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Some Relationships Between Delay Effects and Array Beam

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This report consists of three sections. In the first is derived an elegant but not very useful relationship between the array beam and the surpression of the peak of a point source response by delay effects. In the second is derived an interesting but practically unusable expression for the surpression of mean sidelobes of distant sources. In the third, an analytic example, sufficiently abstract to be inapplicable to the real world, is worked out.

I. Surpression of the beam peak

While it is not strictly accurate to speak of a "delay beam" as the effect is not a convolution, we can use that term to refer to the amount a point source is surpressed in peak amplitude (in some other sense, say rms or main beam power, it will be surpressed by a different amount) by delay effects. At time j the response of a monochromatic interferometer (projected baseline u,v) to a point source at x_o, y_o is

$$A_m = \exp\{-i\omega(u_j x_o + v_j y_o)\} . \tag{1}$$

(Curvature problems are ignored.)

If we split ω into a center frequency ω_o and an offset ω_1, we can describe the response of the wideband interferometer by an integral over the passband $P(\omega_1)$.

$$A = \int P(\omega_1) \exp\{-i(\omega_o + \omega_1)(u_j x_o + v_j y_o)\} d\omega_1 . \tag{2}$$

We now form an array beam by fourier transforming our numbers A, each with weight w, using the kernel $\exp\{i\ \omega_o(u_j\ x + v_j\ y)\}$ for our transform, getting a map B.

$$B(x,y) = \sum_j w_j \exp\{i\ \omega_o(u_j\ x + v_j\ y)\} \int P(\omega_1)\exp\{-i(\omega_o + \omega_1)(u_j\ x_o + v_j\ y_o)\}d\omega_1$$

$$= \sum_j \exp\{i\ \omega_o(u_j(x-x_o) + v_j(y-y_o))\} \int w_j\ P(w_1)\exp\{-i\ \omega_1(u_j\ x_o + v_j\ y_o)\}d\omega_1 . \quad (3)$$

If we have chosen our center frequency ω_o properly, the peak of the map will be at $x = x_o$, $y = y_o$

$$B(x_o,y_o) = \int P(\omega_1) \sum_j w_j \exp\{-i\ \omega_1(u_j\ x_o + v_j\ y_o)\}d\omega_1 . \quad (4)$$

The sum may be written down explicitly in terms of the monochromatic array beam, defined as

$$B_o(x,y) = \sum_j w_j \exp\{i\ \omega_o(u_j\ x_o + v_j\ y_o)\} . \quad (5)$$

Therefore, the peak response is

$$B(x_o,y_o) = \int P(\omega_1)\ B_o\ (\frac{\omega_1}{\omega_o} x_o, \frac{\omega_1}{\omega_o} y_o)\ d\omega_1 . \quad (6)$$

This equation gives an extraordinarily simple and general relationship between the "delay beam" and the array beam.

If $P(\omega_1)$ is a square bandpass, i.e.

$$P(\omega_1) = \begin{cases} \frac{1}{2\pi\nu} & -\pi\nu < \omega_1 < \pi\nu \\ o & \text{otherwise} \end{cases}$$

$$B(x_o,y_o) = \frac{\omega_o}{\pi r_o \nu} \int_o^{\frac{\pi\nu}{\omega_o} r_o} B_o(r\cos\phi,\ r\sin\phi)\ dr \quad (7)$$

where

$$r_o \cos \phi = x_o,\ r_o \sin \phi = y_o \ .$$

The expression tends to be dominated, outside the "main delay beam" by the $\frac{1}{r_o}$ term in front of the integral.

