# VLA SCIENTIFIC REPORT NO. 109

## SUMMARY OF MAPPING PROBLEM WHEN PROJECTION EFFECTS ARE IMPORTANT

R. M. Hjellming

Purpose: To attempt to summarize the consensus of the previous discussions of the problem of obtaining two dimensional maps of observed radiation "on" the sky (the celestial sphere).

### Projecting Maps on Curve Surfaces on to Planes.

In the most general terms the observed visibility functions are related to the apparent intensity distribution on the sky by

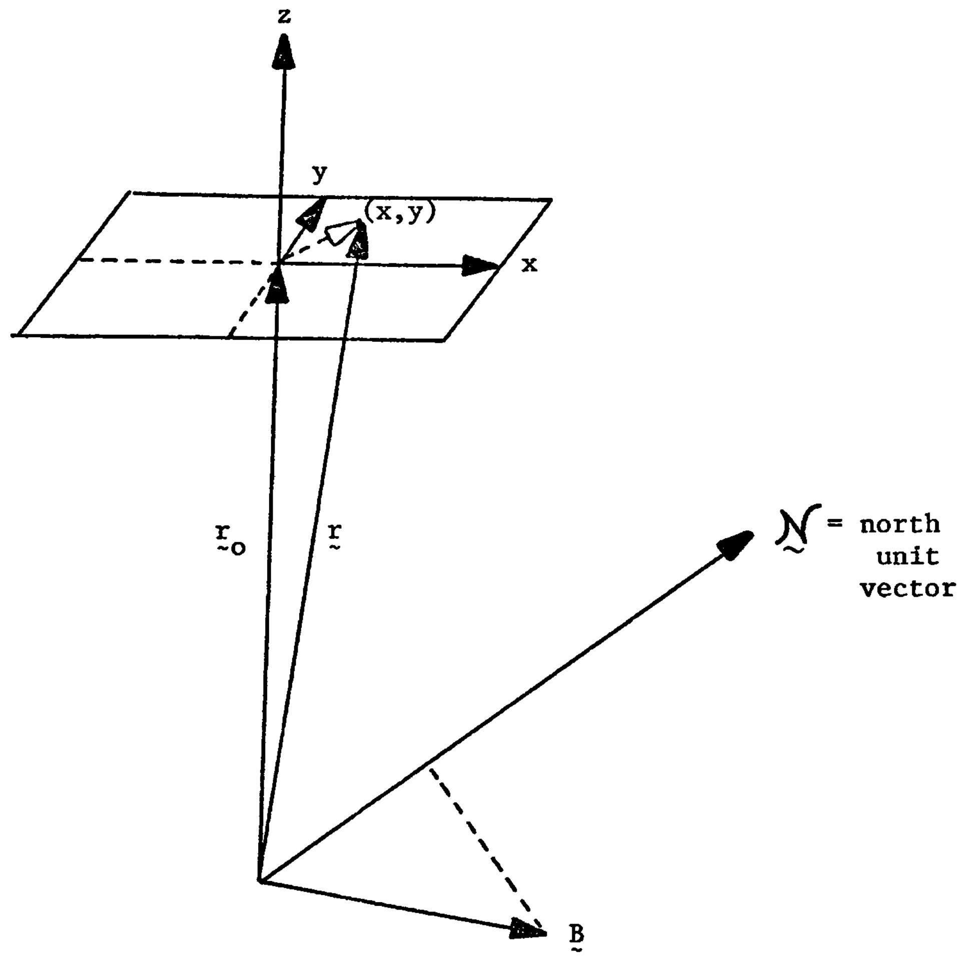
$$V(u,v,w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(\alpha,\delta) e^{2\pi i \underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o)} \, d\alpha \, d\delta \qquad (1)$$

