NATIONAL RADIO ASTRONOMY OBSERVATORY
SOCORRO, NEW MEXICO
VERY LARGE ARRAY PROGRAM

VLA ELECTRONICS MEMORANDUM NO. 192

CLOSURE ACCURACY AND RELATED INSTRUMENTAL TOLERANCES

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February 1980

1.0 INTRODUCTION

The term closure phase is used to denote the sum of the visibility phases observed over a set of baselines involving three or more antennas. For example,

$$\Psi_c = \Phi_{12} + \Phi_{23} + \Phi_{31}$$

where Φ_{mn} denotes the phase measured for antennas m and n and Ψ_c is the closure phase. The components of the measured phases which result from the instrument, the ray paths through the atmosphere, and baselines errors all cancel out of the sum Ψ_c. If the source is unresolved, Ψ_c differs from zero only by the noise in the data, and if the source is resolved, Ψ_c is a measure of the sum of the corresponding phases of the source visibility. This principle was recognized by Jennison (1958) and used in measurements which lead to early models of Cygnus A and Cassiopeia A (Jennison 1959). It has also been used in the interpretation of VLBI data, since the closure phases with various combinations of antennas represent directly observable quantities that depend only on the structure of the source (Rogers et al. 1974). An iterative procedure for obtaining brightness distributions using amplitudes and closure phases and incorporating the CLEAN algorithm has been developed by Readhead and Wilkinson (1978).

In the VLA the closure principle is used in the calibration process. Observations of a calibration source yield values of the complex gains of the antenna pairs,

$$A_{mn} = |A_{mn}| e^{i\Phi_{mn}}$$

for antennas m and n. These values are used to assign a complex gain

$$g_m = |g_m| e^{i\psi_m}$$

to each antenna so that 351 antenna-pair gains can be replaced by 27 antenna gains*. This is done independently for each of the signal channels A to D. The pair gains can be recovered in each case by multiplying the gain, g_m, for the first antenna by the complex conjugate of the gain, g_n^* for the second one. There are several advantages to working with antenna gains rather than pair gains: (1) there is less calibration data to store and interpolate; (2) self-calibration schemes using only long-baseline antenna pairs are possible; (3) antenna channels with incorrect or unstable gains are easily identified.

The antenna gains are overdetermined by the larger number of pair gains, so an exact solution for the g_n values is not possible. Instead, the antenna gains are chosen with the aim of minimizing

$$\sum_{m=1}^{27} \sum_{n=m+1}^{27} |A_{mn} - g_m g_n^*|^2 \qquad (1)$$

The quantity $A_{mn} - g_m g_n^*$ is the complex closure discrepancy for antennas m and n. The amplitude and phase parts are commonly

*The terms antenna gain and antenna-pair gain, as used here, include the characteristics of the entire signal channel from antenna to correlator.

referred to as closure errors, and as used in this memorandum these terms are defined as follows:

$$\text{Amplitude closure error} = \left(\frac{|g_m g_n^*|}{|A_{mn}|} - 1\right) \times 100\ (\%)$$

$$\text{Phase closure error} = \arg(g_m g_n^*) - \arg(A_{mn})$$

The algorithm that solves for the antenna gains in the VLA program ANTSOL is very similar to one described in VLA Computer Memorandum No. 153. This is about to be changed to use a subroutine developed by L. R. D'Addario which more closely minimizes expression (1).

If the frequency responses of the signal channels associated with the various antennas are not identical closure errors can result. The examination of this effect is the main purpose of this memorandum. If the frequency responses are stable, the resulting closure errors can be calibrated on an antenna-pair basis, but for convenience it is preferable to have to handle the antenna gains only. Tolerances on the matching of the frequency responses are determined by calculating closure errors for various combinations of responses. Similar calculations are given by B. G. Clark in VLA Electronics Memorandum No. 171, but contain some numerical errors. However, the general conclusion of VLA EM 171, that errors occur mainly in closure amplitude and frequency responses have little effect on closure phase, is confirmed.

