VLA ELECTRONICS MEMORANDUM 112

April 9, 1973

TOLERANCES IN THE TWO-BIT SAMPLER

A. R. Thompson

It is proposed to use a digital delay system in the VLA with two bit quantization at the samplers and omitting the low and intermediate level products at the multipliers. Two bit quantization has been analyzed by Cooper (1970) who showed that for the case that we are considering the signal to noise ratio is 0.81 of that with a continuous level correlator (for one bit quantization the corresponding figure is 0.64). This report extends Cooper's analysis to include deviations of the threshold levels from the ideal values and considers the effects on the measured correlation and on the signal to noise ratio. Specifications for the samplers are thus obtained. The notation used by Cooper is followed and all voltages are expressed in units of the rms level of the input signals. Only the case where the low and intermediate products are omitted is considered.

1. Introduction

Consider two samplers which provide the inputs for a correlator. The output of the correlator after receiving N samples of each signal waveform is

$$N\ (P_{22} + P_{\bar{2}\bar{2}} - P_{2\bar{2}} - P_{2\bar{2}}) \tag{1}$$

where P indicates the joint probability of the waveforms being at any instant in the states indicated by the subscripts, 2 referring to high level states and the bar to negative levels. The variance of (1) is equal to the number of positive or negative counts accumulated in the correlator output,

$$N\ (P_{22} + P_{\bar{2}\bar{2}} + P_{2\bar{2}} + P_{\bar{2}2})$$

so the signal to noise ratio after N samples is

$$R = \frac{\sqrt{N}\,(P_{22} + P_{\bar{2}\bar{2}} - P_{2\bar{2}} - P_{\bar{2}2})}{\sqrt{P_{22} + P_{\bar{2}\bar{2}} + P_{\bar{2}2} + P_{2\bar{2}}}} \qquad (2)$$

In Cooper's paper the threshold levels are equal positive and negative voltages in which case $P_{22}=P_{\bar{2}\bar{2}}$ and $P_{2\bar{2}}=P_{\bar{2}2}$. In the present analysis the four threshold levels associated with any correlator are treated independently as illustrated in Fig. 1. The expressions for the joint probabilities are given in Appendix 1 and correspond to Cooper's equation (4). To simplify the discussion two particular situations are discussed; case 1 in which the levels differ from the ideal values by a zero offset and case 2 in which they differ by a scale factor. Any general case can be thought of as a combination of these two cases.

Before proceeding further consider the ideal case in which the four voltage differences in Figure 1 are set for the maximum sensitivity. If they are all equal to υ_0 the signal to noise ratio is proportional to (Cooper p.525)

$$e^{-\upsilon_0^2}/[1-\phi(\upsilon_0)]$$

where $\phi(\upsilon_0) = 2(2\pi)^{-1/2}\int_0^{\upsilon_0} e^{-\upsilon^2/2}\, d\upsilon = \mathrm{erf}\,(\upsilon_0/\sqrt{2})$

Maximum sensitivity occurs at

$$\upsilon_0\,[1-\phi(\upsilon_0)] = (2\pi)^{-1/2}\, e^{-\upsilon_0^2/2} \qquad (3)$$

For which a numerical solution υ_0 = 0.6120 was obtained using the polynomial expression for the error function given in Abramowitz & Stegun (p.299, 7.1.26). The value 0.6 given by Cooper is thus rounded off to one figure. Note that use of the symbol υ_0 in this report refers hereafter to the ideal value 0.612, and $\phi(\upsilon_0)$=0.459.

2. Zero-Level Error in the Sampling Levels (Case 1)

The voltage levels for the two samplers in case 1 are shown in Fig. 2. The mean values of the levels are Δ_1 and Δ_2 instead of zero, but the difference between the levels for each sampler is $2\upsilon_0$ as in the ideal case. The analyses in Appendix 2 leads to the following expressions for the output and the signal to noise ratio;

$$N(P_{22} + P_{\bar{2}\bar{2}} - P_{2\bar{2}} - P_{\bar{2}2}) = \frac{2N}{\pi} e^{-\upsilon_0^2}[\Delta_1 \Delta_2 + \rho [1 - \frac{1}{2}(1-\upsilon_0^2)(\Delta_1^2 + \Delta_2^2)]] \quad \text{(A21)}$$

$$R = \frac{2\sqrt{N}}{\pi} \frac{e^{-\upsilon_0^2}}{[1-\phi(\upsilon_0)]} [\Delta_1 \Delta_2 + \rho [1 - \frac{1}{2}(\Delta_1^2 + \Delta_2^2) - 2\rho\Delta_1 \Delta_2 \upsilon_0^4]] \quad \text{(A22)}$$

where ρ is the true correlation between the two signals.