where the variables have the usual definitions, except we now explicitly recognize the three-dimensional character of the baselines by introducing $w = D_o = \underset{\sim}{B} \cdot \underset{\sim}{s}_o$. As discussed in reports No. 106, 107, and 108, if one attempts to force $\underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o)$ into a form linearly dependent upon only two independent spatial variables, say x and y, non-linear terms in x and y are neglected which can cause errors whenever high resolution VLA data, with band-widths less than about 50 MHz, are mapped. The problems that are then introduced are exactly the projection problems involved in mapping portions of any spherical surface, as for example the surface of the earth, a problem which has never been completely solved in the sense that no simple elegant solution exists. The limitations occasionally described as fundamental in report No. 106 are no more than the fundamental limitations involved in projecting three dimensions onto two dimensions. These limitations are, of course, removed by dealing with a completely three-dimensional problem, as indeed mapping the surface of the earth onto a globe is no problem. As Clark and Wade quite correctly point out in reports 107 and 108 the transform phase [$= 2\pi \underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o)$] is linear in three dimensions. Hence whenever VLA data is gathered with both high resolution and band-width less than 50 MHz, we will be working in a three-dimensional context where projection limitations are not fundamental to the interferometry, but rather force us to more complicated computing schemes and a much more complicated conceptual problem.

Before beginning a discussion of the three-dimensional problem, let us summarize the discussion of the two dimensional problem and its limitations.

## Two Dimensional Formulation.

Jerry Hudson and Campbell Wade have independently derived a two dimensional coordinate system on the sky which somewhat simplifies the two dimensional problem. Because of its simplicity we will at first follow the derivation as formulated by Hudson.




In the preceding figure the x-y coordinate system is assumed to be <u>exactly</u> in the plane of the sky at the point defined by the unit vector $\underset{\sim}{r}_o$, and where each (x,y) point is associated with a vector $\underset{\sim}{r}$ given by

$$\underset{\sim}{r} = (1 + x^2 + y^2)^{1/2} \begin{pmatrix} \cos\delta \cos H \\ -\cos\delta \sin H \\ \sin\delta \end{pmatrix}_{LH}$$

which differs from the usual unit vector $\underset{\sim}{s}$ only by the factor $(1 + x^2 + y^2)^{1/2}$, and $\underset{\sim}{r}_o = \underset{\sim}{s}_o$. As always, $(\alpha_o, \delta_o)$ is the position in the sky being tracked by the interferometer.

We can now re-write equation (1) in the following form

$$V(u,v,w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y) e^{2\pi i(\underset{\sim}{B}\cdot\underset{\sim}{r}/r - \underset{\sim}{B}\cdot\underset{\sim}{r}_o)} \, dx \, dy \tag{2}$$

where we have denoted $r = |\underset{\sim}{r}|$.

Now

$$u = \underset{\sim}{B}\cdot\hat{e}_x$$
$$v = \underset{\sim}{B}\cdot\hat{e}_y$$
$$w = \underset{\sim}{B}\cdot\underset{\sim}{r}_o$$

and

$$\underset{\sim}{r} = \underset{\sim}{r}_o + x\,\hat{e}_x + y\,\hat{e}_y$$

where $\hat{e}_x$ and $\hat{e}_y$ are the unit vectors for the x-y coordinates. Hence

$$\frac{\underset{\sim}{B}\cdot\underset{\sim}{r}}{r} = \frac{(\underset{\sim}{B}\cdot\underset{\sim}{r}_o + x\,\underset{\sim}{B}\cdot\hat{e}_x + y\,\underset{\sim}{B}\cdot\hat{e}_y)}{(1 + x^2 + y^2)^{1/2}}$$

$$= \frac{w + ux + vy}{(1 + x^2 + y^2)^{1/2}}$$

which is an exact expression. Now, expanding this expression to second order in x and y, and subtracting $\underset{\sim}{B}\cdot\underset{\sim}{r}_o$,

$$\frac{\underset{\sim}{B}\cdot\underset{\sim}{r}}{r} - \underset{\sim}{B}\cdot\underset{\sim}{r}_o = ux + vy - \frac{1}{2} w\,(x^2 + y^2).$$

