NATIONAL RADIO ASTRONOMY OBSERVATORY

VLA ELECTRONICS MEMO #107

A. R. Thompson

Tolerances on the Phase and Amplitude Responses of the VLA Receiving System

This report concerns recommendations on the tolerable deviations of the response of the RF and IF components of the receiving system from an ideal rectangular bandpass with identical phase characteristics for all channels. Two forms of amplitude deviation are considered, a slope and sinusoidal ripple. The effects of such deviations are considered in terms of the degradation of the output signal-to-noise ratio and the errors introduced into the fringe visibility values if the response is not constant.

1. The Amplitude Response

A. Degradation of Signal-to-Noise Ratio

Let $A(f)\ e^{i\phi_A(f)}$ and $B(f)\ e^{i\phi_B(f)}$ be the complex voltage transfer functions of the two receiving channels connected to a multiplier. The moduli of these functions, $A(f)$ and $B(f)$, will be referred to as the amplitude responses. The signal output at the peak of a fringe when the correlated components are in phase is proportional to the modulus of

$$\int_0^\infty A(f)\ B(f)\ e^{i\left[\phi_A(f)\ -\ \phi_B(f)\right]}\ df$$

For convenience we consider the amplitude and phase effects separately, and for the former will assume that the phase responses are identical in which case the exponential factor is unity.

The spectrum of the noise in a multiplier output is of the same form as that of a square law detector which is equal to the autocorrelation of the power spectral densities of the two channels (see, for example, Bracewell, 1962). Since we are concerned only with output noise near zero frequency the power per unit bandwidth is proportional to

$$\int_0^\infty A^2(f)\ B^2(f)\ df$$

The ratio of the peak signal to the rms noise at the multiplier output is therefore proportional to

$$\int_0^\infty A(f)\ B(f)\ df \Bigg/ \sqrt{\int_0^\infty A^2(f)\ B^2(f)\ df}$$

We can now define a degradation factor D which is the ratio of the peak signal to rms noise for an arbitrary response with limits f_1 and f_2 divided by the same ratio for a uniform response covering the same bandwidth;

$$D = \int_{f_1}^{f_2} A(f)\ B(f) \Bigg/ \sqrt{(f_2-f_1) \int_{f_1}^{f_2} A^2(f)\ B^2(f)\ df} \qquad (1)$$

This is equal to the ratio of the mean to the rms value of A(f) B(f) and has a maximum of unity when A(f) B(f) is a rectangular function.

B. Effects on the ALC System

It is important to consider also the ability of the ALC system to compensate for variations in the amplitude responses. Suppose that the ALC loops hold constant the rms amplitude levels at the inputs to the multiplier. The output signal voltage is then proportional to

$$\int_{f_1}^{f_2} A(f)\, B(f)\, df \Bigg/ \sqrt{\int_{f_1}^{f_2} A^2(f)\, df \times \int_{f_1}^{f_2} B^2(f)\, df}$$

This expression is unity if A(f) and B(f) are identical or differ only by a constant gain factor. If both responses are initially rectangular but A(f) then changes, the multiplier output changes by a factor F given by

$$F = \int_{f_1}^{f_2} A(f)\, df \Bigg/ \sqrt{(f_2-f_1) \int_{f_1}^{f_2} A^2(f)\, df} \qquad (2)$$

which happens to be the same as the expression for D when B(f) is rectangular. If the response varies more frequenctly than observations of calibration sources are made a corresponding error may be made in the measurement of visibility amplitude.

C. Gain Slope Across the Passband

Consider the case where the response measured in dB is a linear function of frequency,

$$A^2(f) = A_o^2\, e^{-C_1 f}$$

where C_1 is a constant related to α, the difference in the decibel response at the band edges, by

$$C_1 (f_2 - f_1) = \alpha \times 10 \text{ Log}_{10} e = 0.23 \alpha$$

Let D_1 be the degradation in sensitivity when only A(f) has the sloping response and B(f) is constant across the band, and let D_2 be the degradation when both channels have identical slopes. Substitution in equation (1) then gives

$$D_1 = F = 2 \sqrt{\frac{1 - e^{-0.115\alpha}}{0.23\,\alpha\,(1 + e^{-0.115\alpha})}} \quad (3)$$

and

$$D_2 = \sqrt{\frac{1 - e^{-0.23\alpha}}{0.115\alpha\,(1 + e^{-0.23\alpha})}} \quad (4)$$

