

NATIONAL RADIO ASTRONOMY OBSERVATORY
SOCORRO, NEW MEXICO
VERY LARGE ARRAY PROGRAM

VLA COMPUTER MEMORANDUM NO. 150

QUANTIZATION CORRECTION OF
VLA CORRELATION MEASUREMENTS

Frederic R. Schwab

December 1978

Released October 1979

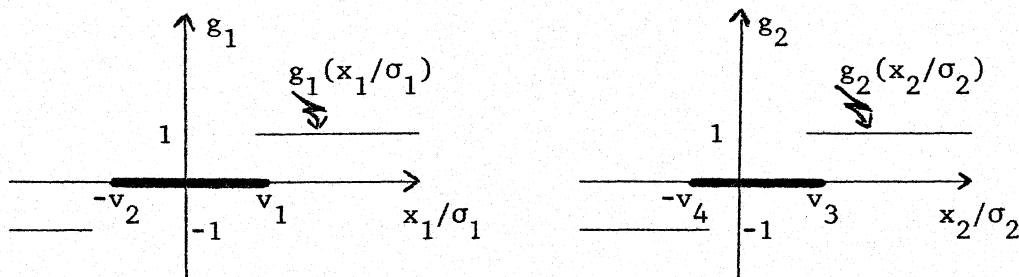
John Granlund, in his analysis [1] of the 3-level by 3-level digital correlator, derives an exact expression, valid for possibly unequal digitizer threshold levels, relating correlator output $r(\tau)$ to true correlation $\rho(\tau)$. Here we will attempt to find a simple approximation to ρ , given r , accurate to 0.1% in relative error for $|\rho| \lesssim 0.9$. Our approximation must be suited to rapid evaluation by the synchronous computing subsystem of the VLA.

Assume the (real, for simplicity) signals, $x_1(t)$ and $x_2(t)$, to be stationary Gaussian random variables with standard deviations σ_1 and σ_2 . For digitizer transfer functions g_1 and g_2 as sketched below, Granlund shows the relation of correlator output r (where $r(\tau) = \langle g_1(x_1(t))g_2(x_2(t+\tau)) \rangle / [\langle g_1^2(x_1(t)) \rangle \langle g_2^2(x_2(t)) \rangle]^{1/2}$) to true normalized correlation ρ ($\rho(\tau) = \langle x_1(t)x_2(t+\tau) \rangle / [\langle x_1^2(t) \rangle \langle x_2^2(t) \rangle]^{1/2}$) to be given by:

$$r(\rho) = L(v_1, v_3, \rho) + L(v_2, v_4, \rho) - L(v_1, v_4, -\rho) - L(v_2, v_3, -\rho), \quad (1)$$

where L is the bivariate normal integral [2, p. 936]. In the case of equal digitizer threshold levels, $v_1 = v_2 = v_3 = v_4$, (1) reduces to the expression given by Cooper [3] or by Hagen and Farley [4].

Because the hardware associated with the IF lines employs phase switching [5], expression (1) does not fully describe the VLA correlation process. Granlund shows that the output r_s of the correlator, given phase switched inputs, is the odd part of (1), viz.:



Digitizer Transfer Functions

$$r_s(\rho) = \frac{r(\rho) - r(-\rho)}{2} . \quad (2)$$

It is the inverse of r_s , $\rho = r_s^{-1}(r)$, which we should like to be able to calculate in order to correct the observed correlations. This calculation cannot be performed because, owing to the hardware configuration, we do not know the settings of the four digitizer thresholds. In [1] Granlund shows that, if the thresholds are maintained to reasonable tolerances, we may make the following simplification: We suppose that g_1 's thresholds are equal and are given by $v_1 = v_2 = v_5$ and similarly that g_2 's thresholds are $v_3 = v_4 = v_6$. Now we assume that the correlation process is governed by the relation

$$r_I(\rho) = 2[L(v_5, v_6, \rho) - L(v_5, v_6, -\rho)] . \quad (3)$$

When v_5 is defined by $v_5 = \sqrt{2} \operatorname{inverfc}\{\langle g_1^2[x_1(t)] \rangle\}$, and similarly for v_6 , digitization with $v_1 = v_2 = v_5$ and $v_3 = v_4 = v_6$ would yield the observed autocorrelations at zero lag, $\langle g_1^2[x_1(t)] \rangle$ and $\langle g_2^2[x_2(t)] \rangle$. Granlund gives numerical evidence that the relative error in approximating r_s^{-1} by r_I^{-1} is never greater than 10^{-3} , provided that the thresholds are held to within 10% of their optimal (with respect to

S/N) setting*, $v_{\text{opt}} \approx .612$. He also proves the inequality $|r_I^{-1}(r)| \leq |r_s^{-1}(r)|$. As the threshold constraints are reasonably well-matched by the VLA correlator hardware, the remainder of the development here will deal with the approximation of r_I^{-1} . In order to compensate for the error in approximating r_s^{-1} by r_I^{-1} (the above inequality) we will favor those approximations which tend not to underestimate $|r_I^{-1}|$.

