}{2B}\right) \int_{-T/2}^{+T/2} \frac{\sin 2\pi B\,(t - \frac{c}{2B})}{2\pi B\,(t - \frac{c}{2B})}\, e^{-2\pi j m \frac{t}{T}}\, dt \qquad (9)$$

Using the same argument that applied for the degree of approximation given by the sampling theorem (6), we see that for $t < -\frac{T}{2}$ or $t > +\frac{T}{2}$ the quantity

$$\frac{\sin 2\pi B\,(t - \frac{c}{2B})}{2\pi B\,(t - \frac{c}{2B})}$$ falls rapidly to zero, whatever $\frac{C}{2B}$ such that:

$$-\frac{T}{2} < \frac{C}{2B} < +\frac{T}{2}$$

condition which is fulfilled in the actual case. Therefore (9) can be written:

$$C_m \doteq \frac{1}{T} \sum_{c=-BT}^{+BT} s\left(\frac{c}{2B}\right) \int_{-\infty}^{+\infty} \frac{\sin 2\pi B\,(t - \frac{c}{2B})}{2\pi B\,(t - \frac{c}{2B})}\, e^{-2\pi j \frac{m}{T} t}\, dt \qquad (10)$$

the limits of the integral being extended to $-\infty$ and $+\infty$

Let:

$$u = 2B\,(t - \frac{c}{2B})$$

$$x = \frac{m}{2BT}$$

Equation (10) becomes:

$$C_m = \frac{1}{BT} \sum_{c=-BT}^{+BT} s\left(\frac{c}{2B}\right) e^{-2\pi jcx} \int_{-\infty}^{+\infty} \frac{\sin \pi u}{\pi u}\, e^{-2\pi jux}\, du$$

In this form, it is clear that the value I(x) of the integral is the Fourier transform of $\frac{\sin \pi u}{\pi u}$. Therefore:

$$I(x) = 1 \text{ for } -\frac{1}{2} \leq x \leq +\frac{1}{2} \text{ i.e } -\frac{1}{2} \leq \frac{m}{2BT} \leq +\frac{1}{2}$$

$$I(x) = 0 \text{ for } x > \frac{1}{2} \text{ i.e } \left|\frac{m}{2BT}\right| > \frac{1}{2}$$

The coefficients C_m and C_{-m} are therefore zero for $m > BT$, and the number of Fourier series components is limited to $m_o = BT$.

More exactly, the preceeding shows that the Fourier series expansion of s(t) in the interval T, limited to $m_o = BT$ components, represents s(t) in this interval with a degree of approximation comparable to the one given by the sampling theorem. This approximation is therefore better when BT is larger.

In figures 1, 2, and 3 are shown a few examples of the degree of approximation obtained in this manner.

Figure 1 represents the function $g(\alpha) = (\frac{\sin \pi B\alpha}{\pi B\alpha})^2$ whose Fourier transform $G(\nu)$ (figure 1a) is zero for $\nu > B$.

It is considered over the ranges $-\frac{4}{B} \leq \alpha \leq +\frac{4}{B}$ and $-\frac{2}{B} \leq \alpha \leq \frac{2}{B}$, and in each case, the Fourier series components are calculated from points at abscissae k/2B within the range of observation (k being an integer). The function $g(\alpha)$ is then restored by adding these Fourier harmonics.

Figure 2 shows similar results in the case of the function $g(\alpha) \neq \frac{\sin \pi B\alpha}{\pi B\alpha}$ whose Fourier transform is $G(\nu) = 1$ for $0 \leq \nu \leq B$, and $G(\nu) = 0$ for $\nu > B$ (fig. 2a).

The first example, in radio astronomyν represents the gain of a continuous linear aperture as a function of α, angle between the direction considered and the plan perpendicular to the antenna.

The second one would represent the gain on a synthesis antenna obtained by multiplying the output signal of the same linear aperture by the output of one element at its end [7].

In both cases, the spatial frequency cut-off of the antenna is equal to $\frac{d}{\lambda}$, the ratio of the length of the aperture and the wavelength of observation.

The third example corresponds very nearly to the actual case of the 85 foot parabolic antenna used at the NRAO.

Due to the tapering of the illumination, the level of the side lobes is appreciably reduced, and in consequence the main lobe is widened; its width at half intensity points is 1.4 times larger than it would be if the illumination were uniform.