Plots of D_1 and D_2 as functions of α are given in Figure 1

It may also be useful to consider the case where the decibel attenuation is proportional to $\sqrt{f}$ since this approximates the differential attenuation of a long cable. The results are more cumbersome than (3) and (4) but are recorded here for possible future reference:

$$A^2(f) = A_0^2\, e^{-C_2 \sqrt{f}}$$

$$D_1 = \frac{4\sqrt{2}\,[1 + \frac{C_2}{2}\sqrt{f_1} - (1 + \frac{C_2}{2}\sqrt{f_2})\, e^{-\frac{C_2}{2}(\sqrt{f_2} - \sqrt{f_1})}]}{C_2 \sqrt{(f_2 - f_1)\,[1 + C_2\sqrt{f_1} - (1 + C_2\sqrt{f_2})\, e^{-C_2(\sqrt{f_2} - \sqrt{f_1})}]}}$$

and

$$D_2 = \frac{2\sqrt{2}\,[1 + C_2\sqrt{f_1} - (1 + C_2\sqrt{f_2})\, e^{-C_2(\sqrt{f_2} - \sqrt{f_1})}]}{C_2 \sqrt{(f_2 - f_1)\,[1 + 2C_2\sqrt{f_1} - (1 + 2C_2\sqrt{f_2})\, e^{-2C_2(\sqrt{f_2} - \sqrt{f_1})}]}}$$

D. Sinusoidal Amplitude Ripple

Another amplitude response likely to be encountered is a sinusoidal ripple which we represent by

$$A(f) = A_o\,(1 + \gamma \cos 2\pi\, nf) \qquad f_1<f<f_2$$

For simplicity we assume that there are an integral number of ripple cycles across the passband, i.e., $n/(f_2 - f_1)$ is an integer. With this last condition the integrals in (1) are independent of the number of ripple cycles. If the ripples occur on one channel only we obtain

$$D_1 = F = \frac{\int_o^1 (1 + \gamma \cos 2\pi f)\, df}{\sqrt{\int_o (1 + \gamma \cos 2\pi f)^2\, df}} = \frac{1}{\sqrt{1 + \gamma^2/2}} \approx 1 - \gamma^2/4, \qquad (5)$$

and if both channels have identical ripples

$$D_2 = \frac{\int_o^1 (1 + \gamma \cos 2\pi f)^2\, df}{\sqrt{\int_o^1 (1 + \gamma \cos 2\pi f)^4\, df}} = \frac{2 + \gamma^2}{\sqrt{4 + 12\gamma^2 + \frac{3}{2}\gamma^4}} \qquad (6)$$

Plots of D_1 and D_2 as functions of γ are given in Figure 2.

2. The Phase Response

The phase response of a receiving channel does not affect the rms noise level at the multiplier output or the action of an ALC loop, and it is necessary to consider only the effect on the fringe amplitude. The differential phase response is

$$\phi_A(f) - \phi_B(f) = kf + \phi_o + \phi_{AB}(f)$$

The linear first term on the right-hand side represents a time delay which can be eliminated by correct adjustment of the delay system. The second term represents a calibratable phase offset which, assuming rectangular amplitude responses, is equal to the mean differential phase across the band. Only the third term should affect the fringe amplitude which, for rectangular amplitude responses, is proportional to the modulus of

$$\int_{f_1}^{f_2} e^{i\phi_{AB}(f)}\, df$$

If ϕ_{AB} is small this becomes

$$\int_{f_1}^{f_2} [1 - \frac{1}{2}\phi_{AB}^2(f)]\, df + \quad i \int_{f_1}^{f_2} \phi_{AB}(f)\, df$$

The first integral is $(f_2 - f_1)\,(1 - \frac{1}{2}\langle\phi_{AB}^2\rangle)$ where $\langle\phi_{AB}^2\rangle$ is the mean square value of ϕ_{AB} and the second integral is zero since ϕ_{AB} has zero mean. The fractional loss in signal amplitude compared to the ideal case of $\phi_{AB} = 0$ is $\frac{1}{2}\langle\phi_{AB}^2\rangle$. In the case of the VLA where there are many receiving channels the tolerable phase response of any one of them must be specified in terms of the root-mean-square of the phase response after we have subtracted from it, first the mean response of all of the channels, second, any linear term that can be eliminated by delay adjustment, and third the mean phase offset. Call this result ϕ_{RMS}. If the remaining phase deviations for any two channels connected to a multiplier combine randomly the fractional decrease in sensitivity resulting from the phase variations is ϕ_{RMS}^2.