2.0 CONDITIONS UNDER WHICH CLOSURE ERRORS OCCUR

Consider solving for the amplitude and phase components of the antenna gains separately. For three antennas with an unresolved source,

$$|A_{12}| = |g_1||g_2| \ , \ |A_{23}| = |g_1||g_3| \ , \ |A_{31}| = |g_3||g_1| \qquad (2)$$

and

$$\Phi_{12} = \psi_1 - \psi_2 \ , \ \Phi_{23} = \psi_2 - \psi_3 \ , \ \Phi_{31} = \psi_3 - \psi_1 \tag{3}$$

An exact solution for (2) can always be found, but for (3) an exact solution implies the following condition:

$$\Phi_{12} + \Phi_{23} - \Phi_{31} = 0 \tag{4}$$

Thus phase closure errors can occur with three antennas, but amplitude closure errors cannot. A full investigation of closure errors requires consideration of at least four antennas. It should also be obvious that any effect which can be represented by (complex) multiplicative factors in the antenna gains does not cause closure errors. A change in the characteristics of a correlator, however, can cause closure errors.

In the absence of noise, the observed response of an antenna pair, A_{mn}, is related to the frequency responses of the two channels, $a_m(f)$ and $a_n(f)$ by

$$A_{mn} = \int_o^\infty a_m(f)\ a_n^*(f)\ df \Bigg/ \left[\int_o^\infty |a_m(f)|^2\ df \ \int_o^\infty |a_n(f)|^2\ df \ \right]^{1/2} \tag{5}$$

The terms in the denominator of (5) result from the action of the ALC which adjusts the total power in each channel to a specified level. From (5)

$$A_{mn} = \frac{\int_o^\infty |a_m||a_n| e^{i(\phi_m - \phi_n)}\ df}{\left[\int_o^\infty |a_m|^2\ df \int_o^\infty |a_n|^2\ df\right]^{1/2}} \tag{6}$$

where $\phi_m = \arg(a_m)$. Now suppose that $\phi_m - \phi_n$ is small so that

$$e^{i(\phi_m-\phi_n)} \simeq 1 + i(\phi_m-\phi_n).$$

Then the real part of A_{mn} is unity and $\arg(A_{mn})$ is approximately equal to the magnitude of the imaginary part. Thus

$$\phi_{mn} = \arg(A_{mn}) = \int_0^\infty |a_m||a_n|(\phi_m-\phi_n)\, df \Bigg/ [\int_0^\infty |a_m|^2\, df \int_0^\infty |a_n|^2\, df]^{1/2}$$

Suppose also that the frequency responses are all identical in amplitude and differ only in phase. Then for n antennas with an unresolved source the closure phase is

$$\phi_{12} + \phi_{23} + \phi_{34} \cdots + \phi_{n1} =$$

$$\int_0^\infty |a_m|^2\ (\phi_1-\phi_2+\phi_2-\phi_3+\phi_3+\cdots\phi_n-\phi_n-\phi_1)\, df \Bigg/ \int_0^\infty |a_m|^2\, df = 0 \quad (7)$$

Thus small-angle differences in phase responses can cause phase closure errors only if the amplitude responses are not identical.

Closure errors depend not only upon the frequency responses but on how they are distributed amongst the antennas. Consider the case of four antennas where the frequency responses differ only because of errors in the settings of the delays. Let antennas 1 and 2 be correctly set, and 3 and 4 have equal and opposite delay errors. Antenna 1 with 2 will produce normal signal amplitudes, 1 or 3 with 2 or 4 will produce decreased amplitudes, and 3 with 4 will produce more severely decreased amplitudes. These results may be quite well represented by assigning normal gains to antennas 1 and 2 and low gains to 3 and 4. However, if the delay errors for antennas 3 and 4 are of the same sign, 3 with 4 will produce normal amplitudes and attempting to describe these results in terms of antenna gains will result in large amplitude closure errors. Thus to find the maximum closure errors resulting from different frequency responses it is often useful to consider a group of antennas with two examples of each response.

3.0 EXAMPLES OF FREQUENCY-RESPONSE MISMATCHES

The following calculations show the closure discrepancies which result from various combinations of frequency responses. In each case $|a_m(f)|$ and $\phi_m(f)$ are defined for four or six antennas and are illustrated in Figure 1. The complex gains A_{mn} are calculated for each of the antenna pairs, the phase for pair 1-2 being arbitrarily chosen as zero. The argument of A_{nm} depends upon the phase difference between responses, so any given phase function can be added to all of the responses without affecting the results. The D'Addario algorithm is used to obtain the best-fit values for the complex gains of the antennas and, finally, the resulting closure errors calculated. The various examples are chosen with the aim of maximizing the closure errors so as to indicate the worst-case tolerances on the frequency responses.

