VLA COMPUTER MEMORANDUM NO. 156

Rational Approximations to Selected 0-order Spheroidal Functions

Frederic R. Schwab

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Criteria by which the spheroidal functions of order 0 turn out to be optimal functions for use in gridding interferometer data were introduced in VLA Scientific Memorandum No. 132. Although a suitable procedure for the computation of these functions, for arbitrary choices of the parameters (support width and weighting exponent), is given there, it is an easy matter, for each specific choice of parameters, to obtain a much simpler approximation to the desired function. Here I have tabulated useful approximations to selected spheroidal functions. These were obtained by computing best rational approximations, rounding the coefficients to single-precision (32-bit floating point), and then ensuring that the approximations could be evaluated in single-precision, with these rounded coefficients, to sufficient accuracy.

These are called rational approximations because each is the quotient of two polynomials. They are called best because (before rounding) each, among all rational approximations of the chosen form, minimizes (approximately -- to about three significant figures) the maximum relative error over the interval of approximation. The proprietary subroutine IRATCU of the IMSL package was used in computing the approximations; this routine is a Fortran version of an Algol routine which is in the public domain and was published in W. Cody et. al. (Numer. Math., 12 (1968) 242-251). I also used a version of this routine in computing best rational approximations to the quantization correction function appropriate for the Green Bank 1024 channel autocorrelator.

The Tables list coefficients of the approximations for five choices of the support width m (4, 5, 6, 7, and 8 cells) and five choices of the weighting exponent α (0, $\frac{1}{2}$, 1, 3/2, and 2). For m>5, two approximations are given for each choice of parameters -- one for small values of the argument η and one for larger values. The reason that the approximations are quotients of polynomials in (η^2-const.) rather than just quotients of polynomials in η^2 is that with the latter, in single-precision arithmetic, one would get 2-3 significant digits of cancellation error in

some cases, whereas with the former there is much less cancellation error. It is possible to maintain a typical maximum relative error of a few parts in 10^6.

Rather than finding the most economical approximating form for each choice of parameters, I chose, for each value of m, to find the most economical approximating form for the most difficult case, $\alpha=0$, and then to use the same approximating form for each of the other weighting exponents. This decision ought to simplify the logic in programs that, for a given support width, allow the selection of any of the five α.

In the Tables ε_1 is the smallest maximum relative error that is achievable with the given approximating form. ε_2 is the maximum relative error that was observed after the coefficients were rounded and the approximation evaluated (using Horner's rule) at a fair number of evenly distributed arguments. Since the best rational approximations could only be approximated, and not computed exactly, the higher-order coefficients are not accurate to all the digits which are shown. I did not feel that it was worthwhile to determine just how many digits are significant, since no computational economy would be achieved by doing so.

Remember that an extra factor is needed: $(1-\eta^2)^\alpha \psi_{\alpha o}(\pi m/2,\eta)$ is the appropriate convoluting function, and $\psi_{\alpha o}$, without the weight factor, is the appropriate function with which to correct the map. Also remember that the variable η is one which ranges from 0 at the center of the convoluting function to unity at its edge (also from 0 at the center of the map to 1 at its edge).

Table I. Approximations for support width m=4, $|\eta| \leq 1$.

$$\psi_{\alpha o}(\pi m/2, \eta) \simeq \sum_{j=o}^{4} p_j(\eta^2-1)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-1)^j$$

a)	α =0	p_0= 1.584774(-2)	q_0= 1.000000(0)
	ε_1=1.18(-6)	p_1=-1.269612(-1)	q_1= 4.845581(-1)
	ε_2=1.9(-6)	p_2= 2.333851(-1)	q_2= 7.457381(-2)
		p_3=-1.636744(-1)	
		p_4= 5.014648(-2)	
b)	α =½	p_0= 3.101855(-2)	q_0= 1.000000(0)
	ε_1=5.42(-7)	p_1=-1.641253(-1)	q_1= 4.514531(-1)
	ε_2=1.6(-6)	p_2= 2.385500(-1)	q_2= 6.458640(-2)
		p_3=-1.417069(-1)	
		p_4= 3.773226(-2)	
c)	α =1	p_0= 5.007900(-2)	q_0= 1.000000(0)
	ε_1=2.69(-7)	p_1=-1.971357(-1)	q_1= 4.228767(-1)
	ε_2=1.1(-6)	p_2= 2.363775(-1)	q_2= 5.655715(-2)
		p_3=-1.215569(-1)	
		p_4= 2.853104(-2)	
d)	α =3/2	p_0= 7.201260(-2)	q_0= 1.000000(0)
	ε_1=1.42(-7)	p_1=-2.251580(-1)	q_1= 3.978515(-1)
	ε_2=9.7(-7)	p_2= 2.293715(-1)	q_2= 4.997164(-2)
		p_3=-1.038359(-1)	
		p_4= 2.174211(-2)	
e)	α =2	p_0= 9.585932(-2)	q_0= 1.000000(0)
	ε_1=7.90(-8)	p_1=-2.481381(-1)	q_1= 3.756999(-1)
	ε_2=6.3(-7)	p_2= 2.194469(-1)	q_2= 4.448800(-2)
		p_3=-8.862132(-2)	
		p_4= 1.672243(-2)	

