

**Relaxing the Isoplanatism Assumption in Self-Calibration;
Applications to Low-Frequency Radio Interferometry**

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1. Introduction. An assumption implicit in the usual scheme for self-calibration of radio interferometer data is one of *isoplanatism*: that over each element of the array, all wavefronts arriving from different parts of the sky to which the interferometer pairs are sensitive are subject to identical tropospheric/ionospheric path delays. Approximate validity of the isoplanatism assumption is a necessary condition for the success of self-calibration. This memorandum is an outline of a means by which the self-calibration algorithm might be modified in order to deal with the anisoplanatic case.

Anisoplanatism is a severe problem with a low-frequency array, such as the one which has been proposed by R. A. Perley and W. C. Erickson [8] for construction at the VLA site. This is because of the extreme magnitude of ionospheric effects at long wavelengths, and the large field of view of such an instrument. An initial attempt at a scheme for self-calibration of low-frequency array data is outlined in Perley and Erickson's proposal; and the need for a generalization of the self-calibration algorithm is reiterated in [2] and [4].

In § 2 is described a method of incorporating an interpolation formula in the self-calibration solution for antenna phases. The idea is to express the phase corruption seen by a given array element, in an arbitrary direction, as a linear combination (i.e., as an interpolation) of the phase corruptions $\{f_i\}_{i=1}^m$ toward the centers of some small number m of "isoplanatic patches". Setting $m = 5$ to 20, or so—with the patches judiciously centered—might suffice in a typical instance. When the source model used for self-calibration is given by a set of CLEAN point source components, it is easy to modify the solution scheme so as to yield the f_i . Choice of an appropriate interpolation formula is discussed in § 3.

Having obtained from the self-calibration solution algorithm a set of n space-variant antenna phases, one for each antenna, the next problem is finding a way to make use of this information in mapping. The usual mapping/deconvolution schemes, such as Fourier inversion combined with CLEAN or with the maximum entropy deconvolution algorithm, are not designed to cope with space-variant effects. A means of utilizing the space-variant antenna phases in a modified, mosaicing version of the usual map/CLEAN combination is outlined in § 4.

A drawback of the method described in § 2 is the increase (by a factor $\approx m$) over the usual number of solution parameters, or degrees of freedom, in the self-calibration solution algorithm. Because of this increase, a better source model, higher signal-to-noise ratio (S/N), or a larger number of antenna elements, (or a combination of all three) becomes desirable. By incorporating assumptions of spatial and temporal correlation of the antenna phases, one may try to hold this larger number of degrees of freedom in check; this idea is pursued in §§ 5–6. Perley and Erickson argue that for the proposed low-frequency array, which is designed to operate at 75 and 150 MHz, simple and accurate source models often will be available (perhaps from 327 MHz observations). And their data suggest that the spatial extent and the velocities of the ionospheric irregularities responsible for the severest phase fluctuations at 75 MHz are such that the techniques of §§ 5–6 would be useful. They report that during this summer they have found the typical case in 327 MHz VLA observations to be one of

near-isoplanatism over 5° to 20° patches, with dominant fluctuation time scales of 5^m to 20^m . They estimate that at 75 MHz the isoplanatic patches would be 1° to 5° in angular extent. The observations often would be of very high S/N with the proposed array design.

2. Incorporating an interpolation formula in the solution for antenna/i.f. gains. In the usual scheme for self-calibration, the visibility measurement obtained on the i - j baseline at time t is assumed to be given by

$$\tilde{V}(u_{ij}(t), v_{ij}(t), w_{ij}(t)) = g_i(t)\bar{g}_j(t)\mathcal{V}(u_{ij}(t), v_{ij}(t), w_{ij}(t)) + \text{error}, \quad (2-1)$$

where \mathcal{V} is the true source visibility and where the spatial frequency coordinates (u, v, w) have been parametrized by time. $g_i\bar{g}_j$ is the systematic "calibration error", and the additive error is assumed to be well-behaved, and not troublesome. The (complex-valued) function $g_k(t)$ is called the *antenna/i.f. gain* for antenna k of the array (or the *complex gain*, for short). g_k may be written as

