VLA ELECTRONICS MEMO #109

DELAY ERRORS IN THE VLA

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This study investigates the tolerances on the setting accuracy of the delay system, considering both analog or digital types, and includes errors resulting from the discrete nature of the delay adjustment and the timing errors of sampling pulses. Maximum values for the smallest delay increment are derived based on a tolerable loss in the mean signal amplitude of 1% and on the peak phase errors.

1. The Effect of a Delay Error on the Interferometer Response

Consider the system shown in Fig. 1. Let $A_1(f)\, e^{i\phi(f)}$ and $A_2(f)\, e^{i\phi_2(f)}$ be the complex voltage transfer functions of the two channels excluding the delay unit. Here f represents a frequency in the IF band. Let f_1 and f_2 be the limits of the IF response and f_{LO} be the local oscillator frequency. For upper sideband reception the multiplier output resulting from correlated signals at the antennas is represented by

$$R = \int_{f_1}^{f_2} A_1(f)\, e^{i2\pi[ft+(f_{LO}+f)\tau_1+\phi_1(f)]}\, A_2\, e^{-i2\pi[ft+f\tau_2+\phi_2(f)]}\, df \qquad (1)$$

If the channel responses are identical (except for the delay) and $A_1=A_2=A$,

$$R = A^2 \int_{f_1}^{f_2} e^{i2\pi[f(\tau_1-\tau_2)+f_{LO}\,\tau_1]}\, df = A^2\, \frac{\sin \pi B \Delta}{B\Delta}\, e^{i2\pi(\Delta f_c+f_{LO}\,\tau_1)} \qquad (2)$$

where $B=f_2-f_1$ is the IF bandwidth, $f_c = \frac{1}{2}(f_2+f_1)$ is the IF center frequency, and

$\Delta = \tau_1 - \tau_2$ is the delay error which is zero if the delay τ_2 is tracking accurately. The factor $e^{2\pi i\, f_{LO}\, \tau_1}$ represents the output fringe pattern.

For the lower sideband case we have

$$R = \int_{f_1}^{f_2} e^{i2\pi[ft - (f_{LO} - f)\tau_1 + \phi_1(f)]}\, e^{-i2\pi[ft + f\tau_2 + \phi_2(f)]}\, df$$

Note that the phase term involving τ_1 has the opposite sign to that in (1); this is because a positive change in τ_1 produces a negative change in the lower sideband phase. Thus

$$R = A^2 \frac{\sin \pi B \Delta}{B \Delta} e^{i2\pi\, (\Delta f_c - f_{LO}\, \tau_1)} \tag{3}$$

To obtain the output waveform from the correlator we take the real part of (2) or (3), and normalizing the expressions for unit fringe amplitude at $\Delta=0$ we have

$$\begin{aligned} R_N &= \text{sinc}\,(B\Delta) \cos 2\pi(\Delta f_c \pm f_{LO}\, \tau_1) \\ &= \text{sinc}\,(B\Delta)\, [\cos\,(2\pi\Delta f_c) \cos\,(2\pi f_{LO}\tau_1) \mp \sin\,(2\pi\Delta f_c) \sin\,(2\pi f_{LO}\tau_1)] \end{aligned} \tag{4}$$

where the upper sign refers to the upper sideband case and we use the notation sinc $x = \sin(\pi x)/\pi x$. For double sideband reception the sum of the two single sideband responses is required which gives

$$R_N = \text{sinc}\,(B\Delta) \cos\,(2\pi\Delta f_c) \cos\,(2\pi f_{LO}\tau_1) \tag{5}$$

In both (4) and (5) the delay error introduces a decrease in fringe amplitude which is a sinc function of $B\Delta$. In addition, in the single sideband case it causes a phase shift $2\pi f_c \Delta$, and in the double sideband case no phase shift but a second amplitude-decrease factor.

2. The Probability Distribution of the Delay Error

During an observation the computer samples the output waveform and fits the samples to a sinusoid of frequency $2\pi f_{LO} \frac{d\tau_1}{dt}$ and determines its amplitude and phase. (The correlator output of course contains phase terms not shown in (4) and (5) which result from the visibility function.) In the fitting process the correlator outputs are averaged over a period expected to be about 10s. To determine the effect of the delay error we require to know its rms value, Δ_{rms}.