An examination of (A21) indicates an unwanted term $\Delta_1\Delta_2$ which produces an offset similar in effect to the self rectified component often encountered in an analog multiplier. This offset is eliminated by the phase switching of the signals at the antennas because the output of the multiplier is then reversed in sign for half of the time. The minimum detectable value of $\rho \sim (B\tau)^{-1/2} = 5.3\text{x}10^{-7}$ where B = 100 MHz is the IF bandwidth and τ = 10 hours is the total observing time. This is less than $\Delta_1\Delta_2$ by a factor which may be as large as $\sim 10^5$. Nevertheless the unwanted term will be accurately cancelled assuming it does not contain significant variations of the phase-switching frequency, since the numbers of samples in each part of the switching cycle are exactly equal. Slow variations in Δ_1 & Δ_2 could cause errors in the measured correlation, and to insure that these do not exceed 1% we require

$\frac{1}{2}(1 - \upsilon_0^2)(\Delta_1^2 + \Delta_2^2) < 0.01$, or $\Delta_1 \approx \Delta_2 < 0.13$. In the expression for the signal to noise ratio (A22) the term $\Delta_1\Delta_2$ is again eliminated by the switching, the term $2\rho\Delta_1\Delta_2\upsilon_0^4$ can be omitted since ρ is usually small, and the main dependence results from the term $\frac{1}{2}(\Delta_1^2 + \Delta_2^2)$. Less than 1% loss in sensitivity requires $\Delta_1 \approx \Delta_2 < 0.1$. This is the more stringent condition and expressed as a fraction of υ_0 the tolerable error is 0.1/0.612 = 16%.

3. Scale-Factor Error in the Sampling Levels (Case 2)

In the second case to be considered the levels are effectively multiplied by a factor which differs from unity by a small amount. The means of the levels remain zero, and the levels become $\pm(\upsilon_0 + \delta_1)$ and $\pm(\upsilon_0 + \delta_2)$ as shown in Fig. 3. This is also equivalent to change in signal levels entering the samplers. The analysis in Appendix 3 results in the following expressions;

$$N(P_{22} + P_{\bar{2}\bar{2}} - P_{2\bar{2}} - P_{\bar{2}2}) = \frac{2N\rho}{\pi} e^{-\upsilon_0^2}[1-\upsilon_0(\delta_1+\delta_2) - \frac{1}{2}(1-\upsilon_0^2)(\delta_1^2+\delta_2^2) + \upsilon_0^2\delta_1\delta_2] \quad \text{(A31)}$$

and

$$R = \frac{2\sqrt{N}\rho}{\pi} \frac{e^{-\upsilon_0^2}}{1-\phi(\upsilon_0)} [1 - \frac{1}{2}(\delta_1^2 + \delta_2^2) + \upsilon_0^2\delta_1\delta_2] \quad \text{(A32)}$$

Note first that in (A31) which is the expression for the output there is a first order dependence on the level errors in the term $\upsilon_0(\delta_1+\delta_2)$. If we require that this produce less than 1% error in ρ, $\upsilon_0(\delta_1+\delta_2) < 0.01$ or $\delta_1 \approx \delta_2 < 0.008$ or <1.3% of υ_0. This is a much more stringent requirement than was found in Case 1. Note that there is no first order dependence on the level errors in the expression for the signal to noise ratio (A32).

4. Measurement of Self Correlation

To reduce the strong dependence on δ_1&δ_2 found in Case 2, S. Weinreb suggests that each IF channel should have a multiplier measuring the self-correlation* of the signal from the sampler, and that the cross correlation data be divided by an appropriate function of the corresponding self correlation values. This arrangement is outlined in Fig. 4. The output of the self correlator fed from the first sampler is $N(P_2+P_{\bar{2}})$ where P_2 & $P_{\bar{2}}$ are the probabilities of finding the input signal outside the positive and negative threshold levels respectively. Appendix 4 shows that the output of the cross correlator divided by the mean of the outputs of the two self correlators is, for Case 2,

$$\frac{P_{22} + P_{\bar{2}\bar{2}} - P_{2\bar{2}} - P_{2\bar{2}}}{\frac{1}{2}\left[(P_2 + P_{\bar{2}})_{\delta_1} + (P_2 + P_{\bar{2}})_{\delta_2}\right]} = \frac{\rho e^{-\upsilon_0^2}}{\pi\left[1 - \phi(\upsilon_0)\right]}\left[1 - \frac{1}{2}(\delta_1^2+\delta_2^2) + \upsilon_0^2\,\delta_1\delta_2\right] \qquad \text{(A42)}$$

The dependence upon δ_1 & δ_2 reduces to second order terms and to determine the maximum possible effect we put $\delta_1 = -\delta_2 = \delta$ so that the dependence becomes $[1 - (1 + \upsilon_0^2)\delta^2]$. A 1% reduction in output then occurs for δ=0.085 which is 14% of υ_0. The technique therefore appears to offer a significant advantage.