$$g_k(t) = a_k(t)e^{i\psi_k(t)}, \quad (2-2)$$

with a_k real, where $\psi_k \equiv \arg(g_k)$ is called the *antenna/i.f. phase* (the *antenna phase*, for short). Various sources of systematic error—in addition to the phase errors arising from differences in the tropospheric and ionospheric path delays—are absorbed into the g_k . But for purposes of the present discussion we shall consider only the phase errors, which, for a low-frequency array, are dominated by the ionospheric effects; and so we shall rewrite Eq. 2-1 as

$$\tilde{V}(u_{ij}(t), v_{ij}(t), w_{ij}(t)) \approx e^{i(\psi_i(t) - \psi_j(t))} \mathcal{V}(u_{ij}(t), v_{ij}(t), w_{ij}(t)). \quad (2-3)$$

Now, given a source *model* whose Fourier transform is V (obtained, say, by Fourier inversion of roughly calibrated data and by deconvolution of the point source response), and given the visibility observations obtained at a given instant of time, one can solve for an approximation to the instantaneous phase corruptions. The least-squares solution (e.g.) for the vector Ψ of phase corruptions is obtained by minimizing the functional

$$S(\Psi) = \sum_{1 \leq j < k \leq n} w_{jk} \left| \tilde{V}_{jk} - e^{i(\psi_j - \psi_k)} V_{jk} \right|^2, \quad (2-4)$$

where the w_{jk} are appropriately chosen weights, n is the number of antennas comprising the array, and $V_{jk} \equiv V(u_{jk}, v_{jk}, w_{jk})$ (see [9] for details).

Since the g 's and ψ 's are written above as functions of time only, an assumption of isoplanatism is implicit in the preceding discussion. To handle the anisoplanatic case, we shall now (with a slight abuse of notation) write the antenna phases as functions of position on the sky and of time: $\psi_k(x, y, t)$, where $(x, y, z \equiv \sqrt{1 - x^2 - y^2} - 1)$ are direction cosines with respect to the visibility phase tracking center. Assume now that the field of view of antenna k can be divided into some small number, m , of "isoplanatic patches" whose centers are $\mathbf{N} = \{(x_l, y_l)\}_{l=1}^m$. By isoplanatic patch we mean a patch of sky over which the isoplanatism assumption is very nearly valid. Now the idea is simply to approximate ψ_k by a function $\hat{\psi}_k$ that interpolates the phase corruptions $\{f_l\}$ over the set \mathbf{N} of *interpolation nodes*, so that (ignoring the time variable):

$$\psi_k(x, y) \approx \hat{\psi}_k(x, y) \equiv \sum_{l=1}^m \omega_l(x, y) f_l, \quad (2-5)$$

where the ω_l are weights prescribed by whatever interpolation formula we choose (see §3). The f_l will be the unknowns in a modified self-calibration solution scheme for antenna phases.

In self-calibration, as it usually is applied, the source model is given by a linear combination of a finite number N of point source components, obtained by some variant of the CLEAN algorithm. In this case, the model source visibility is given by

$$V(u, v, w) = \sum_{q=1}^N p_q e^{2\pi i(u l_q + v m_q + w n_q)}, \quad (2-6)$$

where the p_q are the point component fluxes, and $(l_q, m_q, n_q \equiv \sqrt{1 - l_q^2 - m_q^2 - 1})$ their positions with respect to the visibility phase tracking center. Assuming, for the sake of simplicity, that there are the same number m of interpolation nodes or isoplanatic patches over all antennas of the array, and that the directions $\{(x_l, y_l)\}_{l=1}^m$ to the centers of these patches are the same for each antenna, we can solve for the nm solution parameters f_{kl} by minimizing, in analogy to Eq. 2-4, the functional

$$S(\mathbf{f}) = \sum_{1 \leq j < k \leq n} w_{jk} \left| \tilde{V}_{jk} - \sum_{q=1}^N p_q e^{i(2\pi(u_{jk} l_q + v_{jk} m_q + w_{jk} n_q) + \sum_{l=1}^m \omega_l(l_q, m_q)(f_{j1} - f_{kl}))} \right|^2. \quad (2-7)$$

Here, f_{kl} is the phase corruption for antenna k and the center of the l th isoplanatic patch.