II. Surpression of rms sidelobes

The rms sense of "delay beam" is also worth considering. However, the only case I know of that is amenable to calculation requires additional assumptions. These are 1) square bandpass, and 2) "natural" weighting ($w_j = 1$). We take off directly from equation (2), putting in a square bandpass.

$$A = \frac{\sin \frac{\xi(u_j \cos \phi + v_j - \sin \phi)}{2}}{\frac{\xi(u_j \cos \phi + v_j \sin \phi)}{2}} \exp\{-i\ \omega_o(u_j\ x_o + v_j\ y_o)\} \tag{8}$$

where $\xi = 2\pi r_o \nu$. Take the modulus,

$$AA^* = \frac{2 - 2 \cos[\xi(u_j \cos \phi + v_j \sin \phi)]}{\xi^2(u_j \cos \phi + v_j \sin \phi)^2} \tag{9}$$

which may be put into the form

$$AA^* = \frac{2}{\xi^2} \int_o^{\xi}\int_o^{\eta} \cos\,[\rho\ (u_j \cos \phi + v_j \sin \phi)]\ d\rho\ d\eta \ . \tag{10}$$

With unit weights, the mean square level is given simply by Parseval's theorem by

$$\overline{B^2} = \sum_j AA^*$$

$$= \frac{2}{\xi^2} \int_o^{\xi}\int_o^{\eta} \sum_j \cos \rho\ (u_j \cos \phi + v_j \sin \phi)\ d\rho\ d\eta \tag{11}$$

$$= \frac{2}{\xi^2} \int_o^{\xi}\int_o^{\eta} B_o\ (\frac{\rho}{\omega_o} \cos \phi,\ \frac{\rho}{\omega_o} \sin \phi)\ d\rho\ d\eta \ . \tag{12}$$

The schematic behavior of this integral is sketched in the attached figure. The far field behavior tends to have leading term $\frac{1}{\xi}$ (forms with special properties can modify this, though-see example below), so that the rms sidelobes generated by a source at r_o decrease as $(\nu r_o)^{-1/2}$. This effect is, of course, in addition to the general tapering off of sidelobe levels with distance from the main beam seen even in monochromatic maps, which also tends to go roughly with $r_o^{-1/2}$.

III. An Analytic Example

A simple example which can be easily analysed to sufficient accuracy is given below. Suppose we have a uniform, filled circle of radius u_o in the u,v plane. Then let us ask what is the response at (x = o, y = o) to a source at (x_o, y_o)

$$B(o,o) = \int P(\omega_1) \int \exp \{-i(\omega_o + \omega_1)(u\, x_o + v\, y_o)\}\, dA d\omega_1 \tag{13}$$

$$= \int P(\omega_1)\, \Lambda_1((\omega_o + \omega_1)\, u_o\, r_o)\, d\omega_1 \tag{14}$$