3.1 Delay Errors

A delay error τ produces a linear phase shift $2\pi f\tau$ with frequency. Consider two channels, m and n, both with uniform amplitude responses from 0 to f_0, but n having delay τ relative to m. Then, from (5)

$$A_{mn} = \frac{1}{f_0}\int_0^{f_0} \cos(2\pi f\tau)\, df - \frac{i}{f_0}\int_0^{f_0} \sin(2\pi f\tau)\, df$$

from which

$$|A_{mn}| = \sqrt{2(1-2\pi f_0\tau)}\ /2\pi f_0\tau, \qquad \text{and}$$

$$\phi_{mn} = \arctan\{[\cos(2\pi f_0\tau)-1]\ /\ \sin(2\pi f_0\tau)\}$$

The above expressions are applied to the case of six antennas in which 1 and 2 have zero delay error, 3 and 4 have an error $0.15/f_0$ (equal to 3 ns for f_0 = 50 MHz) and 5 and 6 with error $-0.15/f_0$.

The resulting antenna-pair gains and closure errors are given below.

Pair	$\lvert A_{mn} \rvert$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	1.38%	0^o
1-3	0.963	-27^o	-0.38%	"
1-4	"	"	"	"
1-5	"	27^o	"	"
1-6	"	"	"	"
2-3	"	-27^o	"	"
2-4	"	"	"	"
2-5	"	27^o	"	"
2-6	"	"	"	"
3-4	1.0	0	-9.2%	"
3-5	0.858	$+54^o$	5.8%	"
3-6	"	"	"	"
4-5	"	"	"	"
4-6	"	"	"	"
5-6	1.0	0	-9.2%	"
rms			4.5%	

The computed phase closure values are all of order 10^{-4} degrees or less, and presumably result from the finite number of iterations performed by the program. Such values are listed as zero in results given in this memorandum. Further calculations show that the amplitude closure errors have a dependence on τ which lies between linear and quadratic, and maximum errors of 1% occur for $\tau = \pm 0.05/f_0$ which corresponds to ±1 ns for f_0 = 50 MHz. If antennas 3 and 6 are omitted the maximum amplitude closure error with delay errors of $0.15/f_0$ is only 3% which illustrates the point made at the end of Section 2.0.

3.2 Amplitude Slopes

In this example the phase responses are identical, which results in zero errors in the phase closure, and the amplitude voltage responses slope linearly from $(1-\alpha)$ at $f = 0$ to 1.0 at f_0, or vice versa. For a pair of antennas for which the slopes have opposite signs,

$$A_{mn} = \frac{\int_o^{f_0} (1-\alpha + \frac{\alpha f}{f_0})(1 - \frac{\alpha f}{f_0})\, df}{[\int_o^{f_0} (1-\alpha + \frac{\alpha f}{f_0})^2\, df \int_o^{f_0} (1 - \frac{\alpha f}{f_0})^2\, df]^{1/2}} = \frac{1-\alpha + \frac{\alpha^2}{6}}{1-\alpha + \frac{\alpha^2}{3}}$$

For antenna pairs with slopes of the same sign $A_{mn} = 1$. These results are now applied to four antennas of which 1 and 2 have responses increasing with frequency and 3 and 4 have responses decreasing with frequency. A value of $\alpha = 0.293$ is used which corresponds to a gain difference of 3 dB between the band edges. The antenna-pair gains have zero phase and phase closure errors are zero. The amplitude data are as follows.

Pair	$\lvert A_{mn} \rvert$	Amp Closure
1-2	1.0	-1.33%
1-3	0.980	0.68%
1-4	"	"
2-3	"	"
2-4	"	"
3-4	1.0	-1.33%
rms		0.95%

Even with a 3 dB gain variation the amplitude closure errors are small.

3.3 Differences in Passband Center-Frequency or Cutoff

Suppose that all passbands have rectangular amplitude characteristics and equal bandwidths f_0, but some have the center frequency displaced by Δf from the correct value. For pairs with identical characteristics $\lvert A_{mn} \rvert = 1.0$ and $\Phi_{mn} = 0$.