In ordinary self-calibration, since only pairwise differences of the ψ_k occur in Eq. 2-4, the unknowns are determined only up to an additive constant; similarly, only pairwise differences are required in order to correct the data. One may assume that $\psi_r = 0$ for some choice r of "reference antenna" and, hence, that there are only $n - 1$ rather than n unknowns. Likewise, in the anisoplanatic case $nm - 1$ unknowns are well-determined (assuming there are observations on sufficiently many baselines). This number is $nm - 1$ rather than $nm - m$ because the interpolation formula for the reference antenna provides a 'connection' between the f_{rl} , $l = 1, \dots, m$.

3. Choice of interpolation nodes, and choice of an appropriate interpolation formula. Two considerations might govern the placement of interpolation nodes: (i) the geometry of the array, and (ii) the location of the brighter components of the source model. The ideal situation is that in which the source component which is of primary interest (and of unknown structure) is surrounded by bright components of simple structure, for which a simple and fairly accurate model is known. For a simple array geometry, if (i) were to govern the placement of the nodes, then the nodes might be regularly spaced. However, a necessary condition in order for the phase correction for a given isoplanatic patch to be well-determined is that the antenna must be seeing modeled emission somewhere in the patch. The latter observation, combined with the fact that array geometries generally are somewhat irregular, dictates that in most cases consideration (ii) would govern the placement of interpolation nodes. So, instead of one of the familiar interpolation rules for regularly spaced data, such as a polynomial interpolation formula, an interpolation formula appropriate for scattered data is needed.

A particularly simple scattered data interpolation method, due to Shepard, is described by Gordon and Wixom in [7]. This is an inverse-distance weighted method,

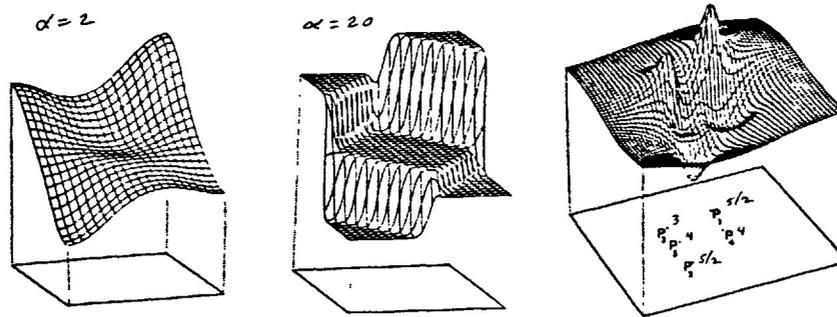


Fig. 1. Qualitative behavior of the Shepard interpolant. At left, $\alpha_j = 2, \forall j$, in Eq. 3-1. At center, $\alpha_j \equiv 20$; the interpolant approaches a piecewise constant function as the parameter is increased. At right is shown the somewhat 'sensible' behavior of the interpolant in areas away from any of the interpolation nodes; here, different values of α_j are used for different nodes—the respective value is noted at the right of each node. Adapted from Gordon and Wixom [7].

for which Eq. 2-5 assumes the form

$$\hat{\psi}_k(x, y) = \frac{\sum_{l=1}^m f_l \prod_{j \neq l} ((x - x_j)^2 + (y - y_j)^2)^{\alpha_j/2}}{\sum_{l=1}^m \prod_{j \neq l} ((x - x_j)^2 + (y - y_j)^2)^{\alpha_j/2}}, \quad (3-1)$$

for some choice of the $\alpha_j > 0$; $\alpha_j = 2, \forall j$, might typically be chosen. Other interpolation schemes which might be suitable are described by Franke in [6]. Eq. 3-1 can be rewritten in the equivalent form

$$\hat{\psi}_k(x, y) = \frac{\sum_{l=1}^m f_l / r_l^{\alpha_l/2}}{\sum_{l=1}^m 1 / r_l^{\alpha_l/2}}, \quad (3-2)$$

where r_l denotes the Euclidean distance between (x, y) and (x_l, y_l) . In our application, we would probably choose $\alpha_l \equiv \alpha = \text{constant}$.