Quantization correction is to be done within the synchronous computing subsystem of the VLA by an FPS AP-120B array processor (AP) driven by a Modcomp miniature computer. The AP performs floating point vector or array arithmetic at a high speed, which it achieves by means of a pipeline adder and a pipeline multiplier. It does scalar operations at much slower speed, using pipelining to advantage only in very carefully programmed (microcoded) computations. In the worst-case spectral line observing modes, the AP must, following each 10-second integration period, correct either 4×351 256-point correlation functions or 351 1024-point correlation functions. The correction task must be completed within 10 seconds, with a portion of that time devoted to other tasks ($\sim 1/2$ second to read data into the AP, ~ 2 seconds to compute FFT's, $\sim 1 1/2$ seconds to send data out). Each of a maximum of 54 IF signals is associated with its own digitizer; hence there are up to 1404 (v_5, v_6) pairs comprised of 54 unique thresholds. The approximations presented below are, save one, polynomials in r whose coefficients are functions of v_5 and v_6 . Much of the work involved in the computation of the coefficients need be done only once per unique threshold, since many terms are separable in v_5 and v_6 . The gain levels likely drift on time scales longer than 10 seconds, so that coefficients might not need be computed at each integration period (considerable storage would be required, though). Computation of coefficients would be done mostly in scalar arithmetic mode. Final correction would readily be done by pipeline mode evaluation of polynomials. One must count on this pipelining in order to achieve sufficient speed; hence one would most likely want to use the same

*and provided $|v_1 - v_2| \lesssim .1 v_{\text{opt}}$ and $|v_3 - v_4| \lesssim .1 v_{\text{opt}}$

approximating form to correct all (256 or 1024, above) measurements of a given correlation function, rather than, say, one form valid for small correlations, and another for large. We will see that correction for $|\rho| \lesssim 0.7$, with relative error less than 10^{-3} , can be accomplished with ease. An approximation valid for $|\rho| \lesssim 0.9$ is also presented, accompanied by a promising timing estimate.

Various computational details are given in the Appendix. v_5 and v_6 may be computed by means of the rational approximation to inverf which is given there.

In each case below, the error in an approximation $\tilde{\rho}(r)$ to $\rho(r) = r_I^{-1}(r)$ is defined as the relative error, $E(r) = [\rho(r) - \tilde{\rho}(r)]/\rho(r)$. Note that for $r > 0$ we should favor an approximation with $E(r) \leq 0$ in order not to underestimate r_I^{-1} . Always an upper limit on $|\rho|$ is quoted*, for which $|E| < 10^{-3}$ when $.551 \leq v_5, v_6 \leq .673$ (this is the approximate range of interest, $v_{\text{opt}} \pm 10\%$). All of the approximations considered are accurate to $|E| < 10^{-2}$ for $|\rho| \lesssim 0.88$. None of the approximations behaves very differently than quoted over a slightly larger range in (v_5, v_6) , say in the range $v_{\text{opt}} \pm 20\%$. The algorithm in [6] was used for the accurate computation of L needed to study the errors in the approximations.

Sine Approximation

A simple approximation to $\rho = r_I^{-1}$ is given by:

$$\tilde{\rho}(r) = \frac{1}{\sqrt{(v_5^2-1)(v_6^2-1)}} \sin \left[\frac{\pi}{2} e^{(v_5^2+v_6^2)/2} \sqrt{(v_5^2-1)(v_6^2-1)} r \right].$$

$|E| < 10^{-3}$ for $|\rho| \leq .62$, $.551 \leq v_5, v_6 \leq .673$. The error curve shown in column 3 of Table II exhibits typical behavior: (for $r > 0$) a single sign change occurs when ρ is between about .8 and .9, with E going

*Conclusions are based on tabulations of the error curves corresponding to several dozen (v_5, v_6) pairs. Behavior of the curves is consistent enough that .01 is very nearly an absolute confidence limit on the upper bounds quoted for the ranges of validity in $|\rho|$.

from positive (the wrong sense) to negative at that point. The first two terms in the Maclaurin expansion of $\tilde{\rho}$ in powers of r are identical to those of the Maclaurin expansion of r_I^{-1} .