For pairs with different center frequencies,

$$|A_{mn}| = 1 - \Delta f/f_0, \text{ and}$$
$$\Phi_{mn} = n\pi \, \Delta f/f_0$$

where $n\pi$ is the phase change across the band. Consider four antennas, 1 and 2 with the correct center frequency and 3 and 4 with a displacement Δf. For $n = 6$, which is a realistic value, and $\Delta f/f_0 = 0.05$, the results are given below.

Pair	$\lvert A_{mn}\rvert$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-3.3%	0^o
1-3	0.95	54^o	1.75%	"
1-4	"	"	"	"
2-3	"	"	"	"
2-4	"	"	"	"
3-4	1.0	0	-3.3%	"
rms			2.4%	"

Phase closure errors are zero, and the amplitude closure errors can be shown to vary approximately linearly with $\Delta f/f_0$. A maximum amplitude error of 1% would result from $\Delta f/f_0 = 0.016$.

If the overall bandwidth is controlled by a final baseband amplifier, which will usually be the case for the VLA with the final-design electronic system, differences in the frequency range of the signal involve only the high-frequency cutoff. If the cutoff is low by Δf, resulting in a reduction in bandwidth, the ALC will increase the signal level, thereby partly compensating for the decreased bandwidth. The resulting values of amplitude closure are therefore only half as large as those calculated for a center-frequency displacement with the same value of Δf.

3.4 Passband Ripple

This example concerns the effect of a reflection in which a fraction of the main signal, of relative amplitude β, is delayed by a time interval τ. The phase of the reflection, θ_m, varies from one antenna to another. Thus

$$a_m(f) = 1 + \beta e^{i(2\pi f\tau+\theta_m)}$$

For antenna pair m-n,

$$A_{mn} = \frac{\{1 - \frac{i\beta}{2\pi R}[e^{i\theta_m}(e^{i2\pi R}-1) - e^{-i\theta_n}(e^{-i2\pi R}-1)] + \beta^2 e^{i(\theta_m-\theta_n)}\}}{\{1+\frac{\beta}{\pi R}[\sin(2\pi R+\theta_m)-\sin\theta_m]+\beta^2\}^{1/2}\ \{1+\frac{\beta}{\pi R}[\sin(2\pi R+\theta_n)-\sin\theta_n]+\beta^2\}^{1/2}} \quad (8)$$

where $R = f_0\tau$ is the number of ripples across the passband. Note that the second term in the numerator of (8) is zero if R is an integer, and it becomes small as R becomes large. The third term is second-order in β. Thus many fine ripples produce only a small closure effect. If $R \lesssim 0.1$ the situation approximates that of linear phase and amplitude variations. In search of large closure errors let us consider the intermediate case of $R = \frac{1}{2}$. Then the numerator of (8) becomes

$$\{1 - \frac{2\beta}{\pi}(\sin\theta_m-\sin\theta_n) + \beta^2\cos(\theta_m-\theta_n)$$

$$+ i\ [\frac{2\beta}{\pi}(\cos\theta_m-\cos\theta_n) + \beta^2\sin(\theta_m-\theta_n)]\}$$

If $\theta_m = \frac{\pi}{2}$ and $\theta_n = -\pi/2$, A_{mn} is purely real, so the phase closure errors will be zero. Instead try $\theta_m = 0$ and $\theta_n = \pi$ in which case $A_{mn} = (1 - \beta^2 + \frac{i4\beta}{\pi})/(1+\beta^2)$.

A value of β = 0.172 corresponds to 3 dB peak-to-peak ripple, and this is used for the case of four antennas for which $\theta_1 = \theta_2 = 0$, $\theta_3 = \theta_4 = \pi$. The antenna-pair gains and closure errors are as follows.

Pair	$\lvert A_{mn} \rvert$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0^o	-2.27%	0^o
1-3	0.966	12.7^o	1.17%	"
1-4	"	"	"	"
2-3	"	"	"	"
2-4	"	"	"	"
3-4	1.0	0^o	-2.27%	"
rms			1.62%	"

The phase closure errors are zero and the amplitude closure errors are small. A maximum closure error of 1% is obtained for α = 0.12 corresponding to 2 dB variation across the passbands.

