This report consists of three sections. In the first is derived an elegant but not very useful relationship between the array beam and the suppression of the peak of a point source response by delay effects. In the second is derived an interesting but practically unusable expression for the suppression of mean sidelobes of distant sources. In the third, an analytic example, sufficiently abstract to be inapplicable to the real world, is worked out.

I. Suppression of the beam peak

While it is not strictly accurate to speak of a "delay beam" as the effect is not a convolution, we can use that term to refer to the amount a point source is suppressed in peak amplitude (in some other sense, say rms or main beam power, it will be suppressed by a different amount) by delay effects. At time \( j \) the response of a monochromatic interferometer (projected baseline \( u, v \)) to a point source at \( x_o, y_o \) is

\[
A_m = \exp \{-i \omega (u_j x_o + v_j y_o)\} .
\]  \hspace{1cm} (1)

(Curvature problems are ignored.)

If we split \( \omega \) into a center frequency \( \omega_o \) and an offset \( \omega_1 \), we can describe the response of the wideband interferometer by an integral over the passband \( P(\omega_1) \).

\[
A = \int P(\omega_1) \exp\{-i(\omega_o + \omega_1)(u_j x_o + v_j y_o)\} \, d\omega_1 .
\]  \hspace{1cm} (2)
We now form an array beam by fourier transforming our numbers $A$, each with weight $w$, using the kernel $\exp\{i \omega_0 (u_j x + v_j y)\}$ for our transform, getting a map $B$.

$$B(x, y) = \sum_j w_j \exp\{i \omega_0 (u_j x + v_j y)\} \int P(\omega) \exp\{-i(\omega_0 + \omega_1)(u_j x_0 + v_j y_0)\} d\omega_1$$

$$= \sum_j \exp\{i \omega_0 (u_j(x-x_0) + v_j(y-y_0))\} \int w_j P(\omega) \exp\{-i \omega_1(u_j x_0 + v_j y_0)\} d\omega_1. \quad (3)$$

If we have chosen our center frequency $\omega_0$ properly, the peak of the map will be at $x = x_0$, $y = y_0$

$$B(x_0, y_0) = \int P(\omega_1) \sum_j w_j \exp\{-i \omega_1(u_j x_0 + v_j y_0)\} d\omega_1. \quad (4)$$

The sum may be written down explicitly in terms of the monochromatic array beam, defined as

$$B_0(x, y) = \sum_j w_j \exp\{i \omega_0(u_j x_0 + v_j y_0)\}. \quad (5)$$

Therefore, the peak response is

$$B(x_0, y_0) = \int P(\omega_1) B_0\left(\frac{\omega_1}{\omega_0}x_0, \frac{\omega_1}{\omega_0}y_0\right) d\omega_1. \quad (6)$$

This equation gives an extraordinarily simple and general relationship between the "delay beam" and the array beam.

If $P(\omega_1)$ is a square bandpass, i.e.

$$P(\omega_1) = \begin{cases} \frac{1}{2\pi\nu} & -\pi\nu < \omega_1 < \pi\nu \\ 0 & \text{otherwise} \end{cases}$$

$$B(x_0, y_0) = \frac{\omega_0}{\pi R_0 \nu} \int_0^{\pi\nu} B_0(r \cos \phi, r \sin \phi) \, dr. \quad (7)$$
The expression tends to be dominated, outside the "main delay beam" by the \( \frac{1}{r_0} \) term in front of the integral.

II. Suppression of rms sidelobes

The rms sense of "delay beam" is also worth considering. However, the only case I know of that is amenable to calculation requires additional assumptions. These are 1) square bandpass, and 2) "natural" weighting \((w_j = 1)\). We take off directly from equation (2), putting in a square bandpass.

\[
A = \frac{\sin \xi(u_j \cos \phi + v_j \sin \phi)}{2} \frac{\exp\{-i \omega_0 (u_j x_0 + v_j y_0)\}}{\xi(u_j \cos \phi + v_j \sin \phi)}
\]

where \( \xi = 2\pi r_0 \nu \). Take the modulus,

\[
AA^* = \frac{2 - 2 \cos[\xi(u_j \cos \phi + v_j \sin \phi)]}{\xi^2(u_j \cos \phi + v_j \sin \phi)^2}
\]

which may be put into the form

\[
AA^* = \frac{2}{\xi^2} \int_{\xi}^{\eta} \int_{\xi}^{\eta} \cos[\rho (u_j \cos \phi + v_j \sin \phi)] \, d\rho \, d\eta.
\]

With unit weights, the mean square level is given simply by Parseval's theorem by

\[
\overline{B^2} = \sum_j AA^*
\]

\[
= \frac{2}{\xi^2} \int_{\xi}^{\eta} \int_{\xi}^{\eta} \cos \rho (u_j \cos \phi + v_j \sin \phi) \, d\rho \, d\eta
\]

\[
= \frac{2}{\xi^2} \int_{\xi}^{\eta} \int_B \frac{\partial}{\partial \omega_0} \cos \phi, \frac{\partial}{\partial \omega_0} \sin \phi \, d\rho \, d\eta.
\]
The schematic behavior of this integral is sketched in the attached figure. The far field behavior tends to have leading term $\frac{1}{\xi}$ (forms with special properties can modify this, though—see example below), so that the rms side-lobes generated by a source at $r_o$ decrease as $(vr_o)^{-1/2}$. This effect is, of course, in addition to the general tapering off of sidelobe levels with distance from the main beam seen even in monochromatic maps, which also tends to go roughly with $r_o^{-1/2}$.

III. An Analytic Example

A simple example which can be easily analysed to sufficient accuracy is given below. Suppose we have a uniform, filled circle of radius $u_o$ in the $u,v$ plane. Then let us ask what is the response at ($x = 0$, $y = 0$) to a source at ($x_q$, $y_q$)

$$B(o,0) = \int P(a) \int \exp \left\{ -i(u_o + \omega_1)(u x_o + v y_o) \right\} dAd\omega_1$$

$$= \int P(\omega) A_1((\omega_o + \omega_1) u_o r_o) d\omega_1$$

$$= \int P(\omega) \sqrt{\frac{8}{\pi}} \frac{\cos \left[ (\omega_o + \omega_1) u_o r_o - \frac{3\pi}{4} \right]}{[ (\omega_o + \omega_1) u_o r_o ]^{3/2}} d\omega_1.$$  \hspace{1cm} (15)

If the percentage bandwidth is small, we can also neglect the $\omega_1$ with respect to $\omega_o$ in the denominator. Inserting the square passband and evaluating the integral gives

$$B(o,0) = \sqrt{\frac{8}{\pi}} \frac{\cos \left( \omega_o u_o r_o - \frac{3\pi}{4} \right)}{[ u_o r_o ]^{3/2}} \frac{\sin (\pi v u_o r_o)}{\pi v u_o r_o}.$$  \hspace{1cm} (16)

The neglected terms become important when the delay beamwidth divided by $r_o$ is of the same order as the percentage bandwidth.

In this symmetric example, the delay effect is a true multiplicative beam, and the peak and rms sidelobes decrease in a satisfactory way with distance.
\( \frac{1}{5} \sum B_o = \text{mean square sidelite suppression} \)

(distance from center of beam →

\( B_o \rightarrow \)

\( \sum B_o \rightarrow \)

\( R \sum B_o \rightarrow \)