Table II. Approximations for support width m=5, $|\eta| \leq 1$.

$$\psi_{\alpha o}(\pi m/2, \eta) \simeq \sum_{j=o}^{6} p_j(\eta^2-1)^j \;/\; \sum_{j=o}^{1} q_j(\eta^2-1)^j$$

a)	$\alpha = 0$	$p_0 = 3.722238(-3)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 7.78(-7)$	$p_1 = -4.991683(-2)$	$q_1 = 2.418820(-1)$
	$\varepsilon_2 = 1.8(-6)$	$p_2 = 1.658905(-1)$	
		$p_3 = -2.387240(-1)$	
		$p_4 = 1.877469(-1)$	
		$p_5 = -8.159855(-2)$	
		$p_6 = 3.051959(-2)$	
b)	$\alpha = \frac{1}{2}$	$p_0 = 8.182649(-3)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 3.85(-7)$	$p_1 = -7.325459(-2)$	$q_1 = 2.291233(-1)$
	$\varepsilon_2 = 1.1(-6)$	$p_2 = 1.945697(-1)$	
		$p_3 = -2.396387(-1)$	
		$p_4 = 1.667832(-1)$	
		$p_5 = -6.620786(-2)$	
		$p_6 = 2.224041(-2)$	
c)	$\alpha = 1$	$p_0 = 1.466325(-2)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 2.02(-7)$	$p_1 = -9.858686(-2)$	$q_1 = 2.177793(-1)$
	$\varepsilon_2 = 1.2(-6)$	$p_2 = 2.180684(-1)$	
		$p_3 = -2.347118(-1)$	
		$p_4 = 1.464354(-1)$	
		$p_5 = -5.350728(-2)$	
		$p_6 = 1.624782(-2)$	
d)	$\alpha = 3/2$	$p_0 = 2.314317(-2)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 1.11(-7)$	$p_1 = -1.246383(-1)$	$q_1 = 2.075784(-1)$
	$\varepsilon_2 = 1.1(-6)$	$p_2 = 2.362036(-1)$	
		$p_3 = -2.257366(-1)$	
		$p_4 = 1.275895(-1)$	
		$p_5 = -4.317874(-2)$	
		$p_6 = 1.193168(-2)$	

(continued)

Table II. Approximations for support width m=5, $|\eta| \leq 1$ (Continued)

e)	α =2	p_0= 3.346886(-2)	q_0= 1.000000(0)
	ε_1=6.38(-8)	p_1=-1.503778(-1)	q_1= 1.983358(-1)
	ε_2=9.9(-7)	p_2= 2.492826(-1)	
		p_3=-2.142055(-1)	
		p_4= 1.106482(-1)	
		p_5=-3.486024(-2)	
		p_6= 8.821107(-3)	

Table IIIA. Approximations for support width m=6, $|\eta| \leq .75$.

$$\psi_{\alpha o}(\pi m/2,\eta) \simeq \sum_{j=o}^{4} p_j(\eta^2-.5625)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-.5625)^j$$