In Figure 3, the gain of this antenna is approximated by the Gaussian curve having the same width at half intensity points:

$$g(\alpha) = 2.72 \exp \left[- \frac{\alpha^2}{2\left(\frac{0.53}{B}\right)^2} \right]$$

with $B = \frac{d}{\lambda}$

The Fourier transform of a Gaussian function is also Gaussian.
Figure 3A shows that, in the present case, $G(\nu)$ is very small for $\nu > B$, and actually sampling $g(\alpha)$ every $\frac{1}{2B}$ and restoring the function in the corresponding interval of observation, give the curves shown in Figure 3.

C. Random stationary functions and linear filters.

Let $x(t)$ be a random stationary func tion. Its correlation function is by definition:

$$\rho_0(\tau) = \overline{x(t).\ x(t-\tau)}.$$

and its power density spectrum:

$$A_0(\nu) = 4 \int_0^\infty \rho(\tau) \cos 2\pi\nu\tau \, d\tau \tag{11}$$

$A_0(\nu)$ and $\rho_0(\tau)$ are Fourier transforms of each other and one has

$$\rho_0(\tau) = \int_0^\infty A_0(\nu) \cos 2\pi\nu\tau \, d\nu \tag{12}$$

Also:

$$\overline{x^2(t)} = \rho_0(o) = \int_0^\infty A_0(\nu)\, d\nu$$

When $x(t)$ is applied to a linear filter whose complex gain is

$$G(\nu) = g(\nu)\ e^{j\varphi(\nu)}$$

the power density spectrum of the output random function becomes:

$$A(\nu) = A_0(\nu).\ g^2(\nu) \tag{13}$$

[Bochner-Khintchine theorem] [- 8 -]

In a radiometer where the most selective filter is the output low pass filter, the power density spectrum of the noise at the input of the latter can be considered as constant over the filter bandwidth.
If $g(\nu)$ is the modulus of the low pass filter gain, the power density spectrum of the receiver output noise is therefore:

$$A(\nu) = A_0.\ g^2(\nu)\ .$$

III. Sampling of a signal mixed with noise.

1. According to Section II, B, a signal s(t) whose Fourier spectrum is limited to a bandwidth B can be represented with good approximation over an interval T (such that BT >> 1) by a Fourier series expansion limited to $m_o = BT$ components. Amplitude and phase of these harmonics are calculated exactly from values of s(t) sampled at intervals smaller than, or equal to, $\frac{1}{2B}$ (equations 4):

$$a_m = \frac{2}{n} \sum_{p=0}^{n-1} s(p\Theta) \sin 2\pi \frac{m}{T} p\Theta$$

$$b_m = \frac{2}{n} \sum_{p=0}^{n-1} s(p\Theta) \cos 2\pi \frac{m}{T} p\Theta$$

$$b_o = \frac{1}{n} \sum_{p=0}^{n-1} s(p\Theta) \tag{14}$$

When a random noise x(t) is superimposed on s(t), the estimate of the coefficients a_m and b_m from the actual record f(t) = s(t) + x(t) leads to values:

$$a^1_m = \frac{2}{n} \sum_{p=0}^{n-1} [s(p\Theta) + x(p\Theta)] \sin 2\pi \frac{m}{T} p\Theta$$

$$b^1_m = \frac{2}{n} \sum_{p=0}^{n-1} [s(p\Theta) + x(p\Theta)] \cos 2\pi \frac{m}{T} p\Theta$$

$$b^1_o = \frac{1}{n} \sum_{p=0}^{n-1} [s(p\Theta) + x(p\Theta)] \tag{15}$$

and the errors made on a_m and b_m are:

$$\epsilon_{am} = \frac{2}{n} \sum_{p=0}^{n-1} x(p\Theta) \sin 2\pi \frac{m}{T} p\Theta$$

$$\epsilon_{bm} = \frac{2}{n} \sum_{p=0}^{n-1} x(p\Theta) \cos 2\pi \frac{m}{T} p\Theta$$

$$\epsilon_{bo} = \frac{1}{n} \sum_{p=0}^{n-1} x(p\Theta) \tag{16}$$

Since x(t) is random, these errors are also random, and only their statistical effect can be estimated. It is expressed in terms of the mean noise energy

which is added to the energy: $V_m^2 = a_m^2 + b_m^2$ related to the component m.
One has:

$$\overline{V_m^{1\,2}} = \overline{a_m^{1\,2}} + \overline{b_m^{1\,2}} = \overline{(a_m + \epsilon_{am})^2} + \overline{(b_m + \epsilon_{bm})^2} = (a_m^2 + b_m^2) + \left(\overline{\epsilon_{am}^2} + \overline{\epsilon_{bm}^2}\right)$$