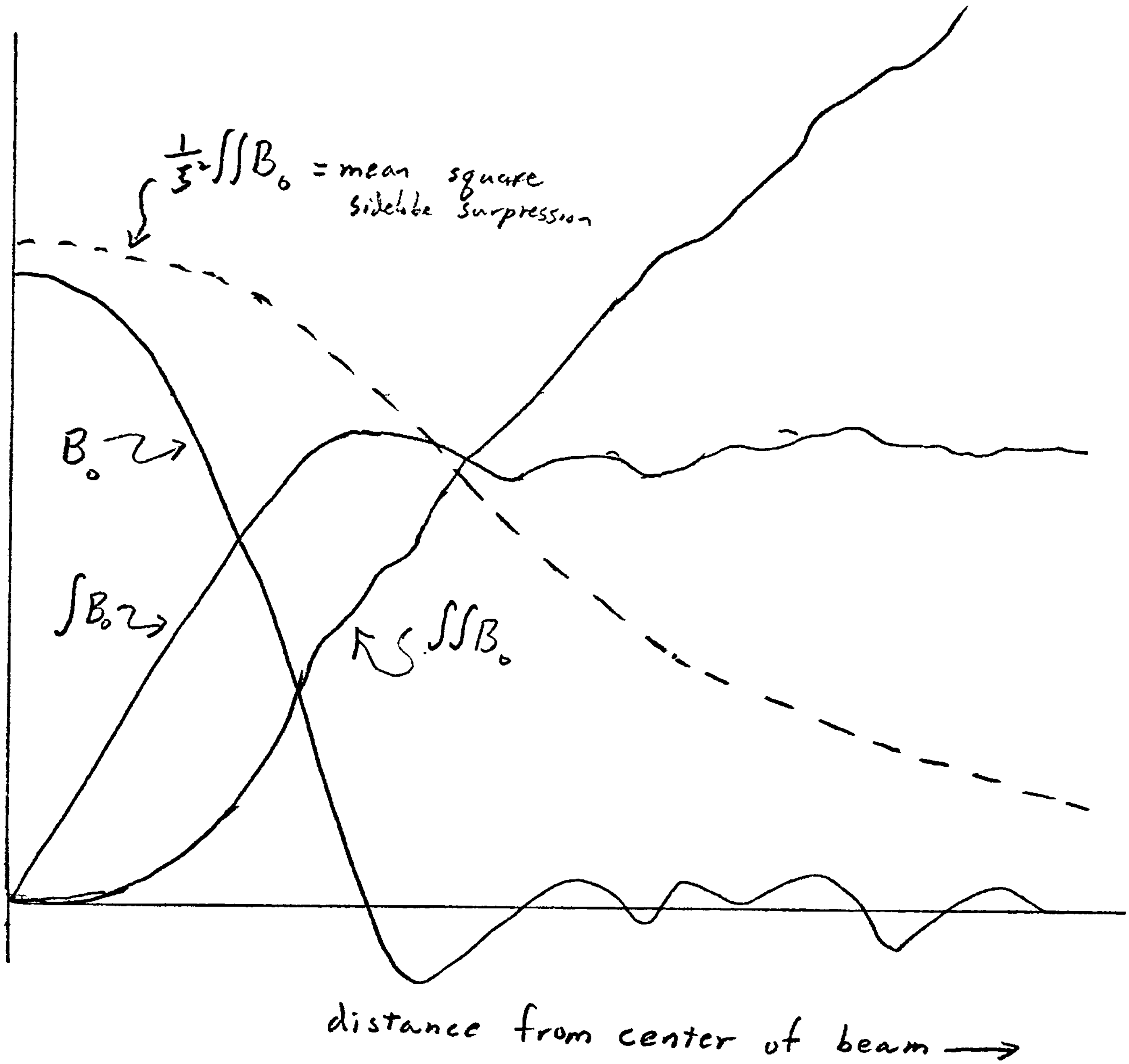
$$\approx \int P(\omega_1) \sqrt{\frac{8}{\pi}}\, \frac{\cos\, [(\omega_o + \omega_1)\, u_o\, r_o - \frac{3\pi}{4}]}{[(\omega_o + \omega_1)\, u_o\, r_o]^{3/2}}\, d\omega_1\ . \tag{15}$$

If the percentage bandwidth is small, we can also neglect the ω_1 with respect to ω_o in the denominator. Inserting the square passband and evaluating the integral gives

$$B(o,o) = \sqrt{\frac{8}{\pi}}\ \underbrace{\frac{\cos\,(\omega_o\, u_o\, r_o - \frac{3\pi}{4})}{(\omega_o\, u_o\, r_o)^{3/2}}}_{\text{Monochromatic Sidelobes}}\ \underbrace{\frac{\sin\,(\pi\nu\, u_o\, r_o)}{\pi\nu\, u_o\, r_o}}_{\text{Delay Beam}}\ . \tag{16}$$

The neglected terms become important when the delay beamwidth divided by r_o is of the same order as the percentage bandwidth.

In this symmetric example, the delay effect is a true multiplicative beam, and the peak and rms sidelobes decrease in a satisfactory way with distance.


$\frac{1}{S^2}\iint B_0$ = mean square sidelobe surpression
B_0
$\int B_0$
$S \iint B_0$
distance from center of beam