For case 1 the corresponding expression for the output is found in Appendix 4 to be

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*The term self-correlation has been used rather than autocorrelation since the latter is usually defined to be unity for zero time difference.

$$\frac{P_{22} + P_{\bar{2}\bar{2}} - P_{2\bar{2}} - P_{\bar{2}2}}{\frac{1}{2}\left[(P_2+P_{\bar{2}})_{\Delta_1} + (P_2+P_{\bar{2}})_{\Delta_2}\right]} = \frac{e^{-\upsilon_0^2}}{\pi\,[1-\phi(\upsilon_0)]} \; \frac{[\Delta_1\Delta_2 + \rho\,[1 - \frac{1}{2}(1-\upsilon_0^2)(\Delta_1^2+\Delta_2^2)]]}{[1 + \frac{1}{2}\upsilon_0^2\,(\Delta_1^2 + \Delta_2^2)]} \qquad \text{(A41)}$$

If we omit the term $\Delta_1\Delta_2$ which is eliminated by the phase switching the dependence on Δ_1 & Δ_2 reduces to a factor $[1 - \frac{1}{2}(\Delta_1^2 + \Delta_2^2)]$. To keep the effects of the level errors less than 1% requires $\frac{1}{2}(\Delta_1^2 + \Delta_2^2) < 0.01$ or $\Delta_1 \approx \Delta_2 < 0.1$. Thus for Case 1, division by the mean of the self correlators slightly increases the dependence on the level errors, but this is only a small effect compared with decreased dependence in Case 2. With the use of the self correlators the most stringent requirement on the level errors is the 14% of υ_0 resulting from equation (A42) above.

What is the effect of the use of the self correlation data on the signal to noise ratio of the cross correlation measurement? The signal to noise ratio in the self correlation data is $[N\,(P_2 + P_{\bar{2}})]^{1/2} = N^{1/2}\,[1 - \phi(\upsilon_0)]^{1/2}$, omitting the effects of level errors. The fractional rms error in the mean of the two self correlation measurements is thus $[2N(1 - \phi(\upsilon_0))]^{-1/2} = 0.96\,N^{-1/2}$. The fractional rms error in the cross correlation measurement is $\pi\rho^{-1}\,N^{-1/2}\,e^{\upsilon_0^2}[1 - \phi(\upsilon_0)]/2 = 1.23\rho^{-1}\,N^{-1/2}$. Since ρ is generally small, the effect of dividing by the mean self correlation has virtually no effect on the relative error in the cross correlation; the latter is increased by less than 0.3% if $\rho < 0.1$.

Note that since the output of the self correlators is not a function of ρ, the linearity of the cross correlator output with respect to ρ is unaffected by division by any function of the self correlator outputs, and remains as indicated in Cooper's Fig. 3.

5. Regions of Indecision

In practice the threshold levels of the samplers are not infinitely sharp, and there are regions of indecision centered on the desired levels in which there is a high probability that the sampler will make an incorrect decision on whethere the signal is above or below the threshold. Suppose that the width of a region of indecision is α and that within it there is a 50% probability of error. The probability of the signal falling within the region associated with either one of the two thresholds is

$$\frac{2\alpha}{\sqrt{2\pi}} e^{-\upsilon_0{}^2/2} \qquad (4)$$

Taking account of the 50% error probability, the error rate will not exceed 1% if (4) is less than 0.02, which requires $\alpha < 0.03$. Expressed as a fraction of υ_0, $\alpha/\upsilon_0 < 0.049$, i.e., if $\upsilon_0 = 1$ volt $\alpha < 49$ mv.

There is a similar effect associated with the time at which the signal goes through the threshold level relative to the sample time. Suppose that if the level crossing occurs within a time interval β associated with the sample time, there will be a 50% probability of error. The rate at which the signal crosses the threshold levels is somewhat less than the rate of crossing the zero level since the signal will not reach the threshold level on every cycle. The probability of the signal crossing within the error interval for any sample is therefore less than $2f_c\beta$ where f_c is the IF center frequency of 25 MHz. Hence $2f_c\beta < 0.02$ or $\beta < 400$ ps insures an error rate of less than 1% from this effect.

6. Count Rates

In designing the correlators it is useful to know the count rates at their outputs for the two extremes of $\rho=0$ and $\rho=1$. The zero correlation case closely approximates the condition for most of the cross correlators, and the rate of occurrence of high levels of the same sign is equal to the rate of occurence of high levels of opposite sign which is $r[1 - \phi(\upsilon_0)]^2/2$ where r is the 100 MHz sample rate. The self correlators operate with full correlation and the rate of occurrence of high levels of the same sign is $r[1 - \phi(\upsilon_0)]$ and the rate for opposite signs is zero. The numerical values of the occurence rates are given in Table 1.