It is the purpose of the following to calculate the standard deviation:

$$\sigma_m = \sqrt{\overline{\epsilon_{am}^2} + \overline{\epsilon_{bm}^2}}$$

from the actual amplitude, $V_m = \sqrt{a_m^2 + b_m^2}$ of each of the Fourier harmonics as a function of the sampling interval Θ (with $\Theta \leq \frac{1}{2B}$) and the statistical properties of x(t).

So far as the phase is concerned, only an estimate of the probable maximum error can be made, by considering the "worst" case where the noise vector and the signal vector are in quadrature.

2. Calculation of $\sigma_m^2 = \overline{\epsilon_{am}^2} + \overline{\epsilon_{bm}^2}$

Let:

$$\eta_m = \epsilon_{am} + j\,\epsilon_{bm}$$

One has:

$$\sigma_m^2 = \overline{\epsilon_{am}^2} + \overline{\epsilon_{bm}^2} = \overline{\eta_m \cdot \eta_m^*}$$

With:

$$\eta_m = \frac{2}{n} \sum_{p=0}^{n-1} x(p\Theta)\, e^{2\pi j \frac{m}{T} p\Theta}$$

Then:

$$\sigma_m^2 = \frac{4}{n^2} \sum_{p=0}^{n-1} \sum_{p^1=0}^{n-1} \overline{x(p\Theta).\, x(p^1\Theta)}\; e^{2\pi j \frac{m}{T}(p^1 - p)\Theta}$$

By definition: $\overline{x(p\Theta).\ x(p^1\Theta)}$ is the value $\rho[(p^1 - p)\Theta]$ of the correlation function $\rho(\tau) = \overline{x(t).\ x(t - \tau)}$.

Let $A(\nu)$ be the power density spectrum of x(t), One has (equ. 12):

$$\rho(\tau) = \int_0^\infty A(\nu)\ \cos 2\pi\nu\tau\ d\nu$$

Then:

$$\overline{x(p\Theta).\ x(p^1\Theta)} = \rho[(p^1 - p)\Theta] = \int_0^\infty A(\nu)\ \cos 2\pi\nu\ (p^1 - p)\Theta\ d\nu$$

and:

$$\sigma_m^2 = \frac{4}{n^2} \sum_{p=0}^{n-1} \sum_{p^1=0}^{n-1} \left[\int_0^\infty A(\nu)\ \cos 2\pi\nu\ (p^1 - p)\Theta\ d\nu\right] e^{2\pi j \frac{m}{T} (p^1 - p)\Theta}$$

$$= \frac{4}{n^2} \int_0^\infty \left[\sum_{p=o}^{n-1} \sum_{p^1=0}^{n-1} e^{2\pi j \frac{m}{T} (p^1 - p)\Theta} \cos 2\pi\nu\ (p^1 - p)\Theta\right] A(\nu)\ d\nu$$

$$= \frac{2}{n^2} \int_0^\infty \sum_{p=o}^{n-1} \sum_{p^1=o}^{n-1} \left\{e^{2\pi j\ (p^1 - p)\Theta\ (\nu + \frac{m}{T})} + e^{-2\pi j (p^1 - p)\Theta(\nu - \frac{m}{T})}\right\} A(\nu)\ d\nu$$

and finally:

$$\sigma_m^2 = 2 \int_0^\infty \left[\left(\frac{\sin n\pi\Theta(\nu + \frac{m}{T})}{n \sin\pi\Theta(\nu + \frac{m}{T})}\right)^2 + \left(\frac{\sin n\pi\Theta(\nu - \frac{m}{T})}{n \sin\pi\Theta(\nu - \frac{m}{T})}\right)^2\right] A(\nu)\ d\nu \tag{18}$$

It has been seen (13) that $A(\nu)$, power density spectrum of x(t) at the output of the receiver, is actually defined by the power gain $g^2(\nu)$ of the output filter:

$$A(\nu) = A_o\ g^2(\nu)$$

where A_o, the power density of the noise before this filter, is constant within the low frequency range considered.