3.5 Combination of Delay Errors and Amplitude Slopes

In this example and the following one the effects of a combination of both phase and amplitude anomalies in the passbands are investigated, in search of larger phase closure errors than found in the previous examples. Six antennas are now considered, all of which have voltage responses which are linear functions of frequency and vary by 3 dB across the band. For antennas 1 to 4 the gain increases with increasing frequency and for 5 and 6 it decreases. Antennas 1 and 2 have a positive delay error and 3 to 6 a negative delay error, and values of both 1 ns and 2 ns are considered. The complex gains for the antenna pairs were in this case estimated numerically by using values at 10 points across the passbands. The results are shown in Table I. The phase closure errors are approximately proportional to the delay error and the largest value is only 1.5°. The amplitude closure errors vary more rapidly with delay error and for 1-ns delay error the maximum closure error of

TABLE I: PAIR GAINS AND CLOSURE ERRORS FOR EXAMPLE 3.5

	Delay = $0.05/f_0$ (1 ns for f_0 = 50 MHz)				Delay = $0.1/f_0$ (2 ns for f_0 = 50 MHz)			
Pair	$\|A_{mn}\|$	Φ_{mn}	Amp Closure	Phase Closure	$\|A_{mn}\|$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-2.3%	0	1.0	0	-7.9%	0
1-3	0.984	20.1°	0.77%	-0.69°	0.938	40.3°	1.5%	-1.4°
1-4	"	"	"	"	"	"	"	"
1-5	0.964	18.0°	1.1%	0.71°	0.918	36.0°	2.6%	1.5°
1-6	"	"	"	"	"	"	"	"
2-3	0.984	20.1°	0.77%	-0.69°	0.938	40.3°	1.5%	-1.4°
2-4	"	"	"	"	"	"	"	"
2-5	0.964	18.0°	1.1%	0.71°	0.918	36.0°	2.6%	1.5°
2-6	"	"	"	"	"	"	"	"
3-4	1.0	0	-0.72%	0	1.0	0	-1.6%	0
3-5	0.980	"	0.28%	-0.70°	0.980	"	-0.62%	-1.4°
3-6	"	"	"	"	"	"	"	"
4-5	"	"	"	"	"	"	"	"
4-6	"	"	"	"	"	"	"	"
5-6	1.0	"	-2.7%	0	1.0	"	-3.6%	0
rms			1.1%	0.63°			2.8%	1.3°

2.3% is close to the sum of the magnitudes of the maximum closure errors for 1-ns delay errors alone in Section 3.1 and for 3 dB passband slopes in Section 3.2.

3.6 Anomalies at the Edge of the Passband

Six antennas are considered, all of which have flat passbands in amplitude, but for antennas 5 and 6 the upper cutoff frequency is reduced to f_0-Δf. For antennas 3 and 4 the phase response is flat up to f_0-Δf and from there increases linearly to π at f_0. The other four antennas have flat phase responses. For antennas 1 to 4 with 5 or 6

$$|A_{mn}| = \sqrt{1-\Delta f/f_0} \qquad \text{and} \qquad \Phi_{mn} = 0,$$

For antennas 1 or 2 with 3 or 4,

$$A_{mn} = 1 + \frac{\Delta f}{f_0} - \frac{i2\Delta f}{\pi f_0}$$

The resulting antenna-pair gains and closure errors for $\Delta f/f_0$ = 0.05 are as follows.

Pair	$\|A_{mn}\|$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-3.5%	0^o
1-3	0.95	-1.92^o	1.6%	0.66^o
1-4	"	"	"	"
1-5	0.975	0	0.25%	-0.63^o
1-6	"	"	"	"
2-3	0.95	-1.92^o	1.6%	0.66^o
2-4	"	"	"	"
2-5	0.975	0	0.25%	0.63^o
2-6	"	"	"	"
3-4	1.0	"	-3.5%	0
3-5	0.975	"	0.25%	-0.63^o
3-6	"	"	"	"
4-5	"	"	"	"
4-6	"	"	"	"
5-6	1.0	"	-1.0%	0
rms			1.55%	0.57^o

For $\Delta f/f_0 = 0.1$ both the amplitude and phase closure errors are approximately doubled, but the cutoff error in the frequency response is then much larger than should occur in practice.