3. Reflection in a Transmission Line

A reflection in a transmission line introduces sinusoidal terms into the amplitude and phase responses, and its effect on the signal-to-noise ratio can be determined by two different approaches which it is instructive to compare. The reflection causes a small component of the signal to suffer a delay τ relative to the main component so the total signal amplitude as measured with a swept signal generator has the form

$$A(f) = A_0 \left(1 + \gamma \cos 2\pi f\tau\right)$$

and the number of ripples cycles across the passband is $(f_2 - f_1)\tau$. The mean squared value of the sinusoidal phase term is $\frac{1}{2}\gamma^2$. If the reflection occurs in one channel only the sensitivity is reduced by a factor $1 - \frac{1}{4}\gamma^2$ from the amplitude effect (equation 5) and $1 - \frac{1}{4}\gamma^2$ from the phase effect. The combined degradation factor is therefore $1 - \frac{1}{2}\gamma^2$.

As an alternative approach we can say that the reflected component will not contribute to the fringes in the multiplier output if it is sufficiently delayed to become substantially decorrelated. For noise with a rectangular power spectrum the autocorrelation function is $\sin \pi\tau (f_2 - f_1)/ \pi\tau (f_2 - f_1)$ which has its first minimum when τ is the reciprocal of the bandwidth. Thus when τ is large enough to produce more than one complete ripple cycle across the band the correlation of the reflected component cannot exceed 0.13, and it is zero if there are an integral number of cycles across the band. The unreflected main component does not have the ripples in its spectrum, but the reflection causes a loss in sensitivity because the power in the main component is reduced by a factor $1 - \gamma^2$ and the fringe amplitude by a factor $\sqrt{1 - \gamma^2} \approx 1 - \frac{1}{2}\gamma^2$ whilst the noise level in the output is unaffected. This result is the same as obtained in the previous paragraph.

A consideration of the power lost in the reflected components in the case where both channels contain a reflection shows that the signal resulting from the main component is decreased by a factor $1 - \gamma^2$. In addition, if the reflected components suffer equal delays they will produce an output signal proportional to γ^2, which could in principle add in phase with the main-component fringes. Unless the delays caused by the two reflections happen to be very close, however, it is more likely that the contribution of the delayed components will combine with the main fringes in some random phase, or be lost altogether if the delay difference is of the order of $(f_2 - f_1)^{-1}$.

Note that sinusoidal amplitude ripples resulting from effects other than reflections, for example from stagger-tuned circuits, also give rise to signal components with differing time delays (see, for example, Goldman, 1954).

4. Proposed Tolerances

As basic figures let us consider the maximum tolerable loss in signal-to-noise ratio to be 5% and the maximum tolerable error in the measured visibility resulting from variation in the response to be about 2%. If the amplitude and phase responses contribute equally to the loss in sensitivity the maximum tolerable degradation is 2-1/2% in each case.

For the linear decibel response where the same slope occurs in both channels a 2-1/2% loss corresponds to a difference of 3.5 dB at the band edges. For a sinusoidal ripple in both channels the corresponding loss occurs for $\gamma = 0.165$ which is a peak-to-peak ripple of 2.9 dB. In both these cases the factor F is less than 1% at the same limits. A total variation not exceeding 3.0 dB across the band is therefore a satisfactory amplitude specification.

A 2-1/2% loss resulting from phase variations corresponds to a mean square contribution from each channel of 0.025 or an rms contribution of 9°. A variation of the phase response from zero to 9° rms could result in a 2-1/2% error in the measurement of fringe visibility which is about the tolerable limit.