Equation (18) can be written:

$$\sigma_m^2 = A_o \int_o^\infty g_{ms}^2(\nu).\ g^2(\nu)\ d\nu \tag{19}$$

with:

$$\frac{1}{2}\, g^2_{ms}(\nu) = \left(\frac{\sin n\pi\Theta(\nu+\frac{m}{T})}{n \sin\pi\Theta(\nu+\frac{m}{T})}\right)^2 + \left(\frac{\sin n\pi\Theta(\nu-\frac{m}{T})}{n \sin\pi\Theta(\nu-\frac{m}{T})}\right)^2 \qquad (20)$$

showing the equivalence of the sampling operation to a linear filtering. A similar calculation applied to the DC component:

$$\sigma^2_o = \overline{\epsilon^2_{bo}} = \frac{1}{n^2} \sum_{p=0}^{n-1} \sum_{p^1=0}^{n-1} \overline{x(p\Theta).\ x(p^1\Theta)}$$

leads to:

$$g^2_{os}(\nu) = \left(\frac{\sin n\pi\nu\Theta}{n \sin\pi\nu\Theta}\right)^2 \qquad (21)$$

3. Graphic Interpretation.

The function $g^2_{ms}(\nu)$, which consists of a double series of lobes, the first two being centered at abscissae $-\frac{m}{T}$ and $+\frac{m}{T}$ (Fig. 4), has the following characteristics:

- Interval between connective lobes: $\frac{1}{\Theta}$
- Total area of one lobe, plus its sideslobes: $\frac{1}{n\Theta}$, slightly inferior to $\frac{1}{T}$
- Surface of the main lobe $\simeq$ 10 times the total surface of the side lobes.
- Width at first zeros: $\frac{2}{n\Theta} \doteq \frac{2}{T}$

In Fig. 5, both $g^2_{ms}(\nu)$, for arbitrary values of m and Θ, and the gain $g^2(\nu)$ of an RC filter are plotted.

The hatched area represents $g^2_{ms}(\nu)$, according to equation (19), and the influence of the length Θ of the sampling interval on σ^2_m can be estimated graphically. One sees that shortening Θ has the effect of rejecting all the lobes, except the first one centered at $+\frac{m}{T}$, toward frequency regions where the low pass filter gain is small. The lobe centered at $+\frac{m}{T}$ cannot be avoided since the filter must allow the signal to pass. Its surface $\frac{A_o}{T} g^2 (\frac{m}{T})$ represents the unavoidable noise energy which affects the component m.

The square root of the ratio of the total hatched area to the area of the lobe centered at $+\frac{m}{T}$ then represents a measure of the contribution of the sampling process to the final standard deviation affecting the component m.

Applied to a somewhat similar problem, the method developed in Section III has been previously proposed by the author [9], helped by the advice of E. Le Roux and J. Arsac. The last mentioned has extended it [10] to the more general context of the Fourier transform, and applying it to a different example has given a result very similar to the first following one.

IV. Application to particular cases

1. Case where the filter has a bandwidth strictly limited to the frequency range 0 - B; the spectrum of the signal is limited to $0-\nu_o$, with $\nu_o < B$.

Figure 6 shows that for every signal component, all lobes except the first one are rejected outside the filter bandwidth when:

$$-\nu_o + \frac{1}{\Theta} \geq B + \frac{1}{T}$$

Or, since by assumption BT >> 1, when:

$$\Theta \leq \frac{1}{B + \nu_o}$$

The optimum sampling interval for the signal mixed with noise, that is the minimum interval for which the sampling process does not introduce any excess error on the signal, is $\Theta_o = \frac{1}{B + \nu_o}$

A sampling interval equal to $\frac{1}{2\nu_o}$ would allow restoration of the signal, only if it were free of noise. On the other hand, by sampling every $\frac{1}{2B}$, one could restore both signal and noise. The relation between these three intervals, each of them optimum at a different point of view, is:

$$\frac{1}{2\nu_o} < \frac{1}{B + \nu_o} < \frac{1}{2B} \tag{22}$$

This result corresponds to the remark made in the introduction: The sampling interval has to be smaller when it is desired to restore identically the original record of signal and noise than when one is interested only in restoring the signal to its best.