Case 1. Consider first the case where the delay unit is able to set exactly when the value called for is an integral multiple of the smallest delay increment, τ_0. As a point in the sky is followed the delay is reset whenever the required delay = $(n + \frac{1}{2})\ \tau_0$ where n is an integer. At the reset time the difference between the actual and required delay values changes sign, and as time progresses it decreases to zero and then increases again as illustrated in Fig. 2a. The probability distribution of this difference function is uniform from $-\tau_0/2$ to $\tau_0/2$. In the VLA the two receiving channels connected to any correlator both contain delay units so the differential delay error, which is what Δ actually represents, is the difference of two sawtooth functions. These have the same probability distribution but periods which for most pairs can be considered to be effectively unrelated, since they depend upon the positions of the two antennas relative to a reference point which will probably be the antenna closest to the source. For a further discussion of the way in which the delay errors can combine, see Section 6; for the present purposes we shall assume that the combining is sufficiently random that the probability distribution of the difference waveform is the convolution of the individual probability distributions which is shown in Fig. 2b and given by

$$P_1(\Delta) = \tau_0^{-2}\ (\tau_0 - \Delta) \quad |\Delta| < \tau_0$$
$$= 0 \quad |\Delta| > \tau_0 \ .$$

In terms of the symbol Λ,

$$P_1(\Delta) = \tau_0^{-1}\ \Lambda(\frac{\Delta}{\tau_0}).$$

The value of Δ_{rms} corresponding to P_1 is 0.408 τ_0.

If the delay units are adjusted so that the rate of change of delay is zero for an antenna at the end of one of the arms the maximum rate of change that can occur is 8.8 ns per second. This means that for some antennas the delay error function will go through many cycles during a 10s averaging period but for others it will not. If we use the value of Δ_{rms} corresponding to P_1 to determine the effect on the signal amplitude it turns out that the result will represent the mean effect over as many 10s periods as are required to establish the delay error variation represented by P_1. This is satisfactory for a consideration of the effect of delay errors on the sensitivity of the array which is the approach used here to establish an upper limit on τ_0.

Case 2. As a second case consider what happens with an ambient-temperature analog system in which the thermal delay changes in the longer acoustic bits is many times τ_0. Some form of delay measuring system is required to set the delay and we assume that this functions well enough to select the combination of bits nearest the value called for. Unlike case 1, one cannot predict what values can be achieved exactly. The sawtooth component of the delay error is not symmetrical about zero as in Fig. 2a but is superimposed upon an additional uncertainty* for which the probability distribution is also uniform from $-\tau_0/2$ to $\tau_0/2$. For a single IF channel the probability distribution for the total delay error is the same a P_1 above, and the differential delay error for two channels has a probability

$$P_2(\Delta) = P_1(\Delta) * P_1(\Delta)$$

where * indicates convolution. The function P_2 is shown in Fig. 3, and an expression for it is derived in Appendix 1 where it is shown that the corresponding value of Δ_{rms} is 0.577 τ_0. The component of the delay error resulting from the long bits not being integral multiples of τ_0 will vary relatively slowly as the long bits are changed, and the remarks given above on the effect of the time taken to establish the range of variation represented by the probability distribution apply here also.

*This is the case when the computer calls for delay settings at time intervals which correspond to changes of τ_0 in the required delay. If the computer calls for the required delay at much more frequent intervals, and the delay sets to within $\tau_0/2$, the delay error distributions are similar to those in Case 1.

3. The Response With an IF Center Frequency Larger Than the Bandwidth

An analog delay system is expected to have an IF bandwith B=100 MHz and center frequency f_c=500 MHz. We assume initially that this band of frequencies is applied to the correlator inputs without further frequency conversion.