The above results suggest specifications of 3.0 dB and 9° on the total response in a receiving channel from the antenna to the multiplier input. The components most likely to introduce unwanted characteristics are those in the waveguide transmission subsystem and the analog delay subsystem (if it is used). If the combined effects of two such subsystems are to remain within the overall specification we must specify for either one of them a maximum amplitude variation of 1.5 dB and a maximum contribution to the rms phase deviation of 6.4°. The initial specification for the delay lines (July 31, 1972) includes both a fixed and a variable phase error. It is suggested that an overall specification of 6.4° rms for the combination of these two effects be adopted.

References

Bracewell, R. N., Handbuch der Physik, Vol. 54, p.49, (Springer Verlag 1962).

Goldman, S., Frequency Analysis, Modulation and Noise, p.102, (McGraw-Hill 1948).

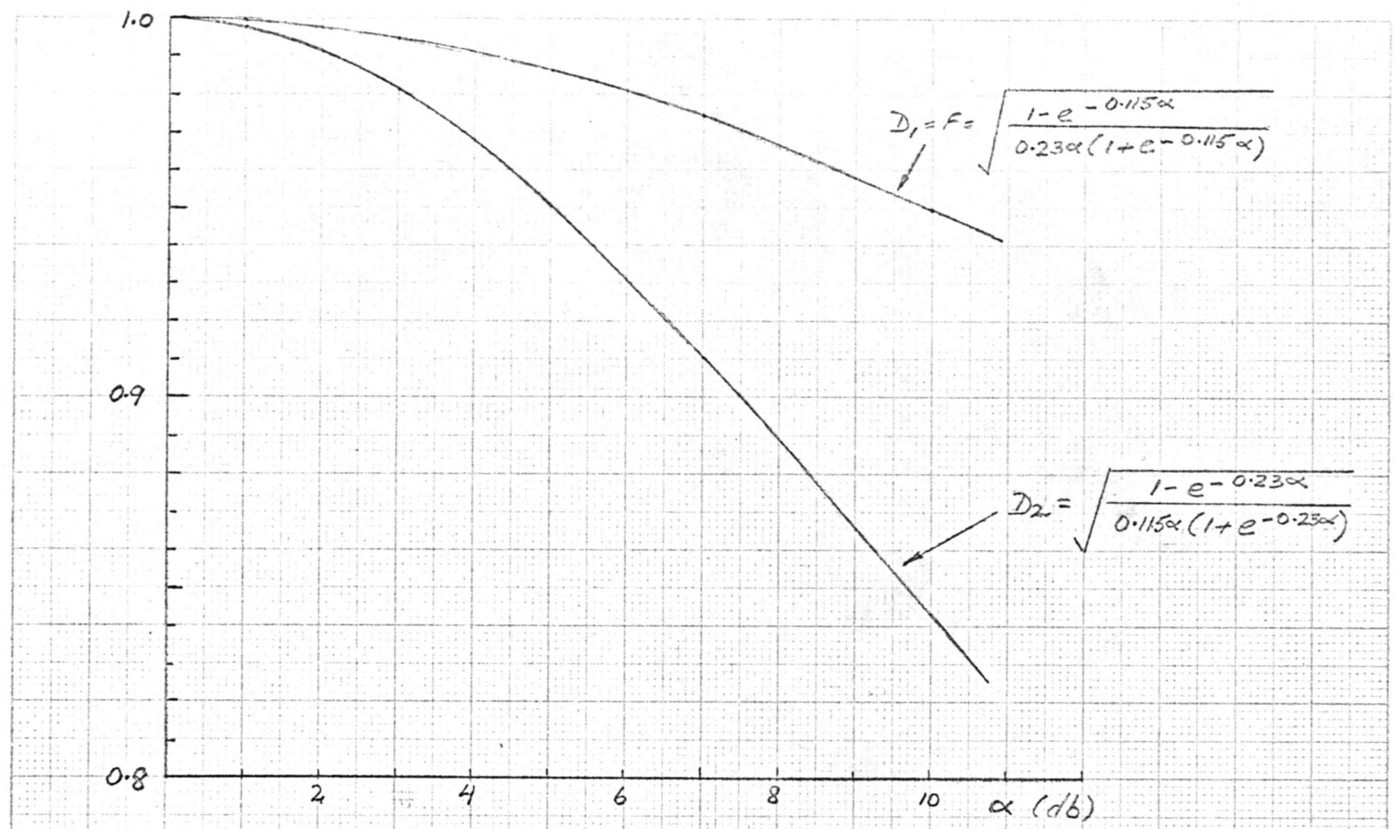

Fig 1 Factors D_1, D_2 and F for decibel response proportional to frequency.