This is due to the fact that limiting the Fourier series expansion to $m_o = \nu_o T$

terms effects by itself a filtering of the noise frequency components above ν_o. Actually, one could use such a tebhnique for smoothing a record of signal and noise, by: (1) Sampling the record at the optimum interval; (2) Calculating the $m_o = \nu_o T$ Fourier components (equ. 15), (3) Adding these harmonics with their respective phases, and (4) plotting the result.

2. Case where both signal and noise have a bandwidth strictly limited to B.

The optimum sampling interval is then: $\Theta_o = \frac{1}{2B}$, relation (22) becoming a double equality.

3. Optimum sampling interval for a DC signal

The amplitude of the DC signal being b_o, the square of the standard deviation from it is, after filtering and sampling (equ. 19 and 21):

$$\sigma_o^2 = A_o \int_o^\infty g_{os}^2 (\nu) .\ g^2(\nu)\ d\nu$$

with:

$$g_{os}^2 (\nu) = \left(\frac{\sin n\pi\nu\Theta}{n\pi\nu\Theta}\right)^2 \doteqdot \left(\frac{\sin \pi\nu T}{\pi\nu T}\right)^2$$

In the case of a rectangular bandwidth filter (Fig. 7), the optimum sampling interval is:

$$\frac{1}{\Theta} = B + \frac{1}{T}$$

or, $\frac{1}{T}$ being by assumption small compared to B:

$$\Theta = \frac{1}{B}$$

V. Filtering, Sampling, and observing time.

After filtering, the noise energy is:

$$\overline{x(t)^2} = \sigma^2 = \int_0^\infty A_0 \, g^2(\nu) \, d\nu$$

and is represented by the area delimited by the curve $A_0 \, g^2(\nu)$ and the coordinate axes.

When signal and noise are observed over a time interval T, the noise energy which affects the amplitude of each of the Fourier series harmonics is equal to:

$$\sigma_m^2 = \int_0^\infty A_0 \, g^2(\nu) . \; g_{ms}^2(\nu) \, d\nu$$

and is shown by the hatched area on the previous figures.

By choosing a sampling interval short enough, all lobes except the first one centered at $+\frac{m}{T}$, can be rejected toward high frequencies where $g^2(\nu)$ is small and make their contribution negligible.

The area of the remaining lobe is:

$$\sigma_{mo}^2 \doteqdot \frac{A_o}{T} \cdot g^2 \left(\frac{m}{T}\right)$$

The standard deviation σ_m, affecting the component m, is inversely proportional to the square root of the observing time (but it must not be forgotten that the amplitude V_m of the component depends itself on the interval within which the signal is observed.)

In the case of a DC signal and of a rectangular bandwidth filter, the ratio of the noise energy after integration over an interval T (the sampling interval being $\Theta \leq \frac{1}{B}$) to the noise energy before integration is (Fig. 7):

$$\frac{\sigma_o^2}{\sigma^2} \doteqdot \frac{\frac{A_o}{2T}}{A_o B} = \frac{1}{2BT}$$

σ_m is also proportional to $g(\frac{m}{T})$; so is the amplitude of the harmonic m which has passed through the filter. The final signal to noise ratio is therefore independent of the characteristics of the filter.

What is gained by a proper choice of the filter, is merely an economy in the amount of data to be sampled and afterward computed.

It seems that there has occasionally been some confusion in this matter. Some authors [12-13] have considered that the choice of the radiometer output time constant τ has to be made in such a way that there is no appreciable loss of information. In this respect the fact of choosing τ small [13], does not necessitate the reduction of a large number of records. Further, if having τ large indeed reduces the amplitude of the receiver output, it must be remembered that the effect is not of importance since the standard deviation is reduced in the same proportion. In the same way, the distortion to which the signal is submitted by the filter is not, theoretically at least, a limiting factor, to the extent that it is always possible to restore the signal as it was before filtering.

Actually, the considerations which lead to the choice of an "optimum" time constant are mainly considerations of convenience: for instance, easiness in interpreting a record directly, or limitation of the degree of computation required, or still technical reasons like the effects of a large time constant on transients (receiver instabilities or interferences).

An example of such a determination of an optimum time constant is given in the following. Since the choice of the filter is of little importance, provided that sampling and computing are made accordingly, we consider the case of the simplest one, the resistance capacity filter.

VI. Application to a concrete case.

We have applied the preceeding results to the problem of the digitalization of the output of a 20-channel extragalactic receiver [14] to be used with the 300-foot telescope whose construction at the NRAO is to be finished [15].