Since it is easier to construct an add-only counter than a bidirectional one the high levels of opposite sign will be counted as zero, low and intermediate products as 1, and high levels of the same sign as 2 (instead of -1, 0 and 1 respectively). Note that to include low and intermediate products with the weighting for optimum sensitivity would require adding numbers as high as 18 at the 100 MHz rate (Cooper p.525). Omitting the low and intermediate products is therefore a worthwhile simplification.

7. Conclusions

If the output of the cross correlators is used without any further correction the tolerance on the threshold levels of the samplers is $\pm 1.3\%$ set by the considerations in section 3. Since this applies to Case 2 where the errors can be considered as a change in scale factor of the voltage levels, it can also be interpreted as requiring less than 1.3% variation in the rms amplitudes of the signals, i.e., better than 0.11 dB level stability. If however, the output of each cross correlator is divided by the arithmetic mean of the two corresponding self correlator outputs the tolerance on the threshold levels is relaxed to ±14%, which corresponds to a signal level stability of 1.1 dB. Note that the 14% figure applies to the combination of the errors in the threshold levels and in the rms signal amplitudes. On the basis of random combination of these two effects, examples of tolerable errors are ±10% in threshold levels and ±0.3 dB in the signals, or ±5% in the thresholds and ±0.7 dB in the signals.

The use of the self correlators is clearly a significant advantage. The number of cross correlators required in the continuum system is 27 (antennas) x 13 (pairs per antenna) x 4 (polarization parameters to be determined from the two cross-polarized channels) x 2 (two 50 MHz bands per 100 MHz IF channel) x 2 (sine and cosine outputs) = 5616. The number of self correlators required is equal to the number of samplers, i.e., 27 x 2 (polarization channels) x 2 (50 MHz bands) x 2 (sine and cosine) = 216, which is only 4% of the number of cross correlators. Assuming that the self-correlator scheme will be used, the following tolerances are recommended for the samplers; threshold-level errors less than ±10% with design goal of less than ±5%, regions of indecision as defined in section 5 to be less than 4% (±2%) in voltage and less than 400 ps (full interval) in time.

REFERENCE

Cooper, B.F.C., "Correlators With Two-Bit Quantization", Aust. J. Phys., 23, 521, 1970.

Appendix 1 Joint Probabilities

The joint probabilities required are P_{22}, $P_{\bar{2}\bar{2}}$ $P_{\bar{2}2}$ & $P_{2\bar{2}}$. For the threshold levels defined in Fig 1 and following p523 of Cooper one obtains:

$$P_{22} = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \int_{V_{1+}}^{\infty}\int_{V_{2+}}^{\infty} e^{-\left(\frac{v_1^2+v_2^2-2\rho v_1 v_2}{2(1-\rho^2)}\right)} dv_1 dv_2$$

Neglecting ρ^2 terms and putting $e^{-\rho v_1 v_2} = 1-\rho v_1 v_2$ one finds

$$P_{22} = \frac{1}{2\pi}\left[\int_{V_{1+}}^{\infty} e^{-v_1^2/2} dv_1\right]\left[\int_{V_{2+}}^{\infty} e^{-v_2^2/2} dv_2\right] + \frac{\rho}{2\pi}\left[\int_{V_{1+}}^{\infty} v_1 e^{-v_1^2/2} dv_1\right]\left[\int_{V_{2+}}^{\infty} v_2 e^{-v_2^2/2} dv_2\right]$$

$$= \frac{1}{4}\left[1-\phi(V_{1+})\right]\left[1-\phi(V_{2+})\right] + \frac{\rho}{2\pi} e^{-V_{1+}^2/2} e^{-V_{2+}^2/2} \qquad \text{A11}$$

where $\phi(V) = 2(2\pi)^{-\frac{1}{2}} \int_0^V e^{-v^2/2} dv = 2\pi^{-\frac{1}{2}} \int_0^{V/\sqrt{2}} e^{-t^2} dt = \operatorname{erf}(V/\sqrt{2})$

Similarly

$$P_{\bar{2}\bar{2}} = \frac{1}{4}\left[1-\phi(V_{1-})\right]\left[1-\phi(V_{2-})\right] + \frac{\rho}{2\pi} e^{-V_{1-}^2/2} e^{-V_{2-}^2/2} \qquad \text{A12}$$

$$P_{2\bar{2}} = \frac{1}{4}\left[1-\phi(V_{1+})\right]\left[1-\phi(V_{2-})\right] - \frac{\rho}{2\pi} e^{-V_{1+}^2/2} e^{-V_{2-}^2/2} \qquad \text{A13}$$

$$P_{\bar{2}2} = \frac{1}{4}\left[1-\phi(V_{1-})\right]\left[1-\phi(V_{2+})\right] - \frac{\rho}{2\pi} e^{-V_{1-}^2/2} e^{-V_{2+}^2/2} \qquad \text{A14}$$

