USEFUL PRECISION FOR THE MEASUREMENT OF A SIGNAL MIXED WITH NOISE

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In general, when one repeats a measurement of a fixed quantity S mixed with some random variable X, in order to reduce the uncertainty of the measured value by averaging, the problem arises of knowing how the precision of the individual readings affect the final result. In other words, how low can the precision of the readings be without introducing a significant inaccuracy in the final value obtained for S? For example, consideration of this matter is essential when the number of quantities to be measured is so large that the elimination of unnecessary digits in individual measurements is worthwhile.

If we consider a signal S, constant, mixed with a gaussian random variable X, the rms of which is σ_0, and if first we assume that S + X is measured with an infinite precision, then when we take the average of n samples of S + X, the random variable added to S becomes: $X' = \frac{1}{n} \sum_{k=1}^{n} X_k$, with a rms σ between σ_0 and $\frac{\sigma_0}{\sqrt{n}}$, according to the degree of correlation of the values of X_k ($\sigma = \frac{\sigma_0}{\sqrt{n}}$ when they are independent).

If now each value S + X is measured with a finite precision, an error of measurement is added to the random variable X, so that the uncertainty of the value of S after averaging n samples is greater than in the ideal case of an infinite precision.

Let Y' be the contribution of the lack of precision of the individual measurements to the final uncertainty of the value of S. The purpose of what follows is:

> To show that Y' is a random variable, the statistical distribution of which tends, when n increases, to a gaussian law.
>
> To calculate its rms ϵ as a function of the precision of the individual measurements.
>
> To define and to evaluate the useful precision as the minimum precision for which ϵ can be considered as negligible with respect to σ.

As a general definition, if one says that an unknown quantity Q is measured with a relative precision $\pm 1/2k$, one means that in its representation by an integer k, the absolute error Y' on Q is certainly between -1/2 and +1/2. Since the exact value of Q is not known, there is an equal probability

of having any value between -1/2 and +1/2.

For instance, if one measures a weight Q by comparison with elementary weights of 1 gr, one knows that Q is greater than k gr, and smaller than (k+1) gr. If one decides to represent Q by k, then the absolute error Y on Q is between 0 and +1. A better representation is to apply to Q the value (k+1/2) and to say that Y is between -1/2 and +1/2, with a probability density p(Y) equal to 1 in this interval:

$$p(Y) = 1 \quad \text{for} \quad -1/2 \leq Y \geq +1/2$$

$$p(Y) = 0 \quad \text{for} \quad |Y| > 1/2 \tag{1}$$

Mean value of Y:

$$\bar{Y} = \int_{-\infty}^{+\infty} Y p(Y) dY = \int_{-1/2}^{+1/2} Y dY = 0$$

Variance:

$$\epsilon_0^2 = \bar{Y^2} = \int_{-\infty}^{+\infty} Y^2 \, p(Y) dY = \int_{-1/2}^{+1/2} Y^2 dY = \frac{1}{12}$$

$$\boxed{\epsilon_0 = \frac{1}{2\sqrt{3}}} \tag{2}$$

Characteristic function:

$$\varphi(\mu) = \int_{-\infty}^{+\infty} e^{j\mu y} \, p(Y) dY = \int_{-1/2}^{+1/2} e^{j\mu y} dY.$$

$$\boxed{\varphi(\mu) = \frac{\sin \frac{\mu}{2}}{\frac{\mu}{2}}} \tag{3}$$

It is obvious that if Q is a constant quantity, the absolute error made in each measurement will be systematic and equal to $(k + 1/2) - Q$.

If now Q is the sum of a constant signal S and a noise X and if the unit of measurement is much smaller than the "range" of X (by range we mean twice the value X_0 such that the probability for X to be bigger than X_0 is less than a given percentage; for instance, for $X_0 = 4\sigma$, this probability is below 0.01% for a gaussian noise), then from one measurement to another Q is represented by a different value of k and the error $(k + 1/2) - Q$ becomes random.