3.7 An Example With VLA Responses

The final example is for four antennas for which the responses are taken from measurements on the VLA. Antennas 1 and 2 have identical responses and so do 3 and 4. The amplitudes are taken from measurements by Percival and D'Addario of the response of the modems and waveguide. The curve in Figure 10 of VLA Electronics Memorandum No. 187 was used for antennas 1 and 2 and the curve in Figure 12 for 3 and 4. Ten points were used across each curve, and in reading the amplitude values the local maximum or minimum nearest the frequency value was taken in each case so as to err in the direction of overestimating the variations. The phase differences between 1 or 2 and 3 or 4 are taken from a measurement of the relative phases of the cosine channels for samplers B11 and B22. The ten values at 5-MHz intervals are as follows: 3.5°, 4.0°, 6.5°, 7.5°, 7.0°, -1.5°, -5.0°, -5.5°, -7.0°, -12.0°. No measurement exists for the phase response of the waveguide, but otherwise the data are believed to represent realistic VLA responses. They were selected as having the greatest deviations from an ideal flat response amongst available measured responses. The results are as follows.

Pair	$\|A_{mn}\|$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-1.46%	0^o
1-3	0.978	-0.136^o	0.75%	"
1-4	"	"	"	"
2-3	"	"	"	"
2-4	"	"	"	"
3-4	1.0	0	-1.46%	"
rms			1.04%	"

4.0 RECOMMENDED TOLERANCES AND CONCLUSIONS

The largest phase closure error in the examples in Section 3.0 is 1.5° in example 3.5 and this is marginally acceptable. Tolerances therefore result mainly from the amplitude responses.

From example 3.1 a delay error of 1 ns for the 50-MHz bandwidth produces a 1% maximum closure error. From example 3.5, the same error coupled with a large amplitude slope results in a 2.3% maximum closure error. Thus 1 ns should be about the maximum allowable error in delay setting. With a baseband center frequency of 25 MHz, 1 ns corresponds to 9° in phase. If the delay setting error enters in such a way that the phase is not compensated for in observing a calibration source, then the phase accuracy requirement is more critical than the closure error. A 1-ns delay error corresponds to 0.4% correlation drop with 50-MHz bandwidth, so the closure error is more critical than the loss in sensitivity.

From examples 3.2 and 3.5 amplitude slopes of 3 dB across the band can cause 1.3% closure errors when alone or 2.7% when combined with 1-ns delay errors. A 3-dB maximum passband slope is recommended in VLA Electronics Memorandum No. 107 from a consideration of loss in sensitivity of the system, and this same criterion appears to be about right for closure errors also.

Sinusoidal ripples have a maximum effect on closure when there is less than one full cycle across the passband, as in example 3.4. The effect is then similar to that of an amplitude slope. With the 3-dB peak-to-peak limit recommended in VLA EM 107 ripples should produce no significant closure problems.

A 5% error in the upper cutoff frequency of some baseband channels can produce maximum closure errors of 1.7% when alone, as in example 3.3 or greater effects if combined with other band edge anomalies as in 3.6. In the procurement of the final stage filters for the VLA, the specification (A13820N5A) calls for tolerances of ±1% on the low-pass cutoff (-3 dB) frequency and ±2% on band-pass center frequency. Closure errors resulting from these tolerances should be negligible.

It does not appear possible to attribute phase closure errors of 5° or more, as observed in the VLA, to unmatched frequency responses. Time variations in the gain factor of an antenna channel, including the atmospheric component of phase, only cause closure errors if the data for the various antenna pairs do not cover identical time periods. It appears that the most probable cause of large phase closure errors is misbehavior of the correlators or the data processing system that follows them.