Thus, to second order in x and y in the transform phase, we can write

$$V(u,v,w) \cong \int_{\infty}^{\infty} \int_{\infty}^{\infty} I(x,y) e^{2\pi i[ux + vy - \frac{1}{2} w (x^2 + y^2)]} dx\, dy \; . \quad (3)$$

In the new x-y coordinate system the non-linear terms (in x and y) in the transform phase, which we will denote by $\Delta\Phi$, are given by the simple form

$$\Delta\Phi = \pi w (x^2 + y^2) \; .$$

In the coordinate system defined by

$$x' = (\alpha - \alpha_o) \cos\delta$$

$$y' = \delta - \delta_o$$

as shown in report No. 106,

$$\Delta\Phi = \pi (x')^2 (w - B_z \sin \delta_o)/\cos^2 \delta_o + \pi w (y')^2 + 2\pi x'y'u \tan \delta_o$$

which has nasty divergence properties at high declinations. Note that the two formulas for $\Delta\Phi$ agree at $\delta_o = 0$.

Thus the new x-y system considerably simplifies the equations of the problem, though the general order of magnitude of $\Delta\Phi$ is the same.

If we wish to evaluate the order of magnitude of $\Delta\Phi$ under different conditions, we can take

$$w \sim B/\lambda$$

where B is the baseline length and $\lambda$ is the observing wavelength. If we define

$$\Theta^2 = x^2 + y^2$$

then

$$\Delta\Phi = \cdot\pi \; w \; \Theta^2 \cong \frac{\pi B}{\lambda} \Theta^2 ,$$

so that

$$\boxed{\Delta\Phi \simeq \frac{53°}{\lambda_{cm}} \left(\frac{B_{km}}{35}\right) \Theta^2_{arcmin}} \; . \qquad (4)$$

If one adopts the standard that

$$\Delta\Phi \lesssim 20°$$

then one is limited to keeping

$$\Theta_{arcmin} \lesssim \left[\frac{20 \; \lambda_{cm}}{53(B_{km}/35)}\right]^{\frac{1}{2}}$$

or

$$\boxed{\Theta_{arcmin} \lesssim 0{:}61 \; \lambda_{cm}^{\frac{1}{2}} \; (35/B_{km})^{\frac{1}{2}}} \qquad (5)$$

for a "field of view" where the transform phase depends only on linear terms in x and y.

For the VLA at maximum resolution ($B_{km} = 35$)

$$\Theta_{arcmin} \lesssim 0{:}61 \; \lambda_{cm}^{\frac{1}{2}}$$

is the region in which the transform phase is linearly dependent only on x and y.

As long as the delay beam is the limiting factor for the sensitivity region of the VLA, since

$$(\mathrm{HPBW})_{\mathrm{delay}} \sim 1\overset{'}{.}3 \left(\frac{45}{\Delta\nu_{\mathrm{MHz}}}\right)\left(\frac{35}{B_{\mathrm{km}}}\right)$$

where $\Delta \nu_{\mathrm{MHz}}$ is the band-width, then

$$\Theta_{\mathrm{max}} \sim 0\overset{'}{.}65 \left(\frac{45}{\Delta\nu_{\mathrm{MHz}}}\right)\left(\frac{35}{B_{\mathrm{km}}}\right)$$

therefore for $\Delta\nu_{\mathrm{max}} \gtrsim 45$ the delay beam, and the region in which $\Delta\Phi$ is negligible, are the same so it is sufficient to assume

$$V(u,v) \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y)e^{2\pi i(ux + vy)}\,dx\,dy \tag{6}$$

so one can perform the two-dimensional Fourier inversion

$$I(x,y) \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(u,v)e^{-2\pi i(ux + vy)}\,dx\,dy\ . \tag{7}$$

<u>When is Two-Dimensional Fourier Inversion Inadequate?</u>

The value of $\Delta\Phi$ can be approximated by equation (4),

$$\Delta\Phi \approx \frac{53^\circ}{\lambda_{\mathrm{cm}}}\left(\frac{B_{\mathrm{km}}}{35}\right)\Theta^2_{\mathrm{arcmin}}\ . \tag{4}$$