To determine the effect of the delay error on the signal amplitude obtained by the computer we use the expression for the instantaneous amplitudes of the cosine and sine components of the fringe pattern given in equation (4). For probability distribution $P(\Delta)$ this gives

$$\langle R_N \rangle = \int_{-\infty}^{\infty} P(\Delta)\ \mathrm{sinc}\ (B\Delta)\ \cos\ (2\pi f_c \Delta)\ d\Delta \qquad (6)$$

The sine component of the fringe pattern in equation (4) makes no contribution if $P(\Delta)$ is an even function. The functions P_1 and P_2 are even but the distribution of Δ over a single 10s period may contain an odd component. The resulting sine components would average to zero over a longer time period, and we shall omit them here. For a further discussion of this effect see Section 7. We are concerned here only with small errors and can therefore substitute in (6) the series expansions for sine and cosine using the first two terms of each:

$$\langle R_N \rangle \simeq \int_{-\infty}^{\infty} P(\Delta)\ [1 - \frac{1}{3!}\ (\pi B\Delta)^2]\ [1 - \frac{1}{2!}\ (2\pi f_c \Delta)^2]\ d\Delta$$

$$\simeq 1 - (\frac{1}{6}\pi^2 B^2 + 2\pi^2 f_c^{\ 2}) \int_{-\infty}^{\infty} P(\Delta)\ \Delta^2\ d\Delta$$

$$= 1 - (\frac{B^2}{6} + 2f_c^{\ 2})\ \pi^2\ \Delta^2_{rms} \qquad (7)$$

Figure 4 curve (a) gives a plot of $\langle R_N \rangle$ against $f_c \Delta_{rms}$ for f_c=5B. A 1% loss in sensitivity occurs with $f_c \Delta_{rms}$=0.0225, or Δ_{rms}=45 ps. with f_c=500 MHz. For case 1 the corresponding value of τ_0 is 110 ps and for case 2, 78 ps.

4. IF Center Frequency = B/2

For B $\sim 5f_c$ the drop in fringe amplitude given by equation (7) depends almost entirely on f_c. This dependence could be greatly decreased if the computer program could take account of the delay error Δ in fitting a sinusoid to the correlator output. In case 1 the delay error is calculable since the computer calls for values that the delay can set to exactly, but in case 2 an unknown error is involved. The increased complexity of the computer program, however, would make this procedure undesirable in either case.

A more practical method of reducing the dependence on f_c is to reduce f_c. The high center frequency of acoustic delay lines is fixed by percentage bandwidth considerations, but the IF can be decreased by heterdyning to a lower frequency after passing through the delay line. The effect on the sensitivity will be the same as decreasing f_c in the above equations if the oscillator used also passes through the delay lines and suffers corresponding phase shifts to the IF. If the oscillator frequency is at the edge of the IF passband, f_c is effectively reduced to B/2. It would also be possible to put the oscillator at the IF center frequency which would further reduce the dependence on Δ but would require using separate correlators for the portions of the IF above and below the oscillator frequency.

In the case where f_c=B/2 the IF frequency range extends from zero to B and the amplitude of the cosine component of the fringes in equation (4) becomes sinc $2B\Delta$. The computer averaged fringe amplitude is thus

$$\langle R_N \rangle = 1 - 2\,(\pi B \Delta_{rms})^2/3 \qquad (8)$$

Figure 4 curve (b) gives a plot of $\langle R_N \rangle$ as a function of $B\Delta_{rms}$. A 1% loss in fringe amplitude occurs at $B\Delta_{rms}$=0.039, or for B=100 MHz, Δ_{rms}=0.39 ns. For case 1 of Section 2 this corresponds to τ_0=0.96 ns and for case 2, 0.68 ns.

5. The Digital Delay System

With the digital delay system the IF passband would extend from near zero frequency to B as discussed in the last section. The bandwidth is limited by the requirement that the sampling rate must be twice the highest intermediate frequency. The proposed sampling rate of 100 MHz thus means B = 50 MHz and for a fringe amplitude loss of 1%, Δ_{rms} = 0.78 ns.