A desirable feature of Shepard's method, which it does not share with the usual polynomial interpolation schemes, is that the interpolant is bounded, even at infinity, and its behavior may be somewhat 'reasonable' even outside of a figure encompassing the nodes. Its extreme values, in fact, are equal to $\min_l f_l$ and $\max_l f_l$. When $\alpha \leq 1$, the interpolant is rather peculiar, generally with a cusp at each of the nodes. However, when $\alpha > 1$, the surface tends to be flat at each of the nodes (the partials with respect to x and y are equal to 0 at the nodes). As $\alpha \rightarrow \infty$, the interpolant approaches a piecewise constant function, constant in some 'domain of influence' of each node. The Shepard interpolant, in fact, reproduces constant functions—i.e., when $f_l \equiv \text{constant}$, $\hat{\psi}_k \equiv \text{constant}$. These properties, which are described in more detail in [7], are illustrated in Fig. 1. Because of the flatness of the Shepard interpolant, this method is not often used in the ordinary applications, such as contour mapping, except with modifications, say to incorporate derivative information [6].

4. Utilization of space-variant antenna phases in inversion. The self-calibration algorithm has two basic components: an algorithm, such as that described in § 2, to solve for an approximation to the antenna/i.f. gains, or the antenna phases, given a source model and given the error-corrupted visibility data; and an inversion algorithm, which, given corrected visibility data, generates a source model. In the isoplanatic case, the inversion algorithm usually is a pair consisting of a Fourier inversion scheme and a deconvolution algorithm—i.e., say, a mapping program combined with either the CLEAN, or the maximum entropy, deconvolution algorithm.

In the anisoplanatic case, because of the departure from the assumption of space-invariance, there is a departure from straightforward Fourier inversion. One way of coping with space-variant effects is by means of mapping in the usual manner over the patches of a mosaic, and then deconvolving from each patch the point source response appropriate to that patch. This must be accomplished by dealing with all patches of the mosaic simultaneously, because sidelobes from a source in any one patch fall into each of the other patches.

Any one of the many sophisticated deconvolution algorithms could, with some degree of effort, be modified to incorporate mosaicing, and to cope with space-variant effects; however, to date, only the CLEAN algorithm has been suitably modified to be capable of making use of space-variant antenna phases. A mosaic version of the "battery-powered" CLEAN algorithm of Cotton and Schwab (see the AIPS Cookbook glossary) has been implemented by Bill Cotton in an AIPS program named MX. Briefly, the program works as follows: It is a variant of the Clark CLEAN algorithm, which incorporates two nested iteration loops. In the inner iterations, each mosaic patch is treated as in the standard CLEAN algorithm. At the outer iterations, the residual maps appropriate to each patch are computed by correcting the model visibility corresponding to the current iterate for sky curvature (the wz term) and possibly for instrumental effects (finite bandwidth, finite integration time, primary beam attenuation, etc.), and re-gridding the (residual) data appropriately for each patch; this is followed by Fourier inversion to obtain a patchwork residual map. The algorithm then goes into more inner iterations. The only modification required is to correct the visibility residuals for the space-variant antenna phases at the same time that the data are re-gridded in the outer iterations.