The preceding applies now to the set of measurements. For each one, the characteristic function of p(Y) is (equ. 3):

$$\varphi(\mu) = \frac{\sin. \frac{\mu}{2}}{\frac{\mu}{2}}$$

For n measurements, the characteristic function of p(Y'), the probability density of the final error is:

$$\varphi_n(\mu) = \left[\varphi(\mu)\right]^n = \left(\frac{\sin. \frac{\mu}{2}}{\frac{\mu}{2}}\right)^n$$

and the variance: $\epsilon^2 = \int_{-\infty}^{+\infty} Y'^2 p(Y') dY'$

is equal to: $\epsilon^2 = j^2 \varphi_n''(0)$

For $\mu \longrightarrow 0$, $\varphi_n(\mu) \simeq \left(1 - \frac{\mu^2}{24}\right)^n$

and after calculation, one has:

$$\varphi_n''(0) \simeq -\frac{n}{12}$$

Therefore:

$$\boxed{\epsilon = \tfrac{1}{2}\sqrt{\frac{n}{3}}} \qquad (4)$$

To prove that the statistical distribution of the random variable Y' tends to a gaussian law with the same rms ϵ one may show that its Fourier transform, the characteristic function:

$$\varphi_n(\mu) = \left(\frac{\sin \mu\sqrt{3/n}\,\epsilon}{\mu\sqrt{3/n}\,\epsilon}\right)^n$$

tends to the Fourier transform:

$$F(\mu) = e^{-1/2\mu^2\epsilon^2}$$

of the gaussian distribution:

$$F(Y') = \frac{1}{\epsilon\sqrt{2\pi}}\, e^{-Y'^2/2\epsilon^2}$$

For that one may write the series expansion of $\log \varphi_n(\mu)$, and see that all the terms, but the first one $\frac{\epsilon^2\mu^2}{2}$, tend to 0 when n tends to the infinity.

Therefore: $\text{Log } \varphi_n(\mu) \longrightarrow \frac{\epsilon^2\mu^2}{2}$ when $n \longrightarrow \infty$

and: $\varphi_n(\mu) \longrightarrow e^{-1/2\mu^2\epsilon^2}$

To see how fast this distribution tends to the gaussian law when n increases, one can use the development of Gram Charlier (1) which permits one to express a given unknown statistical law in terms of a known distribution and the moments of the unknown law. One finds:

$$p(Y') = \frac{1}{\epsilon\sqrt{2}} e^{-Y'^2/2\epsilon^2} \left\{ 1 + \frac{\frac{144}{n}\frac{1}{80} + \frac{n-1}{48} - 3}{4!}\left(\frac{Y'^4}{\epsilon^4} - 6\frac{Y'^2}{\epsilon^2} + 3\right) + \right.$$

$$\left. \frac{1}{6!}\left[\frac{1728}{n^2}\left(\frac{1}{448} + \frac{n-1}{64} + \frac{5(n-1)(n-2)}{576}\right) - 15\left(\frac{144}{n}\right)\left(\frac{1}{80} + \frac{n-1}{48}\right) + 30\right]\left[\frac{Y'^6}{\epsilon^6} - 15\frac{Y'^4}{\epsilon^4} + 45\frac{Y'^2}{\epsilon^2} - 15\right] + \text{-----}\right\} \quad (5)$$

For n = 10, for instance, one has:

$$p(Y') = \frac{1}{\epsilon\sqrt{2}} e^{-Y'^2/2\epsilon^2}\left[1 - 0.005\left(\frac{Y'^4}{\epsilon^4} - 6\frac{Y'^2}{\epsilon^2} + 3\right) + 0.001\left(\frac{Y'^6}{\epsilon^6} - 15\frac{Y'^4}{\epsilon^4} + 45\frac{Y'^2}{\epsilon^2} - 15\right) + \text{------}\right]$$

One can also see this tendency to the gaussian law by plotting the distributions corresponding to n = 1, 2, 3, 4 -----