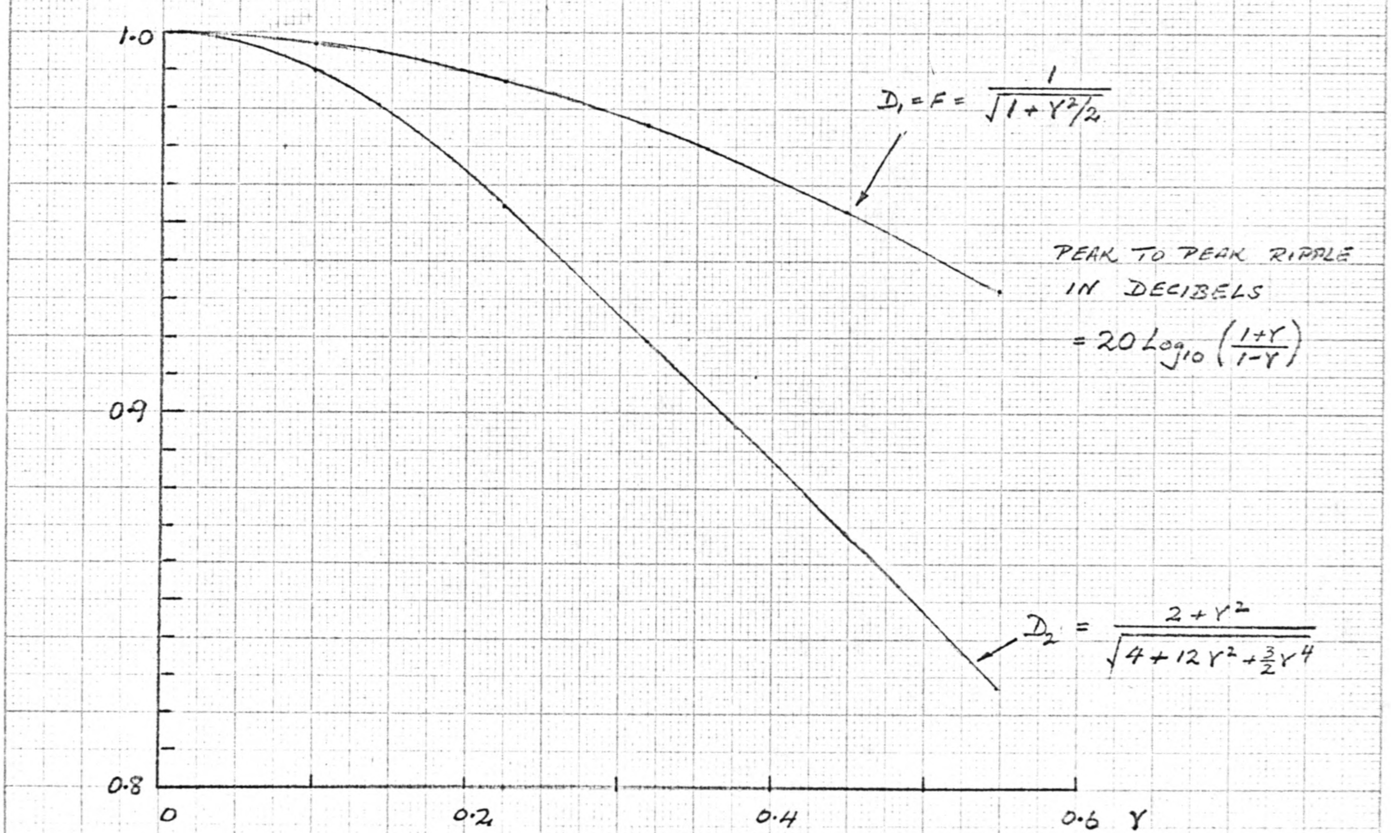

Fig 2 Factors D_1, D_2 and F for response with sinusoidal ripple.

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