Being designed for observations at 21 cm and above, this instrument has a maximum spatial frequency cut-off equal to:

$$\frac{d}{\lambda} = \frac{91.5}{0.21} = 435$$

and since the apparent angular velocity, expressed in solar time, of a source at a declination δ is:

$$\Phi^1 \neq 7.29 \times 10^{-5} \times \cos\delta \ \text{rad/sec}$$

the frequency cut-off of the filter equivalent to the antenna is:

$$\nu_o = 435 \times 7.29 - 10^{-5} \cos\delta = 0.0317 \cos\delta \ \text{cps}$$

Whatever the frequency of observation (1420 Mcs or below) and the declination of the source the highest possible frequency cut-off is therefore:

$$\nu_o = 0.032 \ \text{cps}$$

So far as the signal only is concerned, and for such a frequency bandwidth, the sampling interval must be:

$$\Theta_s \leq \frac{1}{2\nu_o} = 15.6 \ \text{sec}$$

In the present case, the determination of an optimum rate of sampling was particularly useful since we wished to simplify as much as possible the receiver output system by using only one digitalization channel scanning the 20 analog outputs between two consectuive transit recording points [16], and we also desired to reduce to an acceptable minimum the quantity of data to be computed afterwards. According to a previous remark, the low pass filter chosen is of the resistance capacity type. The relative increase of the standard deviation on a given signal frequency component ν has been computed as a function of ν, of the filter time constant τ and of the sampling interval Θ, in the following way. From equation (19), the noise energy which affects the frequency component $\nu_m = \frac{m}{T}$ of the signal is:

$$\sigma_m^2 = A_o \int_o^\infty g_{ms}^2(\nu) . \ g^2(\nu) \, d\nu$$

where A_o is the noise power density before filtering, $g_{ms}^2(\nu)$ the power gain of the sampling equivalent filter, and $g^2(\nu)$ the power gain of the RC filter. The last mentioned is:

$$g^2(\nu) = \frac{1}{1 + (2\pi\nu\tau)^2}$$

where $\tau = RC$ is the filter time constant.

Using equation (20), and taking into account that the area of a lobe is $\frac{1}{T}$, we can approximate the hatched area which represents σ_m^2 (fig. 5) by:

$$\sigma_m^2 \doteq \frac{A_o}{T} \frac{1}{1+(2\pi\tau\nu_m)^2} + \frac{A_o}{T} \sum_{k=1}^{\infty} \frac{1}{1+[2\pi(\tau\nu_m + k\frac{\tau}{\Theta})]^2} + \frac{1}{1+[2\pi(-\tau\nu_m + k\frac{\tau}{\Theta})]^2} \tag{24}$$

since the width of a lobe at zero points, $\frac{1}{2T}$, is small compared to $\frac{1}{\Theta}$ and $\frac{1}{\tau}$.

The first term in equ. (24) represents the minimum unavoidable noise energy σ_{mo} on the component ν_m (which would correspond to a sampling interval tending to zero).

The excess standard deviation on ν_m therefore is:

$$R_m = \frac{\sigma_m - \sigma_{mo}}{\sigma_{mo}} = \sqrt{\frac{\sum_{k=1}^{\infty} \frac{1}{1+[2\pi(\tau\nu_m + k\tau/\Theta)]^2} + \frac{1}{1+[2\pi(-\tau\nu_m + k\tau/\Theta)]^2}}{\frac{1}{1+[2\pi\tau\nu_m]^2}}}$$

and is plotted in Fig. 8 as a function of $\tau\nu_m$, the parameter of the family of curves being $\frac{\tau}{\Theta}$.

The second significant parameter is $\Theta\nu_o$. For different values of this parameter, tables 1, 2, and 3 show the variations of R_m with $\frac{\tau}{\Theta}$ for three frequencies $\nu_m = 0$, $\nu_{o/2}$, and ν_o.

They suggest the few following remarks:

For the DC component of the signal, R_o does not depend on the absolute values of the time constant and of the sampling interval, but only on their ratio. The influence of the value of $\Theta\nu_o$ on $R\nu_m$ increases toward the high frequency components, and for instance, $R\nu_o$ is about five times bigger for $\Theta = \frac{1}{2\nu_o}$ than for $\Theta = \frac{1}{4\nu_o}$, and about three time and a half bigger for $\Theta = \frac{1}{4\nu_o}$ than for $\Theta = \frac{1}{8\nu_o}$ (with $\frac{\tau}{\Theta} = 3$).