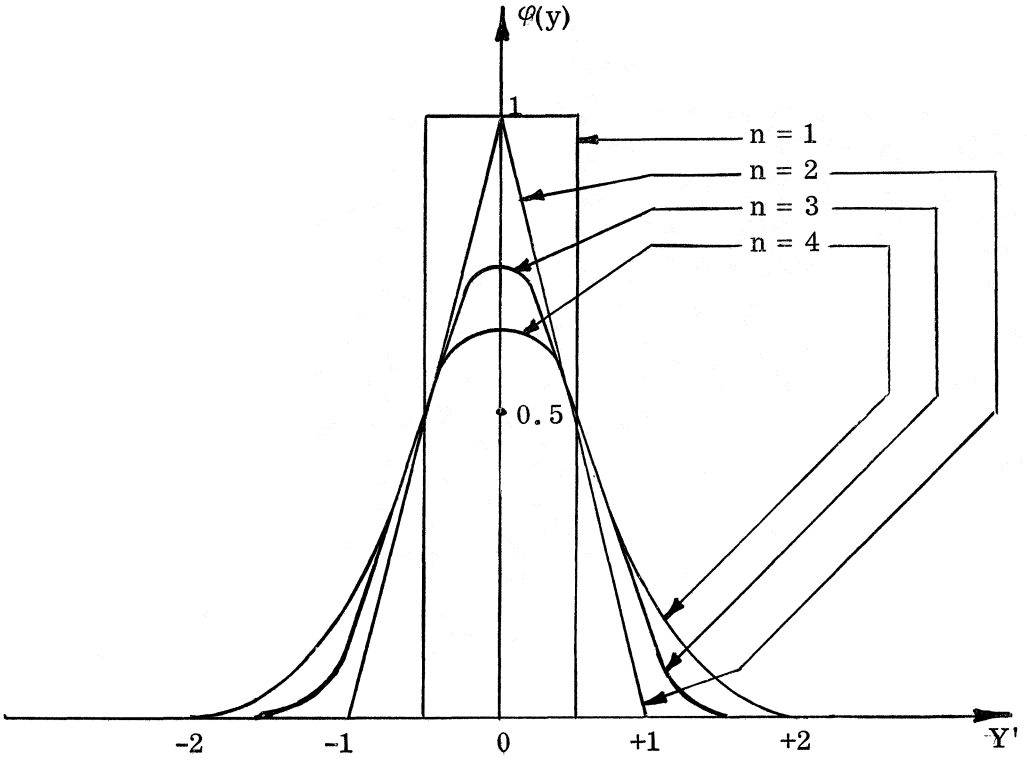

Figure 1

CONCLUSION

We have shown that when a value Q of a signal S mixed with a gaussian noise X(rms σ_0) is measured

with a precision of $\pm 1/2k$, this lack of precision acts as a random variable Y. Its statistical distribution with an rms $\epsilon_0 = \frac{1}{2\sqrt{3}}$ is not gaussian, but linear. Therefore, these two random variables cannot be compared immediately. But, when an average of n measurements is taken, the statistical distribution of $Y' = \frac{1}{n} \sum_{k=1}^{n} Yk$ tends to a gaussian distribution as n increases, and its rms is:

$$\epsilon = \frac{1}{2}\sqrt{\frac{n}{3}}$$

The rms of $X' = \frac{1}{n} \sum_{k=1}^{n} Xk$ is $\sigma = f(n)\ \sigma_0$ with $f(n) = \sqrt{n}$ when the values of the noise are independent. The variables X' and Y' are now comparable; their sum is gaussian and its rms is:

$$\sigma' = \sqrt{\sigma^2 + \epsilon^2} = \sigma\sqrt{1 + \epsilon^2/\sigma^2} \doteq \sigma\left(1 + \frac{\epsilon^2}{2\sigma^2}\right)$$

It is therefore possible to define the useful precision of the measurements in terms of the minimum number of units by which the rms value of the noise must be represented so that $\epsilon^2/2\sigma^2$ can be considered as negligible with respect to 1. In the simple and usual case where $\sigma = \sqrt{n}\ \sigma_0$, one has:

$$\frac{\epsilon^2}{2\sigma^2} = \frac{\frac{n}{12}}{2n\sigma_0^2} = \frac{1}{24\sigma_0^2}$$

Below, the variations of $\frac{\sigma'}{\sigma}$ are plotted versus the number of units by which σ_0 is represented. If one agrees, for instance, to have the final rms increased by a factor of 4%, then it is enough to measure the amplitude of the signal with a precision such that 1 unit corresponds to σ_0. In other words, this means that, for instance, signals up to 100 times the rms of the noise can be represented by only two decimal digits without an increase of more than 4% in the final rms.