Now, as just discussed, whenever the delay beam width is smaller than the antenna beam width

$$\Theta^{\mathrm{delay}}_{\mathrm{max}} \sim 0\overset{'}{.}65 \left(\frac{45}{\Delta\nu_{\mathrm{MHz}}}\right)\left(\frac{35}{B_{\mathrm{km}}}\right)$$

however, whenever the antenna beam width is smaller than the delay beam width, we can take

$$\Theta_{max}^{antenna} \sim 0\overset{'}{.}85\ \lambda_{cm}\ .$$

Therefore, depending upon which regime is applicable

$$\Delta\Phi^{max} \approx \frac{22^\circ}{\lambda_{cm}} \left(\frac{35}{B_{km}}\right)\left(\frac{45}{\Delta\nu}\right)^2 \quad \text{when} \quad \Theta_{max}^{delay} < \Theta_{max}^{antenna} \tag{8}$$

or

$$\Delta\Phi^{max} \approx 38^\circ\ \lambda_{cm} \left(\frac{B_{km}}{35}\right) \quad \text{when} \quad \Theta_{max}^{delay} > \Theta_{max}^{antenna} \tag{9}$$

Note that

$$\Theta_{max}^{delay} = \Theta_{max}^{antenna}$$

when

$$\lambda_{cm}\left(\frac{B_{km}}{35}\right) = 0.76\left(\frac{45}{\Delta\nu_{MHz}}\right)$$

and at this critical point

$$\boxed{\Delta\Phi^{max} \sim 27^\circ \left(\frac{45}{\Delta\nu_{MHz}}\right)} \tag{10}$$

so that for any band-width $\Delta\nu$, equation (10) gives the largest values of $\Delta\Phi$ that can arise.

From equations (8)-(9) we see that $\Delta\Phi^{max}$ is a function of only the product $\lambda_{cm} \cdot B_{km}$ when the primary beam is smaller, but is a function of both the product $\lambda_{cm} \cdot B_{km}$ and $\Delta\nu$ when the delay beam is smaller.

To allow the reader to evaluate the maximum magnitude of $\Delta\Phi$ for any combination of $\lambda$, B, and $\Delta\nu$ the effects of equations (8)-(9) are plotted in Figure 1. By choosing $\Delta\nu$ and a value of $\lambda$ one can immediately tell whether, for a particular B, the delay beam or primary beam limits $\Delta\Phi^{max}$, and one can see what the associated value for $\Delta\Phi^{max}$ will be.

We note from Figure 1 that for $\Delta\nu$ = 45 MHz the value of $\Delta\Phi^{max}$ never becomes too large. However, for $\Delta\nu$ = 3 MHz and B = 35 km, it attains 77° for 2 cm, 230° for 6 cm, and 432° for 11 cm.