The timing of the sampling pulses will be controlled by a master clock system and the phase of the pulses at any sampling circuit can be varied in steps which correspond to the minimum increment τ_0. Errors in the occurrence time of the pulses are expected to consist of a systematic or slowly varying component and a more rapidly varying jitter. The probability distributions of these errors are not known. The contribution to the delay error resulting from the discrete nature of the delay adjustment, as discussed under case 1 of section 2, is 0.41 τ_0. If τ_0 is 1/8*of the sample pulse interval, i.e., 1.25 ns, 0.41 τ_0 = 0.51 ns. A further rms contribution $\sqrt{0.78^2 - 0.51^2}$ = 0.59 ns is allowed from the timing errors before the 1% sensitivity loss is reached. The 0.59 ns rms error is the sum of timing errors in the two channels connected to a multiplier. The allowed rms error in the pulse timing in each sampling unit is therefore 0.42 ns or 0.33 τ_0. This is probably an achievable error level, but a greater margin would be available by making τ_0 equal to 1/16 of the sampling interval or 0.625 ns.

6. Non-Random Combinations of Delay Errors

The assumption was made in Section 2 that the delay-error functions have unrelated frequencies and therefore combine randomly. If a pair of error functions could occur with the same frequency and opposite phase the differential delay error would be a double-amplitude version of the individual error functions. Compared with the randomly combining case the rms delay error would be twice as great and the loss in signal amplitude four times as great. This is an extreme condition which can only be partially approached in practice since the two antennas would have to be on the same arm as the delay reference point and at the same distance from it. A two-to-one harmonic ratio is possible

*Power-of-two fractions, 1/8 or 1/16, give the smallest delay steps for a given number of control bits.

however, as illustrated in Figure 5, and here the loss in signal amplitude would be three times greater than in the randomly combining case. The antiphase condition of the two error functions which results in maximum signal loss would require that the two antennas not be on a straight line with the delay reference point, a condition which could result from ground level irregularities.

If however one takes into account the fact that the antenna positions along the arms are located at irregular multiples of a minimum distance, and that the effects discussed are just as likely to lead to in-phase conditions in which the rms delay errors are less than in the random case, the enhancement of errors from non-random combinations does not appear to be of great importance.

7. Slowly Varying Delays

Thus far the discussion has been based on the effects to be expected when the output of any correlator is averaged over a time interval long enough that the delay errors encountered conform to the distributions P_1 or P_2. The number of 10s intervals over which the data are averaged actually depends upon the rate of change of the projected baseline vector in the (u,v) plane and the dimensions of the synthesized field, and may be as small as one. Since the delay errors over a single 10s period may have a distribution different from that over a longer period, amplitude decreases greater than the 1% average decrease can occur. In addition the delay errors during a 10s period may have a mean value Δ_m which is not close to zero. The sine component omitted in Section 3, which has an amplitude $2\pi f_c \Delta_m$ is then not negligible and results in a phase error $2\pi f_c \Delta_m$ in the measured visibility. If these effects occur in a part of the (u,v) phase where very few 10s data points are averaged they can introduce spurious detail into the measured visibility function.

The most serious deviations from the longer-term error distributions are likely to occur when the rate of change of delay goes to zero as the hour angle of the source becomes equal to that of the direction through the antenna and the delay reference point. For example, for a source at declination 0° and an antenna at D km from the delay reference pont the time taken for the required delay to change by τ_0 ns from the time that the rate of change passes through zero is $\sim 350\sqrt{\tau_0/D}$ seconds. We can therefore expect a number of 10s intervals to occur in which, for two antennas on the same arm as the reference point, the delay error remains essentially constant. The probability distribution of these stationary delay errors is given by P_1 or P_2. After passing through zero the rate of change of delay in the above example will have increased to τ_0 in 10s when the hour angle difference becomes $\sin^{-1}(\frac{0.45\ \tau_0}{D})$, or 1.8 h for τ_0 = 1 ns and D = 1 km.

Consider the effect of the stationary delay errors in the case of the digital delay system. If we omit for the moment the pulse timing errors the delay errors have an rms value of $\tau_0/\sqrt{6}$ and a peak value of τ_0, as in Fig. 2b. For τ_0=1.25ns the corresponding loss in signal amplitude is 0.4% rms and 2.5% peak, and the phase errors are 4.6° rms and 11.2° peak. Estimation of the effects of these errors in a synthesis map is a fairly complex problem and involves consideration of the location of the errors on the (u,v) plane. However, the VLA will also be used in ways which call for observation of a source for only a few minutes, and the possibility of encountering peak errors at certain hour angles is then more serious. The phase errors are the most disturbing, and although not intolerably large they are greater than the phase errors expected from the oscillator system (1° per GHz) in the two lower frequency bands. It therefore appears advisable to consider making τ_0 1/16 of the sampling interval, i.e. 0.625 ns. The rms and peak phase errors then become 2.3° and 5.6° with no contribution from the pulse timing errors; one would hope that the mean of the pulse timing errors over a 10s would be only a small fraction of τ_0. The drop in signal amplitude is very small, and with an rms timing error of 0.3 ns at each sampler amounts to 0.4% rms and 1% peak. Decreasing τ_0 also proportionately increases the rate at which the delay is adjusted and this decreases the hour angle range over which no adjustments occur.

8. Conclusions

The discussions in Sections 3 through 5 consider the effects of the delay errors in terms of their overall probability distributions, and the results are adequate in terms of the loss in overall sensitivity of the array but are less satisfactory in terms of the peak errors. Table 1 summarizes the relationship between τ_0 and the reciprocal bandwidth when the delay errors result solely from the discrete nature of the delay adjustment, i.e. no timing errors, and $f_c=B/2$. A good rule of thumb is $\tau_0=\frac{1}{20}B^{-1}$. For the digital delay system $\frac{1}{8}$ of the sampling interval is $\frac{1}{16}B^{-1}$ which is marginally acceptable. Since some timing errors must be expected, $\frac{1}{16}$ of the sampling interval appears to be a better choice. An rms error in the pulse timing of 0.3 ns is then easily tolerable if the mean over 10s is small, as in the case of a fast jitter, but a slowly varying component causing a mean delay error over 10s of $\sim$ 3 ns would be more serious.

For an analog system the peak delay error for Case 2 is 2 τ_0 and if we allow a corresponding peak phase error of 10°, and the IF bandpass is 0 to 100 MHz τ_0=0.28 ns. This is a little less than half the value obtained from the consideration in Section 4. Note that if the delay measuring system is capable of selecting the nearest combination of bits to any delay called for, delay errors are fully accounted for in the specification of τ_0 and no further specification on delay errors should be required.

The importance of keeping the peak delay errors within tolerable limits is emphasized by considerations of short period observations with the VLA. From this point of view it is clearly undesirable to have the rate of change of delay going through zero at the same time for all of the antennas on one arm. This occurs if the delay reference point is on an arm as has thus far been envisaged. It appears advisable to consider choosing a delay reference point so that the zero rate-of-change times are more uniformly distributed in hour angle.

Appendix 1

The normalized probability distribution $P_2(\Delta)$ is the self-convolution of a function of the form

$$P_1(\Delta) = \frac{1}{\tau_0}\left(1 - \frac{\Delta}{\tau_0}\right),\ |\Delta| < \tau_0 \quad \text{and } P_1(\Delta) = 0,\ |\Delta| > \tau_0$$

It is convenient to work in terms of $\Delta' = \Delta/\tau_0$, and the required convolution is then

$$P_2(\Delta') = \int_{-\infty}^{\infty} P_1(x)\ P_1(x - \Delta')\ dx$$

From Figure A1 one sees that

$$P_2\ (\Delta') = \int_{-1+\Delta'}^{0} (1+x)\ (1+x-\Delta')\ dx + \int_{0}^{\Delta'} (1+x-\Delta')\ (1-x)dx + \int_{\Delta'}^{1} (1-x)\ (1-x+\Delta')\ dx$$

$$= 2/3 - \Delta'^2 + \Delta'^3/3 \qquad \text{for } |\Delta'| < 1,$$

$$P_2\ (\Delta') = \int_{0}^{\Delta'} (1 + x - \Delta')\ (1 - x)\ dx = 4/3 - 2\Delta' + \Delta'^2 - \Delta'^3/6$$

for $1 < |\Delta'| < 2$,

and $P_2\ (\Delta') = 0$ for $|\Delta'| > 2$

A graph of $P_2\ (\Delta')$ is shown in Fig. 3. It resembles a Gaussian - an illustration of the central limit theorem. The rms value of $P_1\ (\Delta')$ is

$$\left[2 \int_{0}^{\infty} \Delta'^2\ P_2\ (\Delta')\ d\Delta'\right]^{\frac{1}{2}} = \frac{1}{\sqrt{3}} = 0.577$$

This rms value can also be determined without calculating the form of P_2 by noting that P_2 is the convolution of four rectangular functions of width τ_0, and that the mean squared values of the abscissas are additive under convolution (Bracewell, The Fourier Transform and Its Applications, p.173).

Criterion	τ_o	
	Case 1	Case 2
Mean sensitivity loss 1%	$0.0955 B^{-1} \simeq \frac{1}{10} B^{-1}$	$0.0675 B^{-1} \simeq \frac{1}{15} B^{-1}$
Peak phase errors 10°	$0.0555 B^{-1} = \frac{1}{18} B^{-1}$	$0.0278 B^{-1} = \frac{1}{36} B^{-1}$
Peak phase errors 5°	$0.0278 B^{-1} = \frac{1}{36} B^{-1}$	$0.0139 B^{-1} = \frac{1}{72} B^{-1}$

Table 1 Relation between τ_o and reciprocal bandwidth when IF response extends from zero to B and only errors resulting from discrete adjustment are included.

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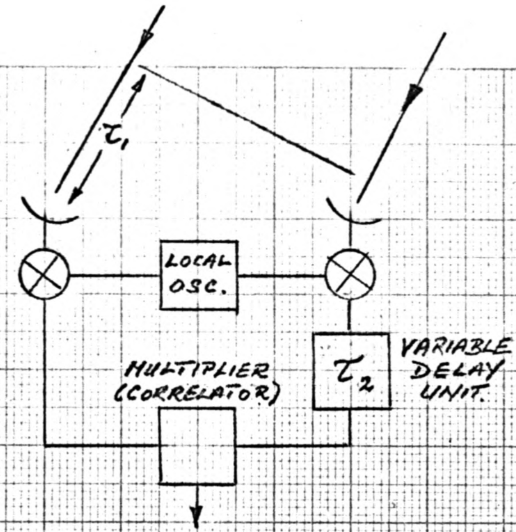

Fig 1 Delays τ_1 & τ_2 in two-antenna system.

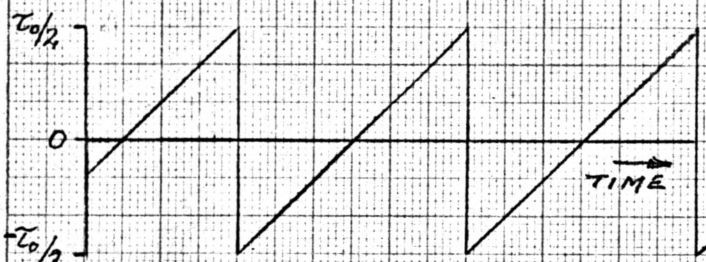

Fig. 2(a) Delay error in one channel for case 1.

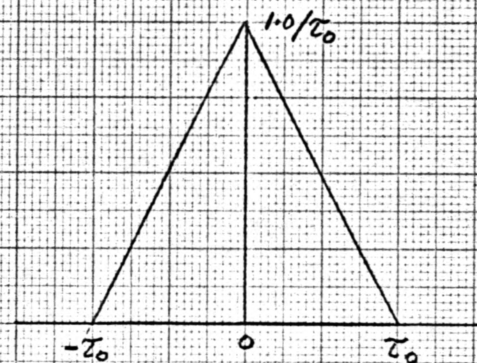

Fig 2(b) Probability distribution $P_1(\Delta) = \tau_0^{-1}\Lambda\left(\frac{\Delta}{\tau_0}\right)$ for difference of two delay errors of form shown in 2(a).

$\Delta_{rms} = \tau_0/\sqrt{6}$

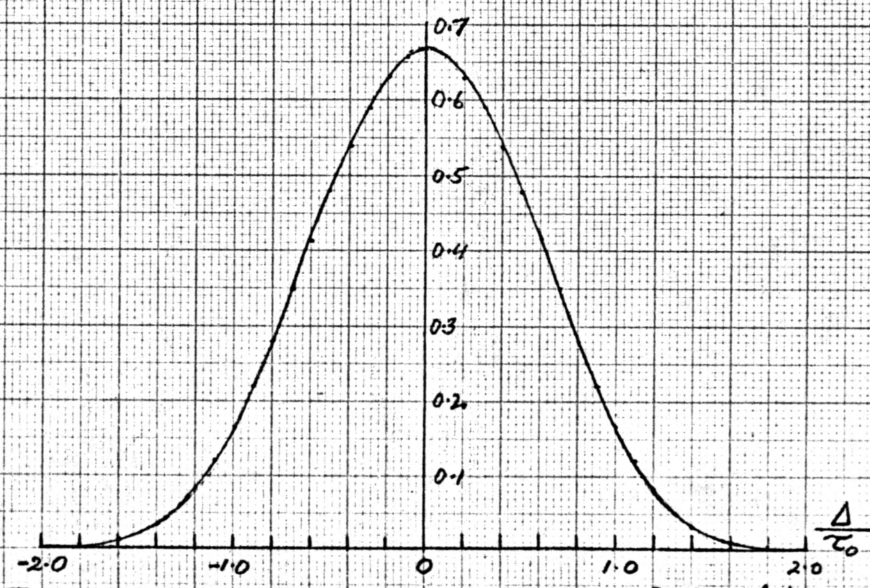

Fig 3. Probability distribution $P_2(\Delta) = \tau_0^{-2}\Lambda\left(\frac{\Delta}{\tau_0}\right) * \Lambda\left(\frac{\Delta}{\tau_0}\right)$.