Maclaurin Expansion

The Maclaurin expansion for r_I can be written

$$r_I(\rho) = 4 \sum_{n=0}^{\infty} \frac{Z^{(2n)}(v_5) Z^{(2n)}(v_6)}{(2n+1)!} \rho^{2n+1}, \quad (4)$$

where $Z^{(m)}$ (which is tabulated in [2]) is the m -th derivative of the function defined by $Z(x) = 1/\sqrt{2\pi} e^{-x^2/2}$.* The Maclaurin series for r_I^{-1} is given by the reversion of (4):

$$\begin{aligned} \rho(r) = r_I^{-1}(r) = \lambda - \frac{Z^{(2)}(v_5) Z^{(2)}(v_6)}{3! Z(v_5) Z(v_6)} \lambda^3 + \\ \frac{1}{5!} \left\{ 10 \left[\frac{Z^{(2)}(v_5) Z^{(2)}(v_6)}{Z(v_5) Z(v_6)} \right]^2 - \frac{Z^{(4)}(v_5) Z^{(4)}(v_6)}{Z(v_5) Z(v_6)} \right\} \lambda^5 + \dots, \end{aligned} \quad (5)$$

where

$$\lambda \equiv \frac{r}{4 Z(v_5) Z(v_6)}.$$

(4) can be rewritten as:

*For completeness, we add that

$$\begin{aligned} r_s(\rho) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} [Z^{(2n)}(v_1) Z^{(2n)}(v_3) + Z^{(2n)}(v_2) Z^{(2n)}(v_4) \\ + Z^{(2n)}(v_1) Z^{(2n)}(v_4) + Z^{(2n)}(v_2) Z^{(2n)}(v_3)] \rho^{2n+1} \end{aligned}$$

$$r_I(\rho) = \frac{2e^{-\frac{1}{2}(v_5^2+v_6^2)}}{\pi} \left[\rho + \frac{(v_5^2-1)(v_6^2-1)}{3!} \rho^3 + \frac{(v_5^4-6v_5^2+3)(v_6^4-6v_6^2+3)}{5!} \rho^5 \right. \\ \left. + \frac{(v_5^6-15v_5^4+45v_5^2-15)(v_6^6-15v_6^4+45v_6^2-15)}{7!} \rho^7 + \dots \right] \quad (6)$$

and (5) can be rewritten as:

$$\rho(r) = \gamma - \frac{1}{3!}(v_5^2-1)(v_6^2-1)\gamma^3 + \frac{1}{5!}[10(v_5^2-1)^2(v_6^2-1)^2 - (v_5^4-6v_5^2+3)(v_6^4-6v_6^2+3)]\gamma^5 \\ + \frac{1}{7!}\{56(v_5^2-1)(v_6^2-1)[(v_5^4-6v_5^2+3)(v_6^4-6v_6^2+3) - 5(v_5^2-1)^2(v_6^2-1)^2] \\ - (v_5^6-15v_5^4+45v_5^2-15)(v_6^6-15v_6^4+45v_6^2-15)\}\gamma^7 + \dots, \quad (7)$$

where

$$\gamma \equiv \frac{\pi}{2} e^{\frac{1}{2}(v_5^2+v_6^2)} r.$$

The coefficients in (4) or (6) are easily evaluated with the aid of the differential recurrence relation $Z^{(m+2)}(x) + xZ^{(m+1)}(x) + (m+1)Z^{(m)}(x) = 0$. Additional coefficients in (5) or (7) may be computed, either numerically or symbolically, from the coefficients in (4) by application of the recursion formulae for reversion of power series (see, e.g. [10]). For polynomials obtained by various truncations of (7) we have $|E| < 10^{-3}$ for:

through terms of order	3,	ρ ≤	.60	}	and	.551	≤	v ₅ ,
	5		.68					
	7		.70					
	13		.80					
	15		.84					
	25		.86					
	35		.90					
						v ₆ ≤	.673.	

A set of error curves is shown in Table II. The curves corresponding to truncations at r^3 , r^5 , r^7 , and r^{13} exhibit behavior which is typical over the range of interest in (v_5, v_6) : the first two underestimate r_I^{-1} over much of the range, whereas the latter two consistently overestimate. Higher-order coefficients in the above (say, ≥ 13 -th order) come at much greater expense (the reversion calculations quickly become excessive) than the coefficients in the seventh-order approximation, good for $|\rho| \lesssim .88$, presented below.

Frequently, approximations superior to a truncated power series can be obtained by converting the truncated series into a corresponding continued fraction or rational function (the latter is called a Padé approximant). A few such approximations were tried, but they were never much superior to the truncated power series. Anyway, the economics of computing on the AP favors polynomial correction over more complex corrections. To illustrate: the time required, on the AP, to evaluate a sixth-order polynomial at each element of a vector is roughly the same as that required to divide two vectors element by element.