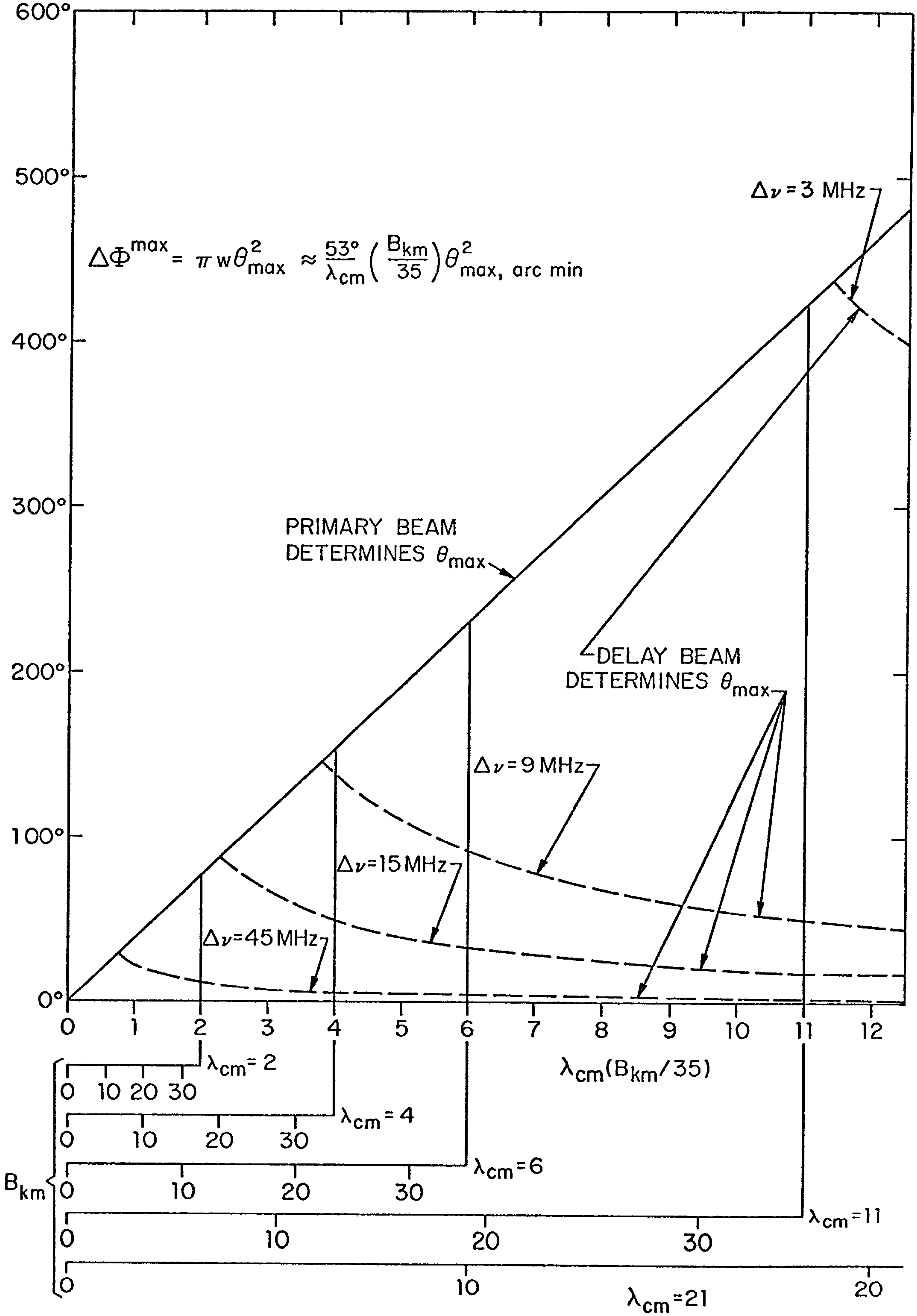
As has been noted by everybody, for anything less than the full band-width VLA one will need to adopt something more complicated than the simple two dimensional Fourier inversion to map high resolution data out to the HPBW.

## A Question of Logical Order.

It is worth imbarking upon a digression to point out something of importance. Observers have frequently been mapping high declination sources without encountering the "divergences" inherent in the "old" x-y coordinate system. This is because of the true logical order of events.

A real interferometer observes the sky and obtains measured visiblity functions according to

$$V(u,v,w,) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} I(\alpha,\delta) e^{2\pi i \underset{\sim}{B} \cdot (\underset{\sim}{s} - \underset{\sim}{s}_o)} \, d\alpha \, d\delta \ ,$$