Table 1. $\Theta = \frac{1}{2\nu_o}$

τ/Θ	0.25	0.50	0.75	1.00	1.25	1.50	2.00	3.00
R_o	32%	7.0%	3.5%	2.0%	1.5%	1.4%	1.0%	0.5%
$R\,\nu_o/2$	50%	24.5%	17.5%	14.5%	13.0%	12.0%	12.0%	11.0%
$R\,\nu_o$	75%	5.5%	5.0%	57.5%	57.5%	56.5%	56.5%	56.5%

Table 2. $\Theta \neq \frac{1}{4\nu_o}$

τ/Θ	0.25	0.50	0.75	1.00	1.25	1.50	2.00	3.00
R_o	32%	7.0%	3.5%	2.0%	1.5%	1.4%	1.0%	0.5%
$R\,\nu_{o/2}$	41%	14.5%	8.5%	6.5%	5.0%	4.0%	3.5%	3.0%
$R\,\nu_o$	50%	24.5%	17.5%	14.5%	13.0%	12.0%	12.0%	11.0%

Table 3. $\Theta = \frac{1}{8\nu_o}$

R_o	0.25	0.50	0.75	1.00	1.25	1.50	2.00	3.00
R_o	32%	7.0%	3.5%	2.0%	1.5%	1.4%	1.0%	0.5%
$R\,\nu_{o/2}$	36%	10.0%	5.5%	4.0%	2.5%	2.4%	2.0%	0.8%
$R\,\nu_o$	41%	14.5%	8.5%	6.5%	5.0%	4.0%	3.5%	3.0%

Lastly, and perhaps the most interesting point, there is little to be gained by having $\tau > \Theta$, whatever the values of $\Theta\nu_o$ and of ν_m.
Therefore, after the length Θ of the sampling interval has been chosen as a function of ν_o and of the excess standard deviations $R\nu_m$, considered as acceptable, an

optimum value of τ is defined from it by the relation: $\tau = \Theta$.

This result is useful to the extent that it is generally desirable to use rather short time constants. One reason is, for instance, that the length of short transients, due either to receiver instabilities or to interferences, is comparable to the filter time constant.

Another reason is that, the larger the time constant the more difficult the direct interpretation of a record becomes (actually, as it has been seen, this consideration applies only when it is desired to avoid further computation).

In figure 9 are plotted R_o, $R\,\nu_o/2$ and $R\,\nu_o$ as a function of $\Theta\nu_o$ for $\frac{\tau}{\Theta} = 1$. Considering the present example of application, we have:

$$\nu_o = 0.032 \text{ cps}$$

to which corresponds, for the signal alone, the optimum sampling interval:

$$\Theta_s = \frac{1}{2\nu_o} = 15.6 \text{ sec}$$

If now an actual record of signal and noise is sampled every $\Theta = 10$ sec. the RC filter time constant being $\tau = \Theta = 10$ sec, the standard deviation on the signal components is increased by a factor varying from 2% for the DC component to 22% for the cut-off frequency.

Choosing $\Theta = 5$ sec, and $\tau = \Theta = 5$ sec, would imply a range of 2% to 8%.

The distortion effects caused by an RC filter on the output signal of a parabolic antenna have been studied, both by Mezzger [12] and by William Howard III, [13]. They can be represented by the following three quantities of the output function relative to the input function: 1) reduction in amplitude, 2) delay in reaching the maximum and 3) increase of the width at half intensity points.

In reference [13], the values of these parameters have been computed in the case where both input function and antenna beam are Gaussian.

Considering, in the present example, a source narrow compared to the antenna beam, and assimilating what would be the beam of the uniformly illuminated 300 foot parabolic antenna to the Gaussian curve having same width at half intensity points, we have:

1. For $\tau = 10$ sec.

   Reduction of amplitude: 25%

   Delay of the output function maximum: 6.7 sec.

   Increase of the width at half intensity points: 28%.

2. For $\tau = 5$ sec.

   Reduction of amplitude: 18%

   Delay of the output function maximum: 4.2 sec.

   Increase of the width at half intensity points: 12%.

Actually, these effects of the low pass filter on the output function can be eliminated in a restoration program which would correct together the filtering effects of the antenna and of the RC filter.

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