Two-Point Taylor Formula

The two-point Taylor formula* [8] [take $n=4$, $a=0$, $b=r_I(\rho_0)$ for

*Given two points, $a < b$, the $(2n-1)$ -degree polynomial defined by

$$p(x) = (x-a)^n \sum_{k=0}^{n-1} \frac{B_k (x-b)^k}{k!} + (x-b)^n \sum_{k=0}^{n-1} \frac{A_k (x-a)^k}{k!} ,$$

$$\text{where } A_k \equiv \frac{d^k}{dx^k} \left[\frac{f(x)}{(x-b)^n} \right]_{x=a} \quad \text{and} \quad B_k \equiv \frac{d^k}{dx^k} \left[\frac{f(x)}{(x-a)^n} \right]_{x=b} ,$$

is the unique polynomial of that degree whose 0-th through $(n-1)$ -th derivatives match those of f at a and b . The remainder is given by

$$f(x) - p(x) = \frac{f^{(2n)}(\xi)}{(2n)!} (x-a)^n (x-b)^n \quad \text{where } \min(x,a) < \xi < \max(x,b).$$

A related method, known as "two-point Padé" can be used to construct rational approximations from the Taylor series about two points.

some fixed $\rho_0 > 0$] yields an approximation of the form:

$$\tilde{\rho}(r) = \text{sign}(r) \cdot [\Delta r^4 (a_1 |r| + a_2 r^2 + a_3 |r|^3) + r^4 (b_0 + b_1 \Delta r + b_2 \Delta r^2 + b_3 \Delta r^3)],$$

where $\Delta r = |r| - \rho_0$. The bracketed expression is just a seventh-order polynomial in $|r|$. Choosing $\rho_0 = .75$, one has $|E| < 10^{-3}$ for $|\rho| \leq .88$ over the range $.551 \leq v_5, v_6 \leq .673$. Additionally, $|E| < 10^{-2}$ for $|\rho| \leq .96$. The error curve shown in the last column of Table II exhibits typical behavior: for $|\rho| \leq .94$ over the range of interest in (v_5, v_6) , $|\tilde{\rho}|$ overestimates $|r_I^{-1}|$. The coefficients for $\tilde{\rho}$ are computed from the first three derivatives of r_I^{-1} evaluated at two points, $r=0$ and $r = r_0 = r_I(\rho_0)$. The bulk of the effort required to find the coefficients is in the computation of r_0 . Computation of 351 sets of coefficients in 32-bit arithmetic (which gives sufficient accuracy) on the IBM 360/65 requires 1.9 seconds. With some streamlining of the computation, one ought to be able to compute 4×351 sets of the coefficients in comparable time either on the Modcomp, or, with microcoding of the scalar arithmetic, on the AP. Evaluation at 256 points of 4×351 seventh-order polynomials requires about 0.66 seconds on the AP. Details of the coefficient computation are given in the Appendix.

Table I presents, in detail, the quality of the approximations, subject to the criterion that $|E| < 10^{-3}$ over the range of interest in (v_5, v_6) . Table II shows a typical set of error curves for the various approximations.

Use of the two-point Taylor approximation appears to be feasible, even in the worst-case observing mode. Computation of the coefficients in that approximation requires much less time than does computation of the number of Maclaurin coefficients needed to give similar accuracy. If high accuracy only to $|\rho|$ somewhat less than .9 is required, use of one of the low-order truncations of the Maclaurin series is recommended.

Much simpler corrections than those given above can be used with correlators whose digitizer thresholds are maintained by feedback at their optimal settings.

REFERENCES

1. Granlund, J., in preparation, VLA Computer Memorandum Series.
2. Abramowitz, M. and Stegun, I. A., Editors, Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, Nat. Bur. Standards Appl. Math. Series, vol. 55, U.S. Government Printing Office, 1964; reprint, Dover, New York, 1966.
3. Cooper, B. F. C., "Correlators with Two-bit Quantization", Aust. J. Phys., 23, 1970, 521-527.
4. Hagen, J. B. and Farley, D. T., "Digital Correlation Techniques in Radio Science", Radio Science, 8, 1973, 775-784.
5. Thompson, A. R., "Phase Switching in the VLA", VLA Electronics Memorandum No. 122, 1974.
6. Donnelly, T. G., "Algorithm 462. Bivariate Normal Distribution", Collected Algorithms from CACM, Association for Computing Machinery.
7. Drezner, Z., "Computation of the Bivariate Normal Integral", Math. Comp., 32, 1978, 277-279.
8. Davis, P. J. and Rabinowitz, P., Methods of Numerical Integration, Academic P., N.Y., 1975, p. 20.
9. Blair, J. M., et al., "Rational Chebyshev Approximations for the Inverse of the Error Function", Math. Comp., 30, 1976, pp. 827-830.
10. Henrici, P., Applied and Computational Complex Analysis, Vol. I, Wiley, N. Y., 1976, ch. 1.