600°
500°
400°
300°
200°
100°
0°
$\Delta\Phi^{max} = \pi w \theta_{max}^2 \approx \frac{53°}{\lambda_{cm}}\left(\frac{B_{km}}{35}\right)\theta^2_{max,\ arc\ min}$
$\Delta\nu = 3$ MHz
PRIMARY BEAM DETERMINES $\theta_{max}$
DELAY BEAM DETERMINES $\theta_{max}$
$\Delta\nu = 9$ MHz
$\Delta\nu = 15$ MHz
$\Delta\nu = 45$ MHz
0 1 2 3 4 5 6 7 8 9 10 11 12
$\lambda_{cm}(B_{km}/35)$
$B_{km}$
$\lambda_{cm} = 2$
$\lambda_{cm} = 4$
$\lambda_{cm} = 6$
$\lambda_{cm} = 11$
$\lambda_{cm} = 21$


Figure 1

exactly, irrespective of the fact that we may have problems handling spherical surfaces in two dimensions. The next step in the logical order of events is our attempt to reconstruct $I(\alpha,\delta)$ from the measurements of $V(u,v,w)$.

If we perform two-dimensional Fourier inversion according to

$$I(x,y) \cong \int_{\infty}^{\infty} \int_{\infty}^{\infty} V(u,v)e^{-2\pi i(ux + vy)}\, du\, dv$$

to obtain maps in "the" x-y plane, it is clear that maps obtained in this way are automatically in the "new" x-y coordinate system in which divergences at high declinations do not occur. One would make significant errors only if one rigidly transformed the $(x,y)$ maps to $(\alpha,\delta)$ using $x = (\alpha-\alpha_o) \cos\delta_o$ and $y = \delta-\delta_o$.

## The General Three Dimensional Problem.

There is no question about the fact that interferometers successfully Fourier transform the apparent intensity distribution on the sky such that

$$V(u,v,w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(\alpha,\delta)e^{2\pi i \underset{\sim}{B}\cdot(\underset{\sim}{s}-\underset{\sim}{s}_o)}\, d\alpha\, d\delta \qquad (1)$$

where we must now be very careful to note that $V$ really depends upon $u,v$ and $w$.

As shown by Wade in report No. 108, one can exactly represent

$$\underset{\sim}{B} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \sin H_o & -\cos H_o & 0 \\ -\sin \delta_o \cos H_o & -\sin \delta_o \sin H_o & \cos \delta_o \\ \cos \delta_o \cos H_o & \cos \delta_o \sin H_o & \sin \delta_o \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \quad (11)$$

and

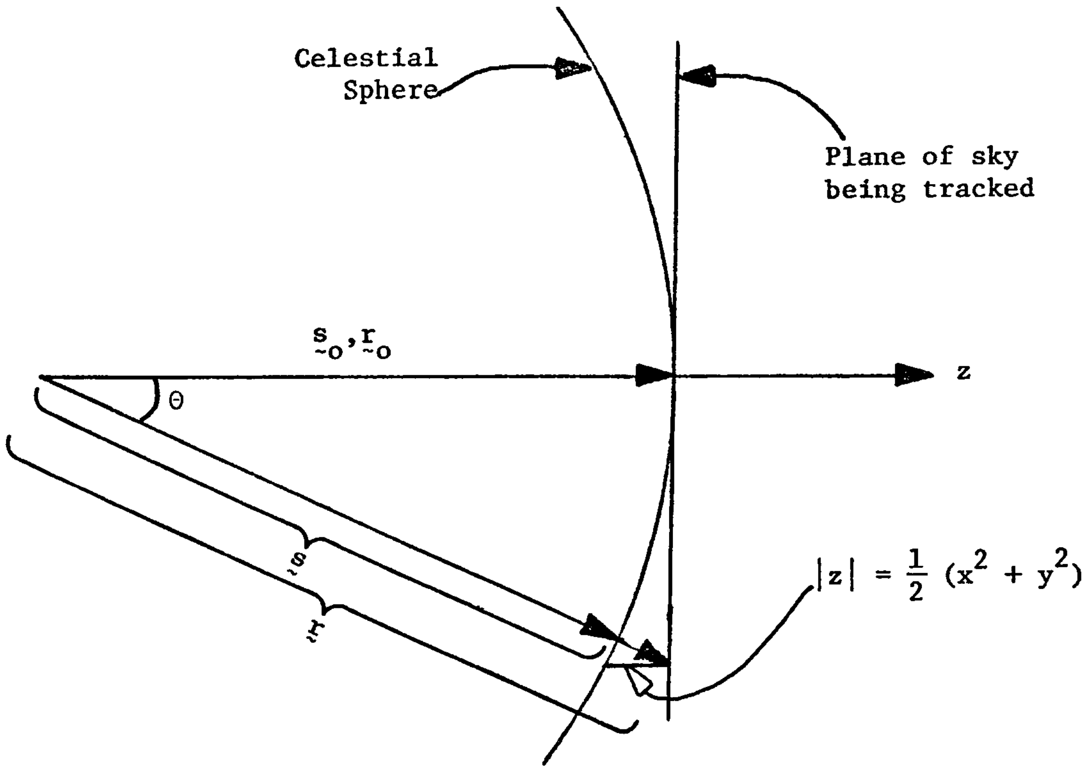
$$\underset{\sim}{s}-\underset{\sim}{s}_o = \frac{\tilde{r}}{r} - \tilde{r}_o = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \delta \sin(\alpha-\alpha_o) \\ -\sin \delta_o \cos \delta \cos(\alpha-\alpha_o) + \cos \delta_o \sin \delta \\ \cos \delta_o \cos \delta \cos(\alpha-\alpha_o) + \sin \delta_o \sin \delta - 1 \end{pmatrix} \quad (12)$$