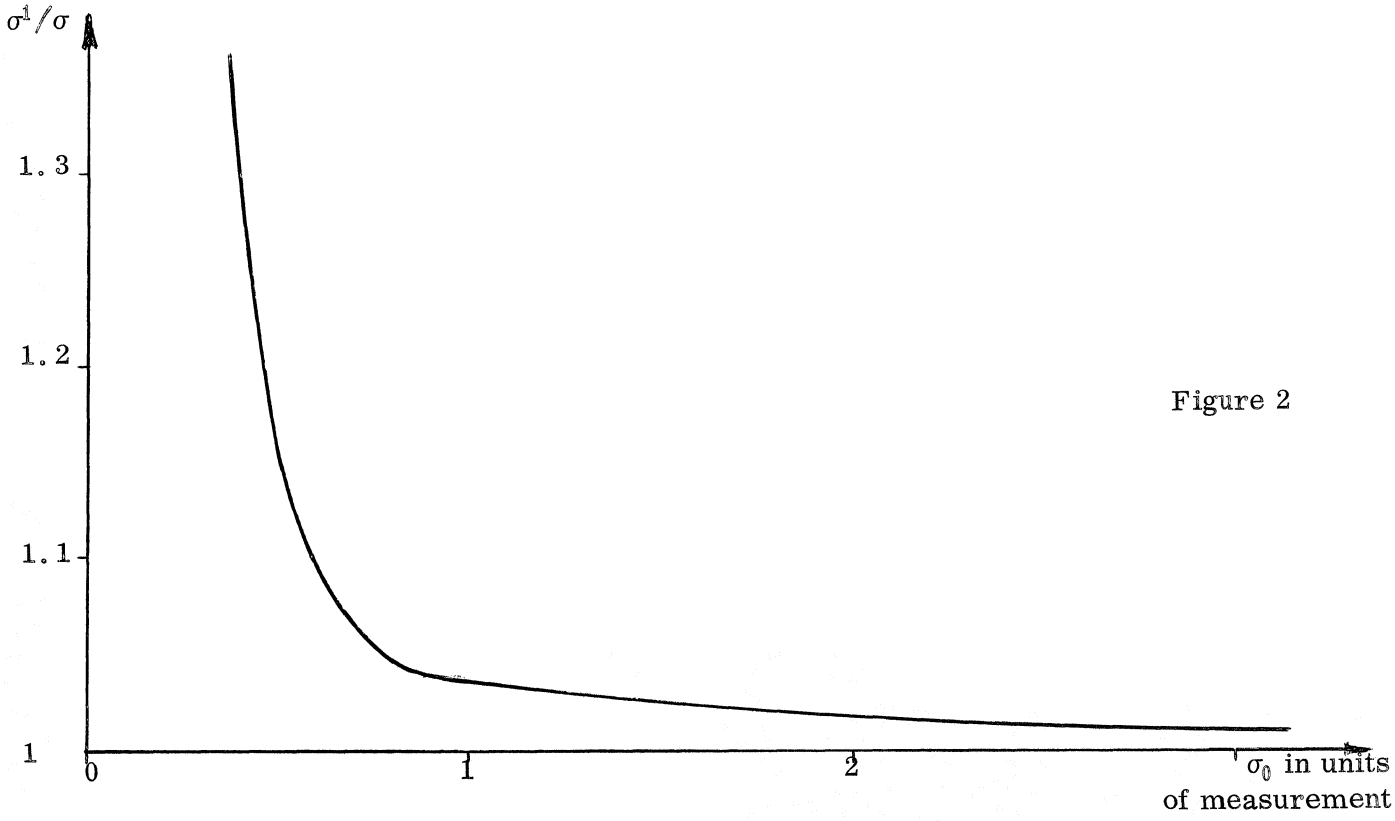

Figure 2

REFERENCE

A. Angot, Complements de Mathematiques. Ed. Revue d' Optique. 1952. Page 606.