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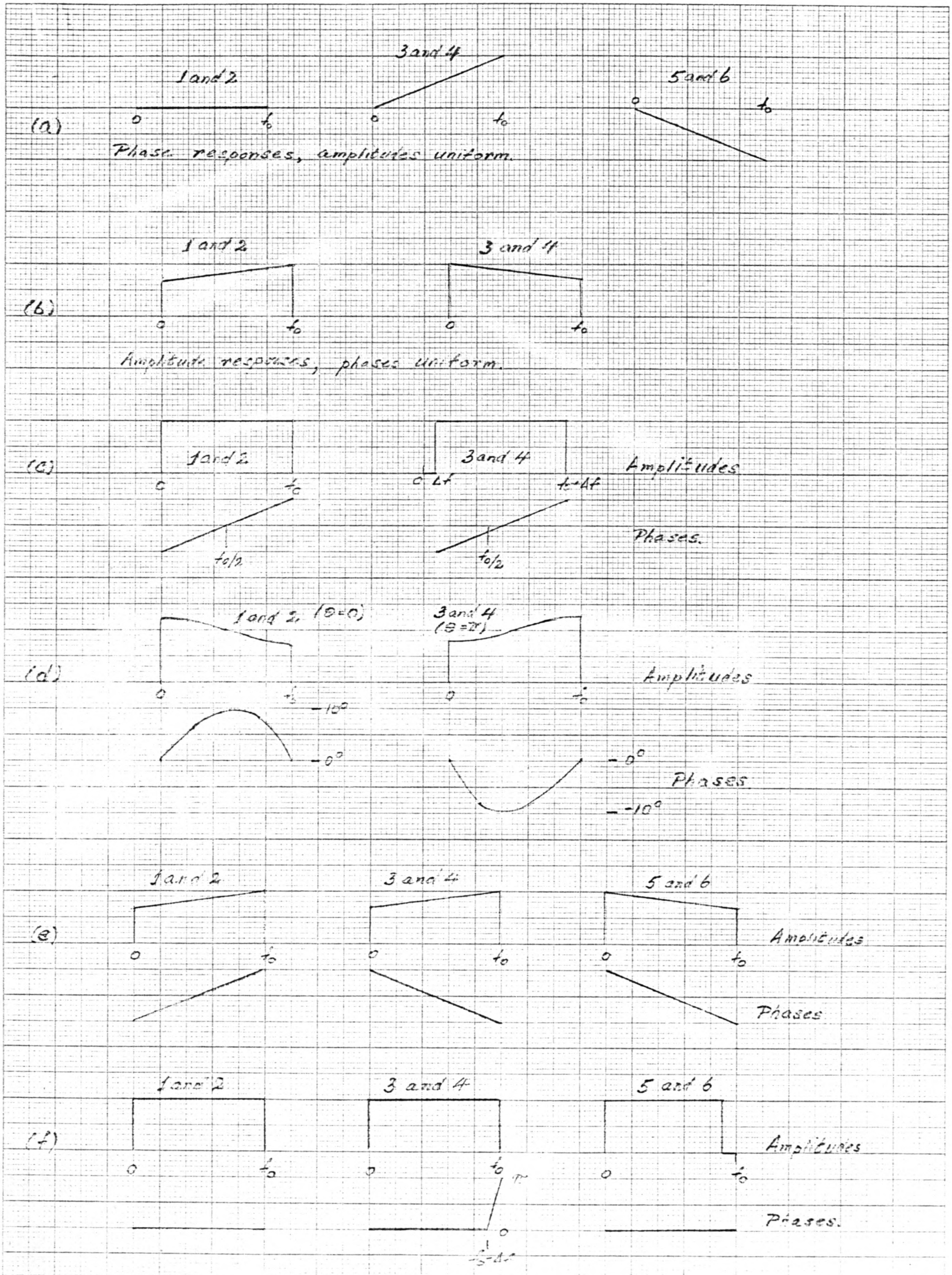

Fig. 1 Frequency responses of examples in section 3.

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ADDENDUM
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March 1980

It has been pointed out by Larry D'Addario that when only two different frequency responses are involved, phase closure errors, as defined in VLA E.M. 192, are always zero although non-zero amplitude errors can occur. Larry expresses the result as the following theorem.

For an array of N antennas in which some have transfer function (frequency response) $a_p(f)$ at frequency f, and all others have transfer function $a_q(f)$ there exists a set of 'antenna complex gains' $\{g_k\}_1^N$ such that the residuals

$$r_{mn} = g_m \, g_n^* - \int_{\infty}^{\infty} a_m(f) \, a_n^*(f) \, df$$

all have zero phase, where a_m and a_n are the transfer functions of the m^{th} and n^{th} antennas and are equal to a_p or a_q. Proof: without loss of generality, assume $a_1 = a_p$. Assign $\arg(g_1) = 0$, and assign

$$\arg(g_m) = \begin{cases} 0 \text{ if } a_m = a_p \\ \arg\int_{-\infty}^{\infty} a_p(f) \, a_q(f) \, df \equiv \psi \text{ if } a_m = a_q \end{cases}$$

for m = 2 to N. To verify that the theorem is satisfied note that there are only three cases: $\arg(g_m g_n^*) = 0, \psi$, or $-\psi$. Direct inspection shows that $\arg r_{mn} = 0$ in each case.

In four examples in VLA E.M. 192, (Sections 3.2, 3.3, 3.4, and 3.7) only two different frequency responses were used. In the case of 3.2 no phase closure error was expected. In the other three cases the calculations have been extended as described below.

<u>Differences in Passband Center Frequency (3.3)</u>

Two farther antennas were added, 5 and 6, with a center frequency offset $\Delta f = -0.03\ f_0$. The results are given below.

Pair	$\lvert A_{mn} \rvert$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-1.6%	0^o
1-3	0.95	54^o	0.93%	"
1-4	"	"	"	"
1-5	0.97	-32.4^o	-0.07%	"
1-6	"	"	"	"
2-3	0.95	54^o	0.93%	"
2-4	"	"	"	"
2-5	0.97	-32.4^o	-0.07%	"
2-6	"	"	"	"
3-4	1.0	0	-6.57%	"
3-5	0.921	-86.4^o	2.55%	"
3-6	"	"	"	"
4-5	"	"	"	"
4-6	"	"	"	"
5-6	1.0	0	-4.5%	"
rms	-		2.52%	"

The calculated phase closure values were all 2×10^{-5} degrees or less, and are attributable to the accuracy of the iterative computation. This result can be explained by noting that the phase responses of the antennas are linear in frequency with the same slope, so that in equation (6) of VLA E.M. 192 the exponential term is independent of frequency, and can be taken outside the integral. Only the amplitudes of the antenna gain factors, g_m, are affected by the differences in the cutoff frequencies of the passbands.

Passband Ripple (3.4)

Antennas 5 and 6, which have uniform amplitudes and zero phases from 0 to f_0, were added. Then for R = ½ and n = 5 or 6,

$$A_{mn} = \frac{1 + \frac{i2\beta e^{i\theta_m}}{\pi}}{1 + \beta^2 - \frac{4\beta \sin\theta_m}{\pi}}$$

The resulting antenna-pair gains and closure errors are as below.

Pair	$\lvert A_{mn} \rvert$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-2.2%	0^o
1-3	0.966	12.7^o	1.2%	-0.068^o
1-4	"	"	"	"
1-5	0.991	6.25^o	-0.07%	-0.066^o
1-6	"	"	"	"
2-3	0.966	12.7^o	1.2%	-0.068^o
2-4	"	"	"	"
2-5	0.991	6.25^o	-0.07%	0.066^o
2-6	"	"	"	"
3-4	1.0	0	-2.2%	0^o
3-5	0.991	-6.25^o	-0.07%	-0.066^o
3-6	"	"	"	"
4-5	"	"	"	"
4-6	"	"	"	"
5-6	1.0	0	0.29%	0^o
rms			1.03%	0.060^o

Example with VLA Responses (3.7)

Again two additional antenna were added, numbers 5 and 6, with uniform amplitude and zero phase responses. The phase curve used in the earlier calculation for antennas 1 and 2 relative to 3 and 4 was assigned as the phase response of antennas 3 and 4. The measured relative phases of the sine channels for samplers B11 and B22 were used as the phase response of antennas 1 and 2, the ten values across the passband being 5.0°, 6.5°, 6.0°, 5.5°, 3.5°, -2.0°, -4.0°,

-4.5°, -5.0°, and -12°. The antenna-pair gains and closure errors are as follows.

Pair	$\lvert A_{mn} \rvert$	Φ_{mn}	Amp Closure	Phase Closure
1-2	1.0	0	-1.14%	0^o
1-3	0.984	0.059^o	0.36%	-0.135^o
1-4	"	"	"	"
1-5	0.987	-0.181^o	0.21%	0.135^o
1-6	"	"	"	."
2-3	0.984	0.059^o	0.36%	-0.135^o
2-4	"	"	"	"
2-5	0.987	-0.181^o	0.21%	0.135^o
2-6	"	"	"	"
3-4	1.0	0	-1.34%	0^o
3-5	0.985	0.165^o	0.31%	-0.135^o
3-6	"	"	"	"
4-5	"	"	"	"
4-6	"	"	"	"
5-6	1.0	0	-1.04%	0^o
rms			0.59%	0.121^o

Conclusions

With the extended examples given in this addendum, all cases considered where phase closure errors are to be expected involve six antennas, two for each of three different frequency responses. This appears to be the smallest sample that will give realistic results. No closure errors greater than found in the VLA E.M. 192 have appeared, and the conclusions stated there remain unchanged.