a)	α =0	p_0= 5.613913(-2)	q_0= 1.000000(0)
	ε_1=6.04(-7)	p_1=-3.019847(-1)	q_1= 9.077644(-1)
	ε_2=1.2(-6)	p_2= 6.256387(-1)	q_2= 2.535284(-1)
		p_3=-6.324887(-1)	
		p_4= 3.303194(-1)	
b)	α =½	p_0= 6.843713(-2)	q_0= 1.000000(0)
	ε_1=3.82(-7)	p_1=-3.342119(-1)	q_1= 8.626056(-1)
	ε_2=1.1(-6)	p_2= 6.302307(-1)	q_2= 2.291400(-1)
		p_3=-5.829747(-1)	
		p_4= 2.765700(-1)	
c)	α =1	p_0= 8.203343(-2)	q_0= 1.000000(0)
	ε_1=2.46(-7)	p_1=-3.644705(-1)	q_1= 8.212018(-1)
	ε_2=9.5(-7)	p_2= 6.278660(-1)	q_2= 2.078043(-1)
		p_3=-5.335581(-1)	
		p_4= 2.312756(-1)	
d)	α =3/2	p_0= 9.675562(-2)	q_0= 1.000000(0)
	ε_1=1.60(-7)	p_1=-3.922489(-1)	q_1= 7.831755(-1)
	ε_2=8.6(-7)	p_2= 6.197133(-1)	q_2= 1.890848(-1)
		p_3=-4.857470(-1)	
		p_4= 1.934013(-1)	
e)	α =2	p_0= 1.124069(-1)	q_0= 1.000000(0)
	ε_1=1.06(-7)	p_1=-4.172349(-1)	q_1= 7.481828(-1)
	ε_2=7.7(-7)	p_2= 6.069622(-1)	q_2= 1.726085(-1)
		p_3=-4.405326(-1)	
		p_4= 1.618978(-1)	

Table IIIB. Approximations for support width m=6, $.75 \leq |\eta| \leq 1$.

$$\psi_{\alpha o}(\pi m/2,\eta) \simeq \sum_{j=o}^{4} p_j(\eta^2-1)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-1)^j$$

a)	α =0	p_0= 8.531865(-4)	q_0= 1.000000(0)
	ε_1=1.18(-6)	p_1=-1.616105(-2)	q_1= 1.101270(0)
	ε_2=1.6(-6)	p_2= 6.888533(-2)	q_2= 3.858544(-1)
		p_3=-1.109391(-1)	
		p_4= 7.747182(-2)	
b)	α =½	p_0= 2.060760(-3)	q_0= 1.000000(0)
	ε_1=5.40(-7)	p_1=-2.558954(-2)	q_1= 1.025431(0)
	ε_2=9.8(-7)	p_2= 8.595213(-2)	q_2= 3.337648(-1)
		p_3=-1.170228(-1)	
		p_4= 7.094106(-2)	
c)	α =1	p_0= 4.028559(-3)	q_0= 1.000000(0)
	ε_1=2.66(-7)	p_1=-3.697768(-2)	q_1= 9.599102(-1)
	ε_2=8.9(-7)	p_2= 1.021332(-1)	q_2= 2.918724(-1)
		p_3=-1.201436(-1)	
		p_4= 6.412774(-2)	
d)	α =3/2	p_0= 6.887946(-3)	q_0= 1.000000(0)
	ε_1=1.39(-7)	p_1=-4.994202(-2)	q_1= 9.025276(-1)
	ε_2=1.2(-6)	p_2= 1.168451(-1)	q_2= 2.575336(-1)
		p_3=-1.207733(-1)	
		p_4= 5.744210(-2)	
e)	α =2	p_0= 1.071895(-2)	q_0= 1.000000(0)
	ε_1=7.67(-8)	p_1=-6.404749(-2)	q_1= 8.517470(-1)
	ε_2=7.0(-7)	p_2= 1.297386(-1)	q_2= 2.289667(-1)
		p_3=-1.194208(-1)	
		p_4= 5.112822(-2)	

Table IVA. Approximations for support width m=7, $|\eta| \leq .775$.

$$\psi_{\alpha o}(\pi m/2,\eta) \simeq \sum_{j=o}^{4} p_j(\eta^2-.775^2)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-.775^2)^j$$

a)	α =0	p_0= 2.460495(-2)	q_0= 1.000000(0)
	ε_1=4.38(-6)	p_1=-1.640964(-1)	q_1= 1.124957(0)
	ε_2=5.2(-6)	p_2= 4.340110(-1)	q_2= 3.784976(-1)
		p_3=-5.705516(-1)	
		p_4= 4.418614(-1)	
b)	α =½	p_0= 3.070261(-2)	q_0= 1.000000(0)
	ε_1=2.93(-6)	p_1=-1.879546(-1)	q_1= 1.075420(0)
	ε_2=3.6(-6)	p_2= 4.565902(-1)	q_2= 3.466086(-1)
		p_3=-5.544891(-1)	
		p_4= 3.892790(-1)	
c)	α =1	p_0= 3.770526(-2)	q_0= 1.000000(0)
	ε_1=1.97(-6)	p_1=-2.121608(-1)	q_1= 1.029374(0)
	ε_2=2.4(-6)	p_2= 4.746423(-1)	q_2= 3.181219(-1)
		p_3=-5.338058(-1)	
		p_4= 3.417026(-1)	
d)	α =3/2	p_0= 4.559398(-2)	q_0= 1.000000(0)
	ε_1=1.34(-6)	p_1=-2.362670(-1)	q_1= 9.865496(-1)
	ε_2=2.2(-6)	p_2= 4.881998(-1)	q_2= 2.926441(-1)
		p_3=-5.098448(-1)	
		p_4= 2.991635(-1)	
e)	α =2	p_0= 5.432500(-2)	q_0= 1.000000(0)
	ε_1=9.26(-7)	p_1=-2.598752(-1)	q_1= 9.466891(-1)
	ε_2=1.7(-6)	p_2= 4.974791(-1)	q_2= 2.698218(-1)
		p_3=-4.837861(-1)	
		p_4= 2.614838(-1)	

Table IVB. Approximations for support width m=7, $.775 \leq |\eta| \leq 1$.

$$\psi_{\alpha o}(\pi m/2,\eta) \simeq \sum_{j=o}^{4} p_j(\eta^2-1)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-1)^j$$

a)	α =0	p_0= 1.924318(-4)	q_0= 1.000000(0)
	ε_1=4.17(-6)	p_1=-5.044864(-3)	q_1= 1.450730(0)
	ε_2=4.8(-6)	p_2= 2.979803(-2)	q_2= 6.578685(-1)
		p_3=-6.660688(-2)	
		p_4= 6.792268(-2)	
b)	α =½	p_0= 5.030909(-4)	q_0= 1.000000(0)
	ε_1=1.98(-6)	p_1=-8.639332(-3)	q_1= 1.353872(0)
	ε_2=2.9(-6)	p_2= 4.018472(-2)	q_2= 5.724332(-1)
		p_3=-7.595456(-2)	
		p_4= 6.696215(-2)	
c)	α =1	p_0= 1.059406(-3)	q_0= 1.000000(0)
	ε_1=1.01(-6)	p_1=-1.343605(-2)	q_1= 1.269924(0)
	ε_2=1.3(-6)	p_2= 5.135360(-2)	q_2= 5.032139(-1)
		p_3=-8.386588(-2)	
		p_4= 6.484517(-2)	
d)	α =3/2	p_0= 1.941904(-3)	q_0= 1.000000(0)
	ε_1=5.41(-7)	p_1=-1.943727(-2)	q_1= 1.196177(0)
	ε_2=1.0(-6)	p_2= 6.288221(-2)	q_2= 4.460948(-1)
		p_3=-9.021607(-2)	
		p_4= 6.193000(-2)	
e)	α =2	p_0= 3.224785(-3)	q_0= 1.000000(0)
	ε_1=3.05(-7)	p_1=-2.657664(-2)	q_1= 1.130719(0)
	ε_2=8.4(-7)	p_2= 7.438627(-2)	q_2= 3.982785(-1)
		p_3=-9.500554(-2)	
		p_4= 5.850884(-2)	

Table VA. Approximations for support width m=8, $|\eta| \leq .775$.

$$\psi_{\alpha o}(\pi m/2,\eta) \simeq \sum_{j=o}^{5} p_j(\eta^2-.775^2)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-.775^2)^j$$

a)	$\alpha = 0$	$p_0 = 1.378030(-2)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 7.69(-7)$	$p_1 = -1.097846(-1)$	$q_1 = 1.076975(0)$
	$\varepsilon_2 = 1.6(-6)$	$p_2 = 3.625283(-1)$	$q_2 = 3.394154(-1)$
		$p_3 = -6.522477(-1)$	
		$p_4 = 6.684458(-1)$	
		$p_5 = -4.703556(-1)$	
b)	$\alpha = \frac{1}{2}$	$p_0 = 1.721632(-2)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 5.17(-7)$	$p_1 = -1.274981(-1)$	$q_1 = 1.036132(0)$
	$\varepsilon_2 = 1.6(-6)$	$p_2 = 3.917226(-1)$	$q_2 = 3.145673(-1)$
		$p_3 = -6.562264(-1)$	
		$p_4 = 6.305859(-1)$	
		$p_5 = -4.067119(-1)$	
c)	$\alpha = 1$	$p_0 = 2.121871(-2)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 3.51(-7)$	$p_1 = -1.461891(-1)$	$q_1 = 9.978025(-1)$
	$\varepsilon_2 = 1.6(-6)$	$p_2 = 4.185427(-1)$	$q_2 = 2.920529(-1)$
		$p_3 = -6.543539(-1)$	
		$p_4 = 5.904660(-1)$	
		$p_5 = -3.507098(-1)$	
d)	$\alpha = 3/2$	$p_0 = 2.580565(-2)$	$q_0 = 1.000000(0)$
	$\varepsilon_1 = 2.40(-7)$	$p_1 = -1.656048(-1)$	$q_1 = 9.617584(-1)$
	$\varepsilon_2 = 1.3(-6)$	$p_2 = 4.426283(-1)$	$q_2 = 2.715949(-1)$
		$p_3 = -6.473472(-1)$	
		$p_4 = 5.494752(-1)$	
		$p_5 = -3.018936(-1)$	