5. Incorporating an assumption of spatial correlation of the antenna phases.

In this section I shall assume that the same set N of interpolation nodes, specified by their direction cosines in the map, $\{(x_l, y_l)\}_{l=1}^m$, is used for each of the n antennas of the array. Closely spaced antennas see a given interpolation node (x_l, y_l) through nearly the same tropospheric/ionospheric path; hence, if antennas j and k are near to one another, then $|f_{jl} - f_{kl}|$ ought to be small. A straightforward way to incorporate this assumption in the solution scheme is by adding a penalty function to the right-hand side of Eq. 2-7. An appropriate choice of penalty function might be an expression of the form

$$\lambda \sum_{\substack{j \neq k \\ 1 \leq l \leq m}} \frac{(f_{jl} - f_{kl})^2}{r_{jk}}, \quad (5-1)$$

where r_{jk} is given by some monotone increasing function of the physical separation of antennas j and k ($r_{jk} = (u_{jk}^2 + v_{jk}^2)^{1/2}$, for example—see §7, Remark 7). The prime denotes that terms corresponding to antenna pairs whose physical separation is so large that f_{jl} and f_{kl} might be very different are excluded from the summation. λ would be chosen so as to achieve a reasonable balance between the influence of the χ^2 error term (Eq. 2-7) and the penalty term—the method of cross-validation, as described in [10], could be used to choose λ .

One could carry this idea a bit further by building into the penalty function some standard norm of the difference between $\hat{\psi}_j$ and $\hat{\psi}_k$; for example, incorporating the square of the L^2 norm, use

$$\lambda \sum_{j \neq k} \frac{1}{r_{jk}} \iint_{\text{f.o.v.}} |\hat{\psi}_j(x, y) - \hat{\psi}_k(x, y)|^2 dx dy. \quad (5-2)$$

Here, f.o.v. denotes the field of view, or region of interest. Or one might choose a

Sobolev norm—say, the H^1 norm:

$$\lambda \sum_{j \neq k} \frac{1}{r_{jk}} \iint_{\text{f.o.v.}} \left(|\delta_{jk}(x, y)|^2 + \left| \frac{\partial}{\partial x} \delta_{jk}(x, y) \right|^2 + \left| \frac{\partial}{\partial y} \delta_{jk}(x, y) \right|^2 \right) dx dy, \quad (5-3)$$

where $\delta_{jk} \equiv \hat{\psi}_j - \hat{\psi}_k$. The latter choice not only encourages the difference between the computed antenna phase surfaces for proximate antennas to be small, but it urges that difference to be smooth, as well. Although Expressions 5-2 and 5-3 are computationally tractable (at least for certain choices of interpolation formula), the added sophistication might not be worth the extra effort required.

6. Incorporating an assumption of temporal correlation. An assumption of temporal correlation of the antenna phases could be incorporated by either of two means: (i) through use of a ‘data (time-) window’; or (ii) via a penalty function method, as in §5, serving to bias the computed solution toward the solution obtained, say, for the next earlier time—or toward a running mean of the solutions obtained for earlier times. In case (i), the summation (Eq. 2-7) defining the χ^2 error term would be extended over time as well as baseline, and, to approximate the phase corruptions corresponding to the center of a time interval $[t - \tau/2, t + \tau/2]$, the data weights would be chosen to be proportional to a real-valued function symmetric about, and peaked at, t . In case (ii), one might assume a Gaussian prior distribution for f_{kl} , of mean μ_{kl} equal, say, to the running mean of previous solutions, and of standard deviation σ . The penalty term to be added to the right-hand side of Eq. 2-7 then would be of the form

$$\lambda' \sum_{\substack{1 \leq k \leq n \\ 1 \leq l \leq m}} \frac{(f_{kl} - \mu_{kl})^2}{\sigma^2}. \quad (6-1)$$

As with the λ of §5, λ' would have to be chosen adaptively, in order to achieve a reasonable balance with the χ^2 error term and, possibly, with the penalty term of §5. Again, in case (ii) the method of cross-validation could be used to choose λ' , or, in case (i), in order to choose the characteristic width of the weight function (or even a parameter defining its shape).

7. Remarks. 1) The technique described in §5 for incorporating an assumption of spatial correlation of the antenna phases could be applied in ordinary self-calibration, as well (via Expