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LIMITATIONS OF THE FOURIER TRANSFORM RELATIONSHIP FOR VLA DATA

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Purpose: To discuss the circumstances in which the standard Fourier transform relationship between V(u,v) and I(x,y) fails for high resolution VLA data.

Following the previous report (#105) discussiong the mapping problem for the VLA, particularly the effects of gridding in the (u,v) plane and the possible usefulness of the direct transform, Ed Fomalont asked whether I had considered the possible importance of higher order terms in the exponential factor involved in the relationship between the observed visibility function V(u,v) and the sought-after apparent intensity, I(x,y), on the sky. In this report the problem is discussed in terms of the phase errors introduced during the Fourier transform process by neglecting these higher order terms. Some possible methods for getting around the limitations imposed by this effect are also discussed.

DERIVATION OF EQUATIONS

If $\underset{\sim}{B}$ is the baseline vector, in wavelengths, for a particular interferometer pair, and $\underset{\sim}{s}$ is a unit vector pointing towards a position in the sky defined by (α,δ), then in general

$$V(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(\alpha,\delta)e^{2\pi i(\underset{\sim}{B}\cdot\underset{\sim}{s}-\underset{\sim}{B}\cdot\underset{\sim}{s}_o)}d\alpha d\delta, \quad (1)$$

where $\underset{\sim}{s}_o$ is the unit vector pointing at the position, (α_o,δ_o), being tracked by the interferometer.

Letting t denote the sidereal time and H the hour angle,

$$H = t - \alpha$$

and

$$\underset{\sim}{s} = \begin{pmatrix} \cos\delta \cos H \\ -\cos\delta \sin H \\ \sin\delta \end{pmatrix}_{LH}$$

in the usual left-handed (LH) coordinate system in which H and δ are defined.

In the right-handed (RH) reference frame on the sky where the origin is at (α_o, δ_o), the x-axis points east, the y-axis points north, and the z-axis is into the plane of the sky, the baseline vector $\underset{\sim}{B}$ is given by

$$\underset{\sim}{B} = \begin{pmatrix} u \\ v \\ D \end{pmatrix}_{RH} = \begin{pmatrix} \sin H & -\cos H & 0 \\ -\sin\delta \cos H & -\sin\delta \sin H & \cos\delta \\ \cos\delta \cos H & \cos\delta \sin H & \sin\delta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{LH} \qquad (2)$$

so u and v are the x- and y-components of $\underset{\sim}{B}$ projected on the sky and D, called the delay, is the component of $\underset{\sim}{B}$ perpendicular to the plane of the sky.

In the reference frame on the sky,

$$\underset{\sim}{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{RH}$$

so that

$$D = \underset{\sim}{B} \cdot \underset{\sim}{s}$$

hence

$$\underset{\sim}{B} \cdot \underset{\sim}{s} - \underset{\sim}{B} \cdot \underset{\sim}{s}_o = D - D_o .$$

Obviously D_o is the delay used to track the point (α_o, δ_o) with the interferometer, and $D - D_o$ is the delay differential for a general point in the sky, (α, δ), or (x,y), where we use the usual definitions

$$x = \Delta\alpha \cos \delta_o$$

and

$$y = \Delta\delta$$

where $$\Delta\alpha = (\alpha - \alpha_o), \; \Delta\delta = (\delta - \delta_o).$$

Thus,

$$V(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y) e^{2\pi i [D(x,y) - D_o]} dxdy. \tag{3}$$

From equation (2),

$$D = B_x \cos \delta \cos H + B_y \cos \delta \sin H + B_z \sin \delta \tag{4}$$

which we can then use to evaluate

$$D(x,y) - D_o = D(H = H_o + \Delta H, \; \delta = \delta_o + \Delta\delta) - D(H_o, \delta_o), \tag{5}$$

where H_o is the hour angle of the tracking position (α_o, δ_o)

Evaluating equation (5) to second order in ΔH and $\Delta\delta$, by use of equation (4), one obtains

$$D(x,y) - D_o = -\Delta H \cos \delta_o \, u + \Delta\delta \, v$$

$$-(\Delta^2 H/2)(D_o - B_z \sin \delta_o) - (\Delta^2 \delta/2) D_o + \Delta\delta \, \Delta H \sin \delta_o \, u$$

so that, changing to $x = \Delta\alpha \cos \delta_o = -\Delta H \cos \delta_o$ and $y = \Delta\delta$,

$$D(x,y)-D_o = ux + vy - \frac{x^2}{2} (D_o-B_z \sin \delta_o)/\cos^2 \delta_o - \frac{y^2}{2} D_o-xy \tan \delta_o u, \quad (6)$$

therefore the equation relating V(u,v) to I(x,y) becomes

$$\boxed{V(u,v)= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y)\exp\left\{2\pi i\left[ux+vy-\frac{x^2}{2}(D_o-B_z\sin \delta_o)/\cos^2\delta_o-\frac{y^2}{2}D_o-xy \tan \delta_o u\right]\right\}dxdy} \quad (7)$$

As we see from equation (7), some nasty quadratic, and still neglected higher order terms, have always infested the relationship between V(u,v) and I(x,y), and unless these non-linear phase terms are negligible the Fourier standard transform relationship between V(u,v) and I(x,y) is invalid.

NUMERICAL EVALUATION

How serious is the problem? As in report #105, where we considered the errors introduced by gridding, we can consider the non-linear terms to introduce a phase error $\delta\Phi$, where

$$\boxed{\delta\Phi = \pi[x^2(D_o-B_z \sin \delta_o)/\cos^2 \delta_o-y^2 D_o-2xyu \tan \delta_o].} \quad (8)$$

First of all we note that as

$$\delta_o \rightarrow 90°$$

$$\delta\Phi \rightarrow \infty$$

because of the terms involving $(\cos \delta_o)^{-2}$ and $\tan \delta_o$. Therefore, for any interferometer the Fourier transform relationship will fail at very high declinations. This effect is not fundamental, however, because a re-definition of u to absorb the $\cos \delta$ factor now included in x will solve the problem.

What about the general magnitude of the phase errors?

At 35 km resolution for the VLA, for an observing wavelength λ_{cm} (in cm),

$$u_{max} \sim v_{max} \sim (D_o)_{max} \sim \frac{3.5 \times 10^6}{\lambda_{cm}} ,$$

therefore for the maximum resolution data of the VLA we can estimate $\delta\Phi$ from equation (8) as

$$\delta\Phi \sim (180°) \frac{3.5\mathrm{x}10^6}{\lambda_{cm}} (2.9\mathrm{x}10^{-4})^2 \left(\frac{x^2_{arcmin}}{\cos^2 \delta_o} + y^2_{arcmin} + 2x_{arcmin}\, y_{arcmin} \tan \delta_o \right)$$

or

$$\boxed{\delta\Phi \sim \frac{53°}{\lambda_{cm}} \left(\frac{x^2_{arcmin}}{\cos^2 \delta_o} + y^2_{arcmin} + 2x_{arcmin}\, y_{arcmin} \tan \delta_o \right)} \;. (9)$$

Taking $x_{arcmin} \sim y_{arcmin}$,

$$\delta\Phi \sim \frac{53°}{\lambda_{cm}} (\cos^{-2} \delta_o + 1 + 2 \tan \delta_o)\, x^2_{arcmin} .$$

Table 1

EVALUATION OF QUADRATIC PHASE ERROR TERMS

δ_o	$\delta\Phi$
$-30°$	$62°\ x^2_{arcmin}/\lambda_{cm}$
$0°$	$106°\ x^2_{arcmin}/\lambda_{cm}$
$30°$	$185°\ x^2_{arcmin}/\lambda_{cm}$
$45°$	$265°\ x^2_{arcmin}/\lambda_{cm}$
$60°$	$449°\ x^2_{arcmin}/\lambda_{cm}$
$80°$	$2411°\ x^2_{arcmin}/\lambda_{cm}$

If we adopt the standard that one must keep

$$\delta\Phi \lesssim 20°$$

then one must restrict the synthesized field such that

$$x_{arcmin} \lesssim \left[\frac{20\ \lambda_{cm}}{53(\cos^{-2}\delta_o + 1 + 2\tan\delta_o)}\right]^{\frac{1}{2}}.$$

Table 2

MAXIMUM SYNTHESIZABLE FIELDS
FOR HIGH RESOLUTION VLA DATA

δ_o	$2x^{maximum}_{arcmin}$
− 30°	$1'\!.1\ \lambda_{cm}^{1/2}$
0°	$0'\!.86\ \lambda_{cm}^{1/2}$
30°	$0'\!.66\ \lambda_{cm}^{1/2}$
45°	$0'\!.54\ \lambda_{cm}^{1/2}$
60°	$0'\!.42\ \lambda_{cm}^{1/2}$
80°	$0'\!.18\ \lambda_{cm}^{1/2}$

The results in Table 2 can be compared with

$$(\text{HPBW})_{\substack{\text{delay}\\ \text{beam}}} \sim 1'\!.3 \quad \text{(45 MHz Bandwidth)}$$

$$(\text{HPBW})_{\text{antenna}} \sim 1'\!.7\ \lambda_{cm}$$

$$(\text{HPBW})_{\substack{\text{synthesized}\\ \text{beam}}} \sim 0''\!.1\ \lambda_{cm}$$

for the VLA at maximum resolution.

NASTY PROBLEM NUMBER 1

We note from Table 2 that at $\lambda = 2$ cm, for the lower declinations, the delay HPBW and the limiting field due to non-linear phase errors are of the same order, but at higher declinations the non-linear phase errors will limit the synthesizable field. Fortunately, the situation improves for longer wavelengths, so it might seem as if all of this discussion is only of academic interest. Unfortunately, unlike the delay beam effect, there is no way to increase the field of view for which non-linear phase errors in the high resolution data are negligible.

For the delay beam effect

$$(\mathrm{HPBW})_{\substack{\text{delay}\\ \text{beam}}} = 1\overset{'}{.}3\ (45/\Delta\nu)$$

for the VLA at maximum resolution, where $\Delta\nu$ is the band-width in MHz. Because of this, people have been able to discuss increasing the size of the synthesized field by decreasing the band-width. Unforunately, because of the non-linear phase terms a decrease of band-width would cause the restriction that $x_{\text{arcmin}} < 0\overset{'}{.}2\ \lambda_{\text{cm}}^{1/2}$ to $1\overset{'}{.}1\ \lambda_{\text{cm}}^{1/2}$, depending upon declination, to dominate. Unless someone can find a flow in this reasoning, the non-linear phase terms will fundamentally limit the VLA's capability to produce high resolution maps (using standard Fourier inversion techniques).

NASTY PROBLEM NUMBER 2

The problem of non-linear phase error terms is somewhat more serious than just limiting the size of the field of view that can be synthesized. Most of the things that we consider to limit the size of the synthesizeable field, the delay beam and the antenna pattern, effectively decrease the sensitivity of the instrument to sources away from the center of the field. That is, the contribution to $V(u,v)$ (see eqn. (1)) due to the brightness $B(x,y)$ is reduced by a factor of $f(x,y)$ so that $I(x,y) = f(x,y)\ B(x,y)$. Thus the amplitude of the contribution to $V(u,v)$ is reduced for far out x and y.

In the case of the errors introduced by the phase terms non-linear in x and y, the magnitude of I(x,y) is unchanged, but its effect is added into the determination of V(u,v) with the "wrong phase" when one attempts to reconstruct I(x,y) by Fourier transforming the data.

