VLA SCIENTIFIC REPORT NO. 107

June 13, 1973

CURVATURE OF THE SKY

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In VLA Scientific Report 106, Hjellming has overestimated the impact of the second order terms in the brightness-spatial frequency relationship. That report may mislead in two directions, implying:

(1) that the effect is a fundamental limitation of interferometry, and

(2) that the effect diverges toward the celestial pole, so that it will cause difficulties at high declinations no matter what circumstances prevail.

Neither is correct. The effect will probably take a small increased increment of computer time to handle properly, and will be a nuisance to explain to outside observers and to maintenance programmers, but need never limit us, at least in the continuum case.

The effect arises because the world is round (Magellan, 1522). We arbitrarily assign all radiation we see to the surface of a unit sphere we call the celestial sphere, so that the emission is touched by the unit source vector, $\underset{\sim}{s}$. An element of emission is recorded at our interferometer with phase

$$\phi = \omega \, \underset{\sim}{B} \cdot \underset{\sim}{s}$$

where B is the baseline in time-of-flight units.

The lobe rotator and delay line reduce the phase observed by a monochromatic interferometer by $\omega \, \underset{\sim}{B} \cdot \underset{\sim}{s}_o$.

$$\phi = \omega \, \underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o) \ .$$

The response of the interferometer to the emission at the vector location $\underset{\sim}{s}$ is therefore the familiar Fourier transform

$$V = \iiint E \, (\underset{\sim}{s}) \exp i \, \omega \left\{ \underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o) \right\} dV$$

where the emission E is zero everywhere but on the surface of the unit sphere.

Given various measurements of V, we can reconstruct the principle solution for E in the familiar inverse transform

$$E = \iiint V(\underset{\sim}{B}) \exp - i\omega \left\{ \underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o) \right\} dV_B$$

where the dV_B indicates the integration is to be carried out over spatial frequency (baseline).

We may then insert our belief that the radio sources are in the far field of the instrument by ignoring E at points other than on the unit sphere.

This is a complete and concise theoretical solution to the problem. It may or may not be the most convenient practical solution.

To talk about practical solutions, we should look in more detail at the conventional development involving (u,v) which is always used as a computational aid.

In this viewpoint, we set up a rectangular coordinate system at the point $\underset{\sim}{s}_o$, with its z axis pointing radially outward and the y axis on a meridional slice of the celestial sphere. Then a point at $(\alpha_o + \Delta\alpha,\ \delta_o + \Delta\delta)$

$$x = \sin\Delta\alpha \cos(\delta_o + \Delta\delta) \simeq \Delta\alpha \cos\delta_o - \Delta\alpha\,\Delta\delta \sin\delta_o$$

$$y = \sin\Delta\delta + (1 - \cos\Delta\alpha)\sin\delta_o \cos(\delta_o + \Delta\delta)$$

$$\simeq \Delta\delta + \frac{1}{2}(\Delta\alpha)^2 \sin\delta_o \cos\delta_o$$

$$z = \cos(\Delta\delta) - 1 - (1 - \cos\Delta\alpha)\cos\delta_o \cos(\delta_o + \Delta\delta)$$

$$\simeq -\frac{1}{2}(\Delta\delta^2 + (\Delta\alpha\cos\delta)^2).$$

Then with the usual definitions of u,v and w, w being the delay,

$$E = \iiint V(u,v,\phi) \exp - i\omega \left\{ ux + vy + zw \right\} du\, dv\, dw\ .$$

Note that z depends only upon the squares of the deviations from the center of the field of view, and the curvature effect is of the same order at all points on the celestial sphere, as one would expect, since it arises from the curvature of the unit sphere.

Note also that the region of interest in z is very small; therefore the quantization in w may be very coarse, and the Fourier transform need only be perhaps two or four units deep in this direction.

Wim Brauw discovered the remarkable fact that with an east-west baseline there exists a coordinate transformation which removes the effect entirely. It is clear that there is no equivalent transformation in the general case, as this is the problem of mapping a sphere without distortion, but transformations to handle higher order terms in (ωwz) may exist.

If we must do the three dimensional transform, there may well be more economical procedures than the simple-minded three dimentional FFT. Since we know that, in the end, we are going to keep only one surface in the output parallel piped, we can for instance gain significantly by doing a DFT in the third dimension, rather than an FFT.

If source sidelobe noise introduced by aliasing is important, a simple mosaic of fields of view is an attractive way to do things. That is, after processing the reduced field of view the data is moved to a new field center by making a phase correction

$$V' = V \exp i\, \omega\, \underset{\sim}{B} \cdot (\underset{\sim}{s}_o - \underset{\sim}{s}_o')$$

and recalculating u and v (only necessary to account of terms of order x^3) by means of a rotation matrix acting on (u,v,w). This procedure has the same, or slightly less, FFT calculation time than acting as if the world were flat, but it does introduce the aliased sidelobe noise. All the reasonable techniques for surpressing it seem to cost a factor of two in computer time which will make the method unattractive for many applications.

In summary, the problem of curvature of the sky is a simple and comprehensible one, and several avenues of handling it are available in the practical case. The various methods differ by factors of order two in required computer time, and the final choice of which one or which ones will be implemented requires further study.