We shall also need to know $[1-\phi(V)]$ when $V = v_0 + \delta$ where δ is small compared with unity. A Taylor series expansion of $\phi(V)$ gives

$$\phi(v_0+\delta) = \phi(v_0) + \delta\phi'(v_0) + \frac{\delta^2}{2!}\phi''(v_0) + \cdots \frac{\delta^3}{3!}\phi'''(v_0)$$

$$\phi'(v_0) = 2(2\pi)^{-\frac{1}{2}} e^{-v_0^2/2} \quad \phi''(v_0) = -2(2\pi)^{-\frac{1}{2}} v_0 e^{-v_0^2/2} \quad \phi'''(v_0) = -2(2\pi)^{-\frac{1}{2}}(1-v_0^2)e^{-v_0^2/2}$$

$$\phi(v_0+\delta) = \phi(v_0) + (2\pi)^{-\frac{1}{2}} e^{-v_0^2/2}\left[2\delta - v_0\delta^2 - \frac{1}{3}(1-v_0^2)\delta^3 + \cdots\right] \qquad \text{A15}$$

For $\delta = 0.15$, which is greater than the values we shall encounter, the three terms in square brackets are 0.3, 1.38×10^{-2} and 7.04×10^{-4}, so working to ~1% we can omit the δ^3 term.

Since v_0 is chosen to give the maximum sensitivity, we have from (3)

$$(2\pi)^{-\frac{1}{2}} e^{-v_0^2/2} = \left[1-\phi(v_0)\right] v_0$$

and thus from A15

$$1-\phi(v_0+\delta) = \left[1-\phi(v_0)\right]\left[1-2v_0\delta + v_0^2\delta^2\right] \qquad \text{A16.}$$

Appendix 2 Case 1

The sampling thresholds are shown in Fig 2 from which we have

$$V_{1+} = v_0 + \Delta_1 \qquad V_{2+} = v_0 + \Delta_2$$
$$V_{1-} = v_0 - \Delta_1 \qquad V_{2-} = v_0 - \Delta_2$$

Thus from A11 - A14 & A16

$$P_{22} = \tfrac{1}{4}\left[1-\phi(v_0)\right]^2\left[1-2v_0\Delta_1+v_0^2\Delta_1^2\right]\left[1-2v_0\Delta_2+v_0^2\Delta_2^2\right] + \frac{\rho}{2\pi}\exp\left[-(v_0^2+v_0(\Delta_1+\Delta_2)+\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]$$

$$P_{\bar{2}\bar{2}} = \tfrac{1}{4}\left[1-\phi(v_0)\right]^2\left[1+2v_0\Delta_1+v_0^2\Delta_1^2\right]\left[1+2v_0\Delta_2+v_0^2\Delta_2^2\right] + \frac{\rho}{2\pi}\exp\left[-(v_0^2-v_0(\Delta_1+\Delta_2)+\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]$$

$$P_{2\bar{2}} = \tfrac{1}{4}\left[1-\phi(v_0)\right]^2\left[1-2v_0\Delta_1+v_0^2\Delta_1^2\right]\left[1+2v_0\Delta_2+v_0^2\Delta_2^2\right] - \frac{\rho}{2\pi}\exp\left[-(v_0^2+v_0(\Delta_1-\Delta_2)+\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]$$

$$P_{\bar{2}2} = \tfrac{1}{4}\left[1-\phi(v_0)\right]^2\left[1+2v_0\Delta_1+v_0^2\Delta_1^2\right]\left[1-2v_0\Delta_2+v_0^2\Delta_2^2\right] - \frac{\rho}{2\pi}\exp\left[-(v_0^2-v_0(\Delta_1-\Delta_2)+\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]$$

First evaluate (1)

$$N(P_{22}+P_{\bar{2}\bar{2}}-P_{2\bar{2}}-P_{\bar{2}2}) = N\Big\{-v_0\Delta_2\left[1-\phi(v_0)\right]^2\left[1-2v_0\Delta_1+v_0^2\Delta_1^2\right]+v_0\Delta_2\left[1-\phi(v_0)\right]^2\left[1+2v_0\Delta_1+v_0^2\Delta_1^2\right]$$
$$+\frac{\rho}{2\pi}\exp\left[-v_0^2-\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]\Big[\exp\left[-v_0(\Delta_1+\Delta_2)\right]+\exp\left[v_0(\Delta_1+\Delta_2)\right]+\exp\left[-v_0(\Delta_1-\Delta_2)\right]+\exp\left[v_0(\Delta_1-\Delta_2)\right]\Big]\Big\}$$
$$= N\Big\{4v_0^2\Delta_1\Delta_2\left[1-\phi(v_0)\right]^2+\frac{\rho}{2\pi}\Big[2\cosh\left[v_0(\Delta_1+\Delta_2)\right]+2\cosh\left[v_0(\Delta_1-\Delta_2)\right]\Big]\exp(-v_0^2)\exp-(\Delta_1^2+\Delta_2^2)/2\Big\}$$
$$= N\Big\{4v_0^2\Delta_1\Delta_2\left[1-\phi(v_0)\right]^2+\frac{2\rho}{\pi}e^{-v_0^2}e^{-(\Delta_1^2+\Delta_2^2)/2}\cosh(v_0\Delta_1)\cosh(v_0\Delta_2)\Big\} \qquad \text{(A+S p84 4.5.39)}$$