In other words, the non-linear phase terms will produce effects in the maps which are analogous to those produced by aliasing, where sources exist just outside the field being mapped. One is then in the interesting situation of wanting the delay beam to suppress these sources at larger x and y. If one did not, one would seriously effect even regions near x,y ~ 0 because of this pseudo-aliasing effect.

Because of this pseudo-aliasing one would not want to decrease the band-width, when working with high resolution data.

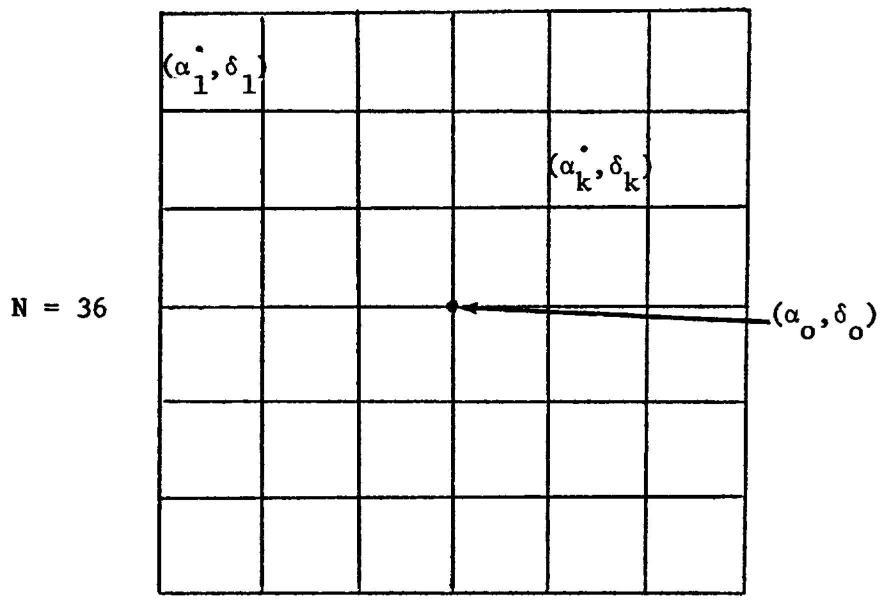
WHAT ARE OUR OPTIONS?

The first possibility is that the VLA will never be able to synthesize fields, with high resolution, which are more than $0'.4\ \lambda_{cm}^{1/2}$ to $2'\ \lambda_{cm}^{1/2}$ in diameter, depending on declination. Indeed, this probably will be the limit for the dual-channel continuum instrument presently planned as the first continuum system. One would then not want to decrease the band-width below 50 MHz. Of course, by observing with the VLA with a factor of two or three less resolution, the synthesizable fields will increase as the square root of the same factors.

Another possibility is that with smaller band-widths one could synthesize larger fields by giving up the Fourier transform and using equation (7) to solve for I(x,y) by methods like the maximum entropy method whereby equation (7) becomes an equation of constraint on I(x,y). A considerable amount of work would have to be done to properly evaluate such a possibility.

The third possibility is a very simple (!) one where an N channel instrument would allow us to synthesize fields a factor of N larger in area, and a factor of $N^{1/2}$ larger in diameter.

Let k be an index identifying the k-th channel where k = 1,...,
N. Then let the k-th channel of the VLA track a point in the sky, (α_k, δ_k) within both the antenna beam and the delay beam.



Each of the N channels would track different points arranged in a grid, with separations corresponding to the size scale on which the quadratic error term, $\delta\Phi$, remains within desirable limits.

It is trivially obvious that with brute force mapping of all N fields one can build up a mosaic map, but there will always be a problem of fitting the pieces together smoothly.

A SIMPLE MOSAIC MAPPING TECHNIQUE

There is a very simple way of carrying out the mapping process when the data are taken in a grid with N different tracking positions within the same delay and antenna beam.

This would work as follows. Collect all the data

$$V_k\ (u,v), \quad k = 1, \ldots, N$$

for each (α_k, δ_k)

then phase shift all the data to a common phase reference position (α_o,δ_o) by transforming

$$V_k(u,v) = V_k(u,v)e^{2\pi i\left[u(\alpha_i-\alpha_o)\cos\delta_o + V(\delta_i-\delta_o)\right]}$$

so that all data taken in the mosaic can be transformed during the same mapping process.

Unlike the usual synthesis mapping problem, one must now introduce weighting according to position. For all k = 1,..., N one is gathering data about each (x,y) point, but of greatly varying quality because of phase errors introduced into data for points too far from the center of each mosaic cell. However, this problem is simply solvable by weighting the data, not only in the (u,v) plane, but also in the (x,y) plane, i.e., one approximates

$$I(x,y) \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(u,v)e^{-2\pi i(ux+vy)}dudv$$

by a slightly modified version of the sum approximations normally used:

$$I(x,y) = \frac{\sum\limits_u \sum\limits_v e^{-2\pi i(ux+vy)}w(u,v) \sum\limits_{k=1}^{N} V_k(u,v)\ W_k(x,y,u,v)}{\sum\limits_u \sum\limits_v w(u,v) \sum\limits_{k=1}^{N} W_k(x,y,u,v)}$$

where w(u,v) is the usual weighting function for the (u,v) plane and $W_k(x,y,u,v,)$ is a weighting function in the for the kth cell of the mosaic.

One could use, for example,

$$W_k(x,y,u,v) = 1 \quad \text{if } |2\pi ux| \text{ or } |2\pi uy| < \Phi_{crit}$$

$$= 0 \quad \text{otherwise,}$$

just to save computing time.

Fancier weighting functions could be used at great cost in computer time. The essential need is to not make use of V(u,v) data that has been too severely contaminated by contributions from data where non-linear phase terms are important.

Independent of the problem currently under discussion, this procedure seems to me to be the only way one should combine data on the many fields of view needed to cover a large source.

NO COOLEY-TUKEY FFT

Unfortunately, there is no way in which this mosaic mapping method can be used in conjunction with the Cooley-Tukey FFT, since there is no way to apply (x,y) weighting before dumping the data into the magic FFT box. However, it can be done by direct transform and any other algorithm one can come up with whereby (x,y) weighting can be applied.

Even single channel mapping should use the weighting and transform method just discussed if very high resolution data is involved.

COST IN COMPUTING TIME

The extra computing time goes both into handling N times as much data, but also in calculating many values of $W_k(x,y,u,v)$ (although most will be zero) and then calculating

$$\sum_{k=1}^{N} W_k(x,y,u,v)$$

and

$$\sum_{k=1}^{n} V_k(u,v)\ W_k(x,y,u,v)$$

for each (u,v) point, before applying the transform. In addition, there will probably be the price of the time it takes to calculate direct transforms as opposed to the FFT.

Obviously the computer load will be much more similar to that envisioned for the line VLA than that planned for the first VLA continuum system.

AN OPINION

In my opinion it will be necessary to map large portions of the antenna beams for some survey problems and all problems involving large (>1') sources. Therefore the existence of the problem of non-linear phase terms for high resolution data makes it even more imperative than before to begin as soon as possible on planning for at least a 25 or 36 channel system that would also be the beginnings of a VLA line system.

HIGH RESOLUTION LINE WORK

We had better hope that no one will ever want to do really high resolution line work with the VLA.

Obviously each line channel will have a mappable field of view with the same limitations as the continuum channels independent of bandwidth. This is no problem for baselines of less than 3 km, which is well within what is expected for line work. If one did want higher resolution data one would have to use something like the mosaic approach discussed above. To use the mosaic approach with N mosaic cells and M line frequency channels would mean $M \cdot N$ total channels. To get high resolution 10' maps for 100 line channels would require roughly 10^4 channels. Let us hope that really high resolution line work never becomes scientifically interesting.