APPENDIX

A.1 Computation of v_5 and v_6

Reference [9] includes tabulations of 84 best uniform rational approximations to inverf . One which is well-suited to our needs is:

$$\operatorname{inverf}(x) \approx x \cdot \frac{p_0 + p_1(x^2 - .5625) + p_2(x^2 - .5625)^2}{q_0 + q_1(x^2 - .5625) + q_2(x^2 - .5625)^2},$$

$$|x| \leq .75, \text{ max. rel error} = 8.3 \times 10^{-7},$$

$$p_0 = 1.591863138$$

$$q_0 = 1.467751692$$

$$p_1 = -2.442326820$$

$$q_1 = -3.013136362$$

$$p_2 = 0.37153461$$

$$q_2 = 1.00000000$$

Rounding the coefficients to 38-bit precision (28 bit mantissa) should do little harm. The next lower precision approximation from [9] was used in the 360 computations.

Compute v_5 according to $v_5 = \sqrt{2} \operatorname{inverf}\{1 - \langle g_1^2[x_1(t)] \rangle\}$, and v_6 correspondingly. The approximation is valid for any measured auto-correlation in the range [.25,1].

A.2 Computation of r_I

$r_I(\rho_0)$ is needed to high accuracy (say 6 digits) for computation of the coefficients of the two-point Taylor approximation. Calculation of $r_I(.75)$ to this accuracy requires, on average, 3.45 ms. on the IBM using a slightly streamlined version of the algorithm in [6] to compute $L(v_5, v_6, .75)$ and $L(v_5, v_6, -.75)$. The streamlining was accomplished by eliminating two of the computations of the error function in each evaluation of L ; the needed values are implicit in the auto-correlation measurements. No routine to compute erf is included in the standard library of the AP, but approximations are readily avail-

able (see Luke: Mathematical Functions and Their Approximation, or Hart: Computer Approximations, or [2]).

On the IBM the algorithm in [7] is not competitive in timing with that in [6]. However, on the AP, if it is microcoded, [6], which is full of branches, would likely be slower than [7].

A.3 Coefficients for the two-point Taylor formula with $n=4$, $a=0$, $b=r_I(\rho_0)$

a) To get A_0, A_1, A_2, A_3 :

Define

$$\hat{\alpha} = \frac{2}{\pi} e^{-\frac{1}{2}(v_5^2 + v_6^2)}, \quad \hat{\gamma} = (v_5^2 - 1)(v_6^2 - 1)\hat{\alpha}, \quad \alpha = 1/\hat{\alpha}, \quad \gamma = -\hat{\gamma}/\hat{\alpha}^4, \quad r_0 = r_I(\rho_0).$$

$$\text{Then } A_0 = 0, \quad A_1 = \alpha/r_0^4, \quad A_2 = 8 A_1/r_0, \quad A_3 = (\gamma/r_0^2 + 60A_1)/r_0^2.$$

b) To get B_0, B_1, B_2, B_3 :

$$\text{Define } \hat{\alpha} = \left. \frac{dr_I}{d\rho} \right|_{\rho=\rho_0}, \quad \hat{\beta} = \left. \frac{d^2 r_I}{d\rho^2} \right|_{\rho=\rho_0}, \quad \hat{\gamma} = \left. \frac{d^3 r_I}{d\rho^3} \right|_{\rho=\rho_0} \quad (\text{see A.4}).$$

$$\text{Set } \alpha = 1/\hat{\alpha}, \quad \beta = -\hat{\beta}/\hat{\alpha}^3, \quad \gamma = (3\hat{\beta}^2 - \hat{\alpha}\hat{\gamma})/\hat{\alpha}^5.$$

$$\begin{aligned} \text{Then } B_0 &= \rho_0/r_0^4, \\ B_1 &= (\alpha - 4\rho_0/r_0)/r_0^4, \\ B_2 &= [\beta + (20\rho_0/r_0 - 8\alpha)/r_0]/r_0^4, \\ B_3 &= \{\gamma + [-12\beta + (60\alpha - 120\rho_0/r_0)/r_0]/r_0\}/r_0^4. \end{aligned}$$

c) Finally, $a_k = A_k/k!$, $b_k = B_k/k!$

A.4 Derivatives of r_I

The first three derivatives of r_I about $\rho \neq 0$ are needed for the two-point Taylor formula with $n=4$.