and using (3) this becomes

$$\frac{2N}{\pi}e^{-v_0^2}\left[\Delta_1\Delta_2+\rho e^{-(\Delta_1^2+\Delta_2^2)/2}\cosh(v_0\Delta_1)\cosh(v_0\Delta_2)\right]$$

$$\simeq \underline{\frac{2N}{\pi}e^{-v_0^2}\left[\Delta_1\Delta_2+\rho\left[1-\tfrac{1}{2}(1-v_0^2)(\Delta_1^2+\Delta_2^2)\right]\right]} \qquad \text{A21}$$

Now evaluate (2)

$$R = \frac{\sqrt{N}\,(P_{22}+P_{\bar{2}\bar{2}}-P_{2\bar{2}}-P_{\bar{2}2})}{\sqrt{P_{22}+P_{\bar{2}\bar{2}}+P_{\bar{2}2}+P_{2\bar{2}}}}$$

$$P_{22}+P_{\bar{2}\bar{2}}+P_{\bar{2}2}+P_{2\bar{2}} = \tfrac{1}{2}\left[1-\phi(v_0)\right]^2\Big\{\left[1-2v_0\Delta_1+v_0^2\Delta_1^2\right]\left[1+v_0^2\Delta_2^2\right]+\left[1+2v_0\Delta_1+v_0^2\Delta_1^2\right]\left[1+v_0^2\Delta_2^2\right]\Big\}$$
$$+\frac{\rho}{2\pi}\exp(-v_0^2)\exp\left[-(\Delta_1^2+\Delta_2^2)/2\right]\Big[2\cosh\left[v_0(\Delta_1+\Delta_2)\right]-2\cosh\left[v_0(\Delta_1-\Delta_2)\right]\Big]$$
$$= \left[1-\phi(v_0)\right]^2\left[1+v_0^2\Delta_1^2\right]\left[1+v_0^2\Delta_2^2\right]+\frac{2\rho}{\pi}e^{-v_0^2}e^{-(\Delta_1^2+\Delta_2^2)/2}\sinh(v_0\Delta_1)\sinh(v_0\Delta_2)$$
$$\simeq \left[1-\phi(v_0)\right]^2\left[1+v_0^2(\Delta_1^2+\Delta_2^2)\right]+\frac{2\rho}{\pi}e^{-v_0^2}\left[1-\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]v_0^2\Delta_1\Delta_2$$
$$= \left[1-\phi(v_0)\right]^2\Big\{\left[1+v_0^2(\Delta_1^2+\Delta_2^2)\right]+4\rho\Delta_1\Delta_2v_0^4\left[1-\tfrac{1}{2}(\Delta_1^2+\Delta_2^2)\right]\Big\} \qquad \text{(again using (3))}$$

Ignoring powers of Δ greater than 2

$$R \simeq \frac{2\sqrt{N}}{\pi}\,\frac{e^{-v_0^2}}{\left[1-\phi(v_0)\right]}\,\frac{\left[\Delta_1\Delta_2+\rho\left[1-\tfrac{1}{2}(1-v_0^2)(\Delta_1^2+\Delta_2^2)\right]\right]}{\left[1+\tfrac{1}{2}v_0^2(\Delta_1^2+\Delta_2^2)+2\rho\Delta_1\Delta_2v_0^4\right]}$$

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$$\simeq \frac{2\sqrt{N}}{\pi} \frac{e^{-v_0^2}}{[1-\phi(v_0)]} \left[\Delta_1\Delta_2 + \rho\left[1 - \tfrac{1}{2}(\Delta_1^2+\Delta_2^2) - 2\rho\Delta_1\Delta_2 v_0^4\right]\right] \qquad \text{A22}$$

Inserting numerical values $v_0 = 0.612$, $\phi(v_0) = 0.459$

$$R = 0.809\sqrt{N}\left[\Delta_1\Delta_2 + \rho\left[1 - \tfrac{1}{2}(\Delta_1^2+\Delta_2^2) - 0.281\rho\Delta_1\Delta_2\right]\right]$$

Appendix 3 Case 2

The sampling thresholds are shown in Fig 3 from which

$$V_{1+} = V_{1-} = v_0 + \delta_1 \qquad V_{2+} = V_{2-} = v_0 + \delta_2$$

and thus from A11 - A14 & A16

$$P_{22} = P_{\bar{2}\bar{2}} = \tfrac{1}{4}[1-\phi(v_0)]^2[1-2v_0\delta_1+v_0^2\delta_1^2][1-2v_0\delta_2+v_0^2\delta_2^2] + \frac{\rho}{2\pi}\exp\left[-[v_0^2 + v_0(\delta_1+\delta_2) + \tfrac{1}{2}(\delta_1^2+\delta_2^2)]\right]$$

$$P_{2\bar{2}} = P_{\bar{2}2} = \tfrac{1}{4}[1-\phi(v_0)]^2[1-2v_0\delta_1+v_0^2\delta_1^2][1-2v_0\delta_2+v_0^2\delta_2^2] - \frac{\rho}{2\pi}\exp\left[-[v_0^2 + v_0(\delta_1+\delta_2) + \tfrac{1}{2}(\delta_1^2+\delta_2^2)]\right]$$

First evaluate (1)

$$N(P_{22}+P_{\bar{2}\bar{2}} - P_{\bar{2}2} - P_{2\bar{2}}) = \frac{2N\rho}{\pi} e^{-v_0^2} e^{-v_0(\delta_1+\delta_2)} e^{-(\delta_1^2+\delta_2^2)/2}$$

$$\simeq \frac{2N\rho}{\pi} e^{-v_0^2}\left[1 - v_0(\delta_1+\delta_2) - \tfrac{1}{2}(1-v_0^2)(\delta_1^2+\delta_2^2) + v_0^2\delta_1\delta_2\right] \qquad \text{A31}$$

Now evaluate (2)

$$P_{22} + P_{2\bar{2}} + P_{\bar{2}2} + P_{\bar{2}\bar{2}} = [1-\phi(v_0)]^2[1-2v_0\delta_1+v_0^2\delta_1^2][1-2v_0\delta_2+v_0^2\delta_2^2]$$

$$\simeq [1-\phi(v_0)]^2[1-2v_0(\delta_1+\delta_2) + v_0^2(\delta_1^2+\delta_2^2))$$

$$R \simeq \frac{2\sqrt{N}\rho}{\pi} \frac{e^{-v_0^2}}{[1-\phi(v_0)]}\left[1 - v_0(\delta_1+\delta_2) - \tfrac{1}{2}(1-v_0^2)(\delta_1^2+\delta_2^2) + v_0^2\delta_1\delta_2\right]\left[1+v_0(\delta_1+\delta_2) - \tfrac{1}{2}v_0^2(\delta_1^2+\delta_2^2)\right]$$

$$\simeq \frac{2\sqrt{N}\rho}{\pi} \frac{e^{-v_0^2}}{1-\phi(v_0)}\left[1 - \tfrac{1}{2}(\delta_1^2+\delta_2^2) + v_0^2\delta_1\delta_2\right] \qquad \text{A32}$$

Appendix 4. Self-Correlation Measurements.

The outputs of the two self correlators are $N(P_2 + P_{\bar{2}})_{\rho=1}$ evaluated for the appropriate threshold levels. Since each correlator is fed by the same data at both inputs the probabilities P_2 and $P_{\bar{2}}$ are given by $1-\phi$ (see Cooper p. 524). Thus

$$P_2 = \tfrac{1}{2}[1-\phi(V_+)] \qquad P_{\bar{2}} = \tfrac{1}{2}[1-\phi(V_-)]$$

Case 1

$$V_+ = v_0 + \Delta_1 \qquad V_- = v_0 - \Delta_1 \qquad \text{for channel 1.}$$

Using A16

$$N(P_2 + P_{\bar{2}}) = 2N[1-\phi(v_0)][1 + v_0^2\Delta_1^2]$$

and similarly for channel 2, $N(P_2 + P_{\bar{2}}) = 2N[1-\phi(v_0)][1+v_0^2\Delta_2^2]$
The mean of the outputs of the two self correlators is thus

$$2N[1-\phi(v_0)][1+\tfrac{1}{2}v_0^2(\Delta_1^2+\Delta_2^2)]$$

The output of the cross correlator, A21, divided by the mean output of the two self correlators is

$$\frac{1}{\pi}\frac{e^{-v_0^2}}{[1-\phi(v_0)]}\frac{[\Delta_1\Delta_2 + \rho[1-\frac{1}{2}(1-v_0^2)(\Delta_1^2+\Delta_2^2)]]}{[1+\frac{1}{2}v_0^2(\Delta_1^2+\Delta_2^2)]} \qquad \text{A41}$$