the latter of which, though not pretty, is exact. Using the x-y-z coordinate system defined by equation (12), equation (1) then is <u>exactly</u> represented by

$$\boxed{V(u,v,w) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} I(x,y,z) e^{2\pi i(ux + vy + wz)} dx\, dy\ .} \quad (13)$$

Equation (13) contains the condition that z is explicitly dependent upon x and y. As can be gathered from previous discussions, an approximate expression for the relation between z, and x and y, that will suffice for all practical purposes is

$$\boxed{z \cong -\frac{1}{2} (x^2 + y^2)}\ . \quad (14)$$




The above figure shows how z fits into the coordinate system. Obviously, when $\Theta$ is small

$$z \doteq (\underset{\sim}{r} - \underset{\sim}{r}_o)_z$$

to very high accuracy, sufficient at least for VLA usage.

Three-Dimensional Fourier Inversion.

The Fourier inversion of equation (13) contains one bothersome problem. The original $(\alpha,\delta)$ coordinate system contained only two independent coordinates, and in equation (13) only x and y are independent coordinates. However, the general three-dimensional

Fourier inversion equation gives

$$I(x,y,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(u,v,w)e^{-2\pi i(ux + vy + wz)} dudvdw. \quad (15)$$

Although I have written it down, and I will proceed on the assumption that it is valid, the lack of reciprocity in the number of independent variables bothers me.

The practical problem of evaluating equation (15) by direct transform is straightforward and allows one to easily evaluate I only for the z of interest, that is

$$I[x,y,z= -\frac{1}{2}(x^2+y^2)] = \frac{\sum_{data} W(u,v,w,)\ V(u,v,w,)e^{-2\pi i[ux+vy - \frac{1}{2}w(x^2+y^2)]}}{\sum_{data} W(u,v)}$$

explicitly including only the sampled data points. That is, non-sampled u,v,w cells are assumed to have V = 0.

If one carries out a Cooley-tukey FFT inversion one actually processes an $N_u \times N_v \times N_w$ network of data points (though most are zero) to obtain a three dimensional map of I(x,y,z), where in general $z \neq -\frac{1}{2}(x^2 + y^2)$. One then must reconstruct the map of radiation on the celestial sphere by solving (by interpolation) for

$$I(\alpha,\delta) = I[x(\alpha,\delta),\ y(\alpha,\delta),\ z\ (\alpha,\delta)]\ .$$

Either mapping approach will be costly in computing time. We are going to have to put a considerable amount of work into considering these practical problems and various alternative computing schemes.