$$\frac{dr_I}{d\rho} = \frac{2 e^{-\frac{1}{2}(v_5^2+v_6^2)/(1-\rho^2)}}{\pi\sqrt{1-\rho^2}} \cosh\left(\frac{\rho v_5 v_6}{1-\rho^2}\right) .$$

$$\text{Define } y = \rho(1-\rho^2-v_5^2-v_6^2) + v_5 v_6(1+\rho^2) \tanh\left(\frac{\rho v_5 v_6}{1-\rho^2}\right) .$$

$$\text{Then } \frac{d^2 r_I}{d\rho^2} = \frac{y}{(1-\rho^2)^2} \frac{dr_I}{d\rho} .$$

Finally,

$$\begin{aligned} \frac{d^3 r_I}{d\rho^3} = & \left[\frac{(1-\rho^2)^2(1+2\rho^2)-2(1-\rho^2)(1-\rho)^2(1-2\rho)v_5 v_6 - (1-\rho^2)(1+5\rho^2)(v_5-v_6)^2}{\pi(1-\rho^2)^{9/2}} \right. \\ & \left. + \frac{[(1-\rho)^2 v_5 v_6 - \rho(v_5-v_6)^2]^2}{\pi(1-\rho^2)^{9/2}} \right] e^{-\frac{(v_5^2-2\rho v_5 v_6+v_6^2)}{2(1-\rho^2)}} \\ & + \left[\frac{(1-\rho^2)^2(1+2\rho^2)-2(1-\rho^2)(1+\rho)^2(1+2\rho)v_5 v_6 - (1-\rho^2)(1+5\rho^2)(v_5-v_6)^2}{\pi(1-\rho^2)^{9/2}} \right. \\ & \left. + \frac{[(1+\rho)^2 v_5 v_6 + \rho(v_5-v_6)^2]^2}{\pi(1-\rho^2)^{9/6}} \right] e^{-\frac{(v_5^2+2\rho v_5 v_6+v_6^2)}{2(1-\rho^2)}} \end{aligned}$$

Note that the lengthiest parts of these expressions involve ρ , which would be held fixed at ρ_0 for correction by the two-point Taylor formula.

Table I.

max |p| such that $|e| < 10^{-3}$

$$(e = \frac{f - \tilde{f}}{f})$$



sine		Maclaurin series (7) truncated beyond:							two-point
α_5, α_6	approximation	r^3	r^5	r^7	r^{13}	r^{15}	r^{25}	r^{35}	Taylor formula, $n=4$
.551, .551	.66	.60	.82	.80	.84	.86	.92	.92	.88
.575, .551	.66	.60	.80	.78	.82	.84	.92	.92	.88
.575, .575	.66	.60	.78	.78	.82	.84	.92	.92	.88
.600, .551	.64	.60	.80	.78	.82	.84	.92	.92	.88
.600, .575	.66	.60	.78	.76	.82	.84	.90	.92	.88
.600, .600	.68	.62	.76	.76	.82	.84	.90	.92	.88
.624, .551	.64	.60	.78	.78	.82	.84	.90	.92	.88
.624, .575	.66	.60	.76	.76	.82	.84	.90	.92	.88
.624, .600	.68	.62	.76	.74	.82	.84	.90	.92	.88
.624, .624	.82	.64	.74	.74	.82	.84	.88	.94	.88
.649, .551	.64	.60	.78	.76	.82	.84	.90	.92	.88
.649, .575	.66	.62	.76	.76	.82	.84	.90	.92	.88
.649, .600	.82	.62	.74	.74	.82	.84	.88	.96	.88
.649, .624	.82	.66	.72	.74	.80	.84	.88	.94	.90
.649, .649	.80	.80	.72	.72	.80	.84	.88	.92	.90
.673, .551	.62	.60	.78	.76	.82	.84	.90	.92	.88
.673, .575	.64	.60	.74	.74	.82	.84	.90	.92	.88
.673, .600	.82	.64	.74	.74	.80	.84	.88	.92	.90
.673, .624	.80	.82	.72	.72	.80	.84	.88	.92	.90
.673, .649	.78	.80	.70	.72	.80	.84	.88	.92	.90
.673, .673	.76	.78	.68	.70	.80	.84	.86	.90	.94

Table II. A Typical Set of Error Curves.