FUNDAMENTAL LIMITATIONS

The non-linearity of $\underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o)$ is a fundamental limitation of interferometry. I believe we must develop methods of radio interferometry in the non-linear phase regime.

It would be nice if someone could find a coordinate system in which either the phase was linear, or else the non-linear terms become important at a higher level of baseline separations.

APPENDIX

NON-LINEAR PHASE TERMS FOR THE

NRAO INTERFEROMETER

δ_o	$\Phi_{\text{non-linear}}$ (2700m., x=HPBW/2)		$\Phi_{\text{non-linear}}$ (2700m., x=HPBW/4)	
	3.7 cm.	11 cm.	3.7 cm.	11 cm.
-30°	10°	30°	3°	8°
0°	16°	49°	4°	12°
30°	28°	85°	7°	21°
45°	41°	122°	10°	31°
60°	71°	213°	18°	53°
80°	376°	1129°	94°	282°

POST-SCRIPT

Let us ask the simplest of all questions. What do the non-linear phase terms do to the map of a point souce?

Assume a point source of flux density S is located at a position x_s, y_s, then the true intensity map should appear to be

$$I(x,y) \;=\; S\, f(x_s,y_s)\, \delta(x-x_s)\;\; \delta(y-y_s)\;.$$

Since

$$V(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y)\, \exp\left\{2\pi i\left[(ux + vy) - \frac{x^2}{2}\,(D_o - B_z\, \sin\, \delta_o)/\cos^2\,\delta_o - \frac{y^2}{2}\, D_o - x\, y\, u\, \tan\, \delta_o\right]\right\} dudv$$

this means that for this case

$$V(u,v) = S\, f(x_s,y_s)\, \exp\left\{2\pi i\left[(ux_s + vy_s - \frac{x_s^2}{2}\,(D_o - D_z\, \sin\, \delta_o)/\cos^2\,\delta_o - \frac{y_s^2}{2}\, D_o - x_s\, y_s\, u\, \tan\, \delta_o\right]\right\}.$$

If we then attempt to "map" with this visibility function "data"

$$I(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(u,v) e^{-2\pi i(ux + vy)} dudx$$

$$= S\, f(x_s,y_s) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{2\pi i\left[u(x_s - x - x_s y_s\, \tan\, \delta_o + v(y_s - y) - \frac{x_s^2}{2}\,(D_o - B_z\, \sin\, \delta_o)/\cos^2\,\delta_o - \frac{y_s^2}{2}\, D_o\right]\right\} dudv$$

without the extra terms we retrieve

$$I(x,y) \;=\; S\, f(x_s,y_s)\;\delta(x-x_s)\;\delta(y-y_s)$$

but with them

$$I(x,y) = S\, f(x_s,y_s)e^{-\pi i[x_s^{\,2}\,(D_o-B_z\,\sin\,\delta_o)/\cos^2\,\delta_o-y_s^{\,2}\,D_o]}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i[u(x_s-x-x_s y_s\,\tan\,\delta_o)\,+\,v(y_s-y)]}\,dudv$$

$$\boxed{I(x,y) = S\, f(x_s,y_s)\delta(y_s-y)\delta(x_s-2x_s y_s\,\tan\,\delta_o-x)e^{-\pi i[x_s^2(D_o-B_z\sin\delta_o)/\cos^2\delta_o-y_s^2\,D_o]}}$$

First of all, as we might have expected from the non-Hermitian nature of the V(u,v), the map is not real to the extent that the non-linear terms in $x_s^{\,2}$ and $y_s^{\,2}$ contribute a significant phase, and secondly the position of the source will be shifted by

$$\Delta x \;=\; 2\,x_s\,y_s\,\tan\,\delta_o$$

in the wrong direction.