$\Delta_{rms} = \tau_0/\sqrt{3}$

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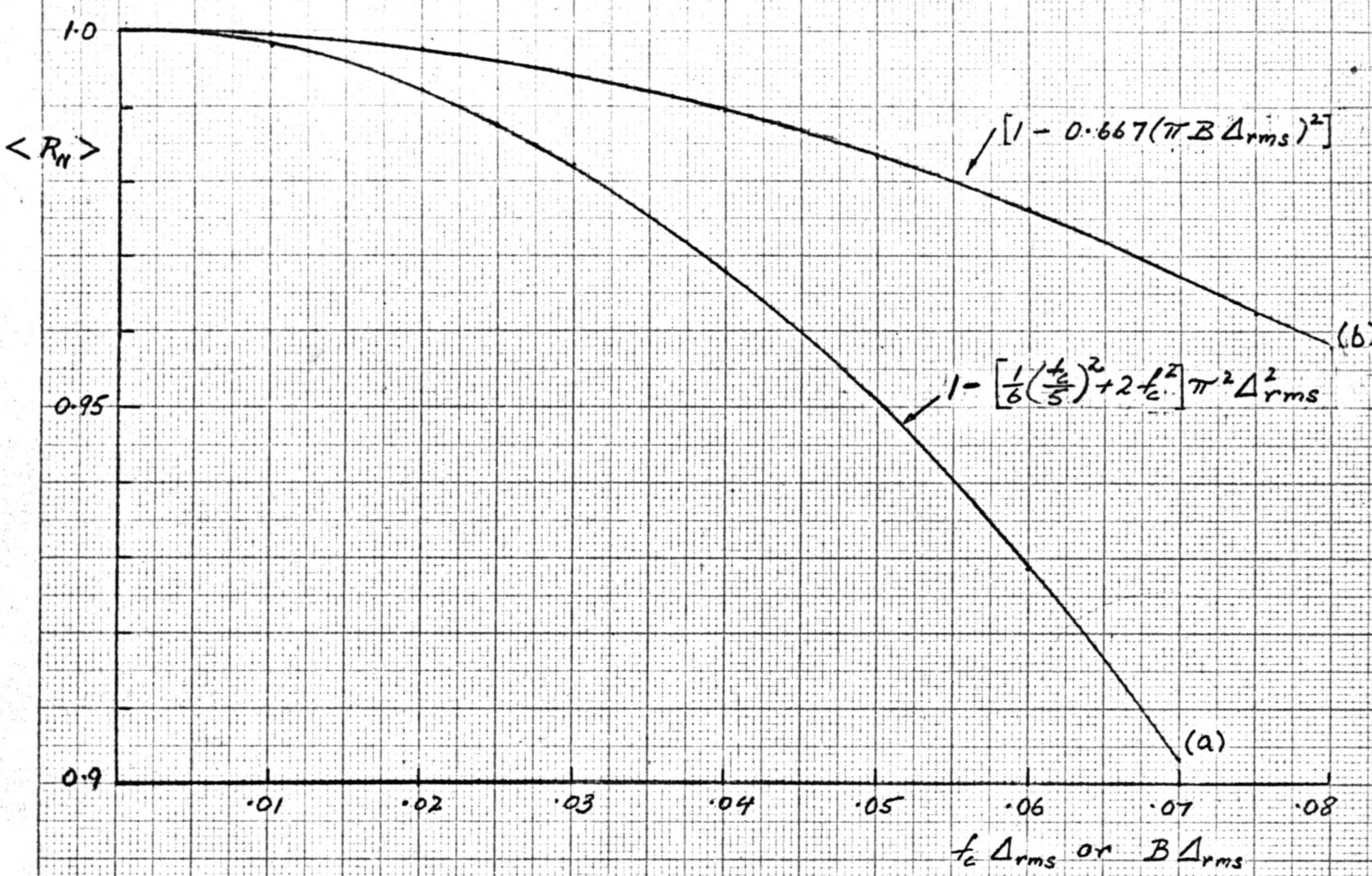
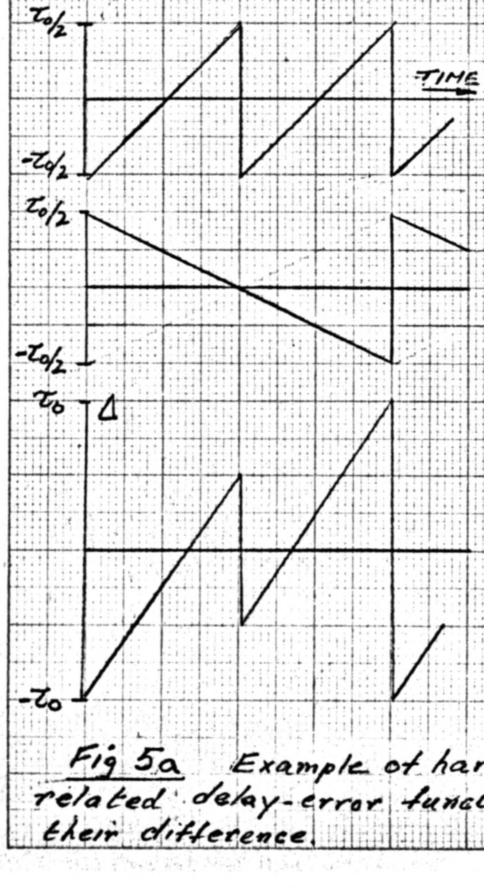


Fig 4. Curves of normalised fringe amplitude from equations 7+8.



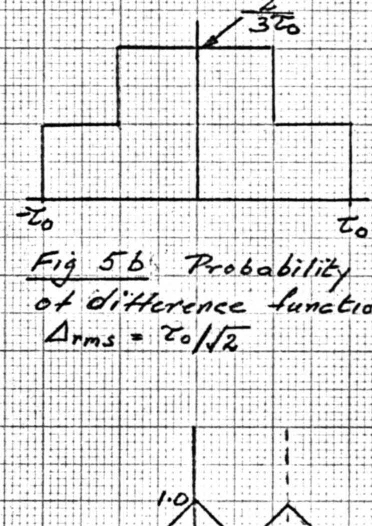


Fig 5b Probability distribution of difference function in Fig 5a. $\Delta_{rms} = \tau_0/\sqrt{2}$

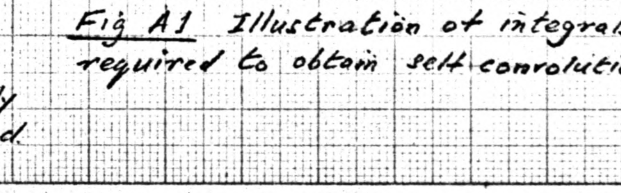


Fig A1 Illustration of integrals required to obtain self convolution of P_1.

Fig 5a Example of harmonically related delay-error functions and their difference.

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