V5 = C.64E6CC V6 = 0.5510CC

Maclaurin series (?) truncated beyond:

Two-point Taylor formula, $m=4, \rho=.75$

V	$\rho = \sqrt{V^2(r)}$	size approx. max.	r^3	r^5	r^7	r^{13}	r^{15}	r^{25}	r^{35}	Two-point Taylor formula, $m=4, \rho=.75$
0.0000000000	0.0000000000	C.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.0086432267	0.0086432267	E.47E-10	5.23E-14	1.56E-15	1.60E-15	1.60E-15	1.60E-15	1.60E-15	1.60E-15	1.60E-15
0.0177300841	0.0177300841	C.34E-08	1.70E-09	1.10E-14	2.84E-15	2.84E-15	2.84E-15	2.84E-15	2.84E-15	2.84E-15
0.0265497042	0.0265497042	6.87E-08	3.68E-11	2.11E-13	1.32E-15	1.32E-15	1.32E-15	1.32E-15	1.32E-15	1.32E-15
0.0354716211	0.0354716211	2.18E-07	2.73E-07	2.06E-10	2.11E-12	1.73E-16	1.73E-16	1.73E-16	1.73E-16	1.73E-16
0.0443502728	0.0443502728	5.32E-07	6.88E-07	7.83E-10	2.27E-11	2.78E-16	2.78E-16	2.78E-16	2.78E-16	2.78E-16
0.0532361015	0.0532361015	1.11E-06	1.39E-06	2.33E-09	5.51E-11	5.78E-16	3.47E-16	3.47E-16	3.47E-16	3.47E-16
0.0621305557	0.0621305557	2.65E-06	1.29E-08	5.65E-10	1.93E-14	4.34E-16	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.0710350507	0.0710350507	3.51E-06	7.08E-06	1.47E-09	1.02E-13	3.08E-15	1.54E-16	1.54E-16	1.54E-16	1.54E-16
0.0799511697	0.0799511697	6.32E-06	4.82E-08	3.45E-09	4.51E-13	1.60E-14	2.78E-16	2.78E-16	2.78E-16	2.78E-16
0.0888302655	0.0888302655	1.27E-05	1.59E-05	8.42E-08	7.64E-09	1.74E-12	7.62E-14	2.52E-16	2.52E-16	2.52E-16
0.0977278612	0.0977278612	1.81E-05	2.26E-05	1.40E-07	1.57E-08	6.00E-12	3.12E-13	2.31E-16	2.31E-16	2.31E-16
0.1067834512	0.1067834512	2.50E-05	3.13E-05	2.22E-07	3.05E-08	1.98E-11	1.14E-12	1.07E-16	1.07E-16	1.07E-16
0.1157605425	0.1157605425	3.38E-05	4.23E-05	3.39E-07	5.67E-08	5.41E-11	3.83E-12	9.91E-17	9.91E-17	9.91E-17
0.1247566559	0.1247566559	4.47E-05	5.60E-05	5.01E-07	1.01E-07	1.45E-10	1.18E-11	4.63E-17	4.63E-17	4.63E-17
0.1337723270	0.1337723270	5.82E-05	7.29E-05	7.17E-07	1.75E-07	3.68E-10	3.41E-11	3.04E-16	2.60E-16	2.60E-16
0.1428121073	0.1428121073	7.46E-05	9.34E-05	9.98E-07	2.95E-07	8.84E-10	9.24E-11	2.04E-16	0.00E+00	0.00E+00
0.1519745056	0.1519745056	9.43E-05	1.18E-04	1.35E-06	4.83E-07	2.02E-09	2.37E-10	5.40E-16	3.85E-16	3.85E-16
0.1609627393	0.1609627393	1.18E-04	1.47E-04	1.78E-06	7.74E-07	4.45E-09	5.82E-10	3.65E-15	2.92E-16	2.92E-16
0.1700768858	0.1700768858	1.45E-04	1.82E-04	2.29E-06	1.22E-06	9.42E-09	1.37E-09	1.50E-14	1.73E-16	1.73E-16
0.1792159844	0.1792159844	1.78E-04	2.23E-04	2.85E-06	1.98E-06	1.93E-08	3.09E-09	5.51E-14	6.61E-17	5.56E-04
0.1884932387	0.1884932387	2.16E-04	2.70E-04	3.45E-06	2.48E-06	3.84E-08	6.75E-09	1.88E-13	3.15E-16	5.20E-04
0.1978583291	0.1978583291	2.59E-04	3.24E-04	4.02E-06	2.88E-06	7.44E-08	1.43E-08	6.11E-13	7.24E-16	4.76E-04
0.2073656567	0.2073656567	3.05E-04	3.86E-04	4.47E-06	6.35E-06	1.41E-07	2.95E-08	1.88E-12	3.53E-15	4.26E-04
0.2161108986	0.2161108986	3.65E-04	4.57E-04	4.64E-06	9.30E-06	2.60E-07	5.52E-08	5.52E-12	1.58E-14	3.72E-04