Case 2

$$V_+ = v_0 + \delta_1 \qquad V_- = v_0 + \delta_1 \qquad \text{for channel 1}$$

$$N(P_2 + P_{\bar{2}}) = 2N[1-\phi(v_0)][1-2v_0\delta_1 + v_0^2\delta_1^2]$$

The mean of the outputs of the two self correlators is thus

$$2N[1-\phi(v_0)][1 - v_0(\delta_1+\delta_2) + \tfrac{1}{2}v_0^2(\delta_1^2+\delta_2^2)]$$

The output of the cross correlator, A31, divided by the above is

$$\frac{\rho}{\pi}\frac{e^{-v_0^2}}{[1-\phi(v_0)]}\frac{[1-v_0(\delta_1+\delta_2) - \frac{1}{2}(1-v_0^2)(\delta_1^2+\delta_2^2) + v_0^2\delta_1\delta_2]}{[1-v_0(\delta_1+\delta_2)+\frac{1}{2}v_0^2(\delta_1^2+\delta_2^2)]}$$

$$\simeq \frac{\rho}{\pi}\frac{e^{-v_0^2}}{[1-\phi(v_0)]}[1-\tfrac{1}{2}(\delta_1^2+\delta_2^2) + v_0^2\delta_1\delta_2] \qquad \text{A42}$$

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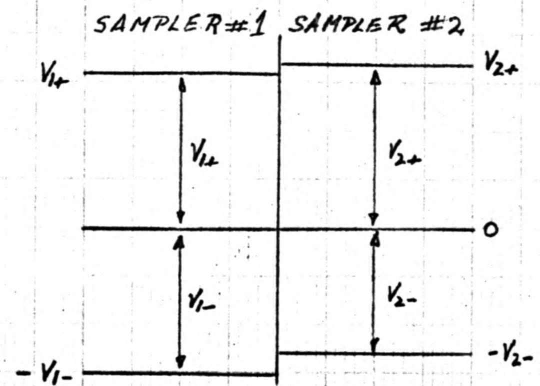

Fig. 1 Threshold levels of samplers in general case.

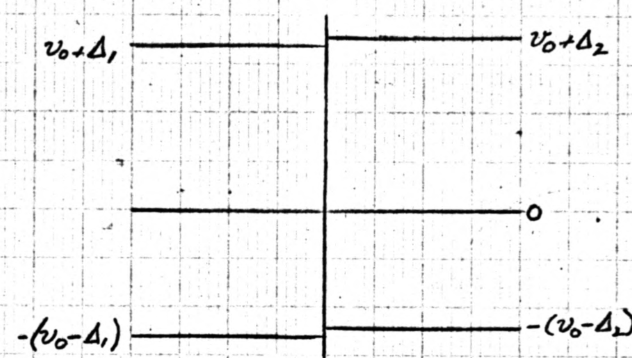

Fig. 2 Case 1, zero offsets in levels.

Fig. 3. Case 2, variation in scale factor of levels.

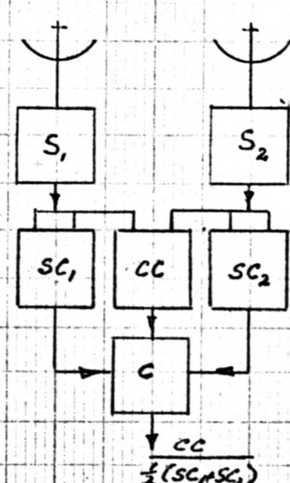

Fig. 4. System using self-correlators.

S_1, S_2 samplers.
SC_1, SC_2 self-correlators.
CC cross correlator.
C computer.

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	HIGH LEVEL PRODUCTS OPPOSITE SIGN	LOW AND INTERMEDIATE PRODUCTS	HIGH LEVEL PRODUCTS SAME SIGN.
RATE OF OCCURANCE, $P=0$	$r[1-\phi(v_0)]^2/2 = 1.46\times10^7\ s^{-1}$	$r[1-(1-\phi(v_0))^2] = 7.07\times10^7 s^{-1}$	$r[1-\phi(v_0)]^2/2 = 1.46\times10^7\ s^{-1}$
RATE OF OCCURANCE, $P=1$	0	$r\,\phi(v_0) = 4.59\times10^7\ s^{-1}$	$r[1-\phi(v_0)] = 5.41\times10^7 s^{-1}$

Table 1. Rates of Occurance of the Various Products for 100 MHz Sample Rate.

A.R.T. 4/2/73

[redacted] K+E [redacted] 7 X 10 TO INCHES [redacted] O TO [redacted] CH [redacted] 333 [redacted]
KEUFFEL & ESSER CO. MADE IN U.S.A.