(Continued)

Table VA. Approximations for support width m=8, $|\eta| \leq .775$ (Continued).

e)	α =2	p_0= 3.098251(-2)	q_0= 1.000000(0)
	ε_1=1.65(-7)	p_1=-1.854823(-1)	q_1= 9.278774(-1)
	ε_2=1.1(-6)	p_2= 4.637398(-1)	q_2= 2.530051(-1)
		p_3=-6.359482(-1)	
		p_4= 5.086794(-1)	
		p_5=-2.595588(-1)	

Table VB. Approximations for support width m=8, $.775 \leq |\eta| \leq 1$.

$$\psi_{\alpha o}(\pi m/2,\eta) \simeq \sum_{j=o}^{5} p_j(\eta^2-1)^j \;/\; \sum_{j=o}^{2} q_j(\eta^2-1)^j$$

a)	$\alpha = 0$	$p_0 =$ 4.290460(-5)	$q_0 =$ 1.000000(0)
	$\varepsilon_1 =$7.24(-7)	$p_1 =$-1.508077(-3)	$q_1 =$ 1.379457(0)
	$\varepsilon_2 =$1.1(-6)	$p_2 =$ 1.233763(-2)	$q_2 =$ 5.786953(-1)
		$p_3 =$-4.091270(-2)	
		$p_4 =$ 6.547454(-2)	
		$p_5 =$-5.664203(-2)	
b)	$\alpha = \frac{1}{2}$	$p_0 =$ 1.201008(-4)	$q_0 =$ 1.000000(0)
	$\varepsilon_1 =$3.46(-7)	$p_1 =$-2.778372(-3)	$q_1 =$ 1.300303(0)
	$\varepsilon_2 =$9.2(-7)	$p_2 =$ 1.797999(-2)	$q_2 =$ 5.135748(-1)
		$p_3 =$-5.055048(-2)	
		$p_4 =$ 7.125083(-2)	
		$p_5 =$-5.469912(-2)	
c)	$\alpha = 1$	$p_0 =$ 2.698511(-4)	$q_0 =$ 1.000000(0)
	$\varepsilon_1 =$1.76(-7)	$p_1 =$-4.628815(-3)	$q_1 =$ 1.230436(0)
	$\varepsilon_2 =$6.4(-7)	$p_2 =$ 2.470890(-2)	$q_2 =$ 4.593779(-1)
		$p_3 =$-6.017759(-2)	
		$p_4 =$ 7.566434(-2)	
		$p_5 =$-5.202678(-2)	
d)	$\alpha = 3/2$	$p_0 =$ 5.259595(-4)	$q_0 =$ 1.000000(0)
	$\varepsilon_1 =$9.38(-8)	$p_1 =$-7.144198(-3)	$q_1 =$ 1.168075(0)
	$\varepsilon_2 =$7.7(-7)	$p_2 =$ 3.238633(-2)	$q_2 =$ 4.135871(-1)
		$p_3 =$-6.946769(-2)	
		$p_4 =$ 7.873067(-2)	
		$p_5 =$-4.889490(-2)	

(Continued)

Table VB. Approximations for support width m=8, $.775 \leq |\eta| \leq 1$ (Continued).

e)

α =2	p_0= 9.255826(-4)	q_0= 1.000000(0)
ε_1=5.21(-8)	p_1=-1.038126(-2)	q_1= 1.111893(0)
ε_2=7.3(-7)	p_2= 4.083176(-2)	q_2= 3.744076(-1)
	p_3=-7.815954(-2)	
	p_4= 8.054087(-2)	
	p_5=-4.552077(-2)	