0.2254219144	0.2254219144	4.29E-04	5.39E-04	4.83E-06	1.35E-05	4.72E-07	1.16E-07	1.55E-11	6.94E-14	3.15E-04
0.2347178559	0.2347178559	5.01E-04	6.29E-04	5.14E-06	1.94E-05	8.40E-07	2.23E-07	4.17E-11	2.92E-12	2.58E-04
0.2441626313	0.2441626313	5.81E-04	7.29E-04	6.10E-07	2.78E-05	1.47E-06	4.20E-07	1.08E-10	1.18E-12	2.04E-04
0.2535522336	0.2535522336	6.69E-04	8.41E-04	4.00E-06	3.95E-05	2.54E-06	7.77E-07	2.67E-10	4.56E-12	1.53E-04
0.2630747689	0.2630747689	7.64E-04	9.64E-04	1.18E-05	5.57E-05	4.31E-06	1.41E-06	6.33E-10	1.70E-11	1.09E-04
0.2726004954	0.2726004954	8.70E-04	1.10E-03	2.41E-05	7.83E-05	7.24E-06	2.54E-06	1.43E-09	6.10E-11	7.20E-05
0.2821758809	0.2821758809	9.81E-04	1.24E-03	4.32E-05	1.10E-04	1.20E-05	4.49E-06	3.08E-09	2.12E-10	4.32E-05
0.2918026850	0.2918026850	1.10E-03	1.40E-03	7.21E-05	1.53E-04	1.58E-05	7.84E-06	6.21E-09	7.14E-10	2.26E-05
0.3014870804	0.3014870804	1.22E-03	1.56E-03	1.15E-04	2.13E-04	3.22E-05	1.35E-05	1.14E-08	2.34E-09	5.68E-06
0.3112258298	0.3112258298	1.33E-03	1.72E-03	1.78E-04	2.96E-04	5.20E-05	2.31E-05	1.77E-08	7.48E-09	2.95E-06
0.3210765490	0.3210765490	1.44E-03	1.87E-03	2.69E-04	4.12E-04	8.34E-05	3.91E-05	1.78E-08	2.33E-08	4.46E-07
0.3309130592	0.3309130592	1.53E-03	2.02E-03	4.02E-04	5.72E-04	1.33E-04	6.56E-05	1.89E-08	7.12E-08	6.45E-09
0.3408671799	0.3408671799	1.58E-03	2.13E-03	5.93E-04	7.97E-04	2.10E-04	1.09E-04	1.93E-07	2.13E-07	7.56E-09
0.3509052324	0.3509052324	1.57E-03	2.19E-03	8.70E-04	1.11E-03	3.32E-04	1.81E-04	9.10E-07	6.26E-07	7.18E-07
0.3610252369	0.3610252369	1.47E-03	2.16E-03	1.27E-03	1.56E-03	5.21E-04	2.57E-04	2.74E-06	1.81E-06	6.50E-06
0.3713219319	0.3713219319	1.22E-03	2.00E-03	1.85E-03	2.21E-03	8.16E-04	4.84E-04	8.47E-06	5.17E-06	2.54E-05
0.3817441461	0.3817441461	7.46E-04	1.62E-03	2.69E-03	3.05E-03	1.27E-03	7.86E-04	2.45E-05	1.46E-05	9.45E-05
0.3923667311	0.3923667311	9.00E-04	3.92E-03	4.40E-03	1.99E-03	1.77E-03	6.88E-05	4.06E-05	4.06E-05	2.49E-04
0.4026055303	0.4026055303	-7.59E-05	5.75E-03	6.31E-03	3.10E-03	2.07E-03	1.90E-04	1.90E-04	1.90E-04	5.72E-04
0.4134536622	0.4134536622	-1.43E-03	3.40E-03	5.49E-03	9.16E-03	4.84E-03	3.26E-03	5.27E-04	3.10E-04	1.19E-03
0.4246273339	0.4246273339	-2.62E-03	-2.40E-03	8.49E-03	9.16E-03	4.84E-03	3.26E-03	5.27E-04	3.10E-04	1.19E-03
0.4356060521	0.4356060521	-7.13E-03	-5.74E-03	1.26E-02	1.34E-02	7.54E-03	5.14E-03	1.44E-03	8.48E-04	2.30E-03
0.4463050612	0.4463050612	-1.27E-02	-1.11E-02	1.90E-02	2.00E-02	1.17E-02	7.96E-03	4.07E-03	2.30E-03	4.14E-03
0.4572355700	0.4572355700	-2.18E-02	-1.99E-02	2.91E-02	3.04E-02	1.82E-02	1.19E-02	1.20E-02	6.04E-03	6.52E-03
0.4685685184	0.4685685184	-3.49E-02	-3.49E-02	4.61E-02	4.77E-02	2.82E-02	1.65E-02	3.91E-02	1.36E-02	1.05E-02
0.4800000000	0.4800000000	-6.20E-02	-5.91E-02	7.34E-02	7.57E-02	3.99E-02	1.45E-02	1.51E-01	7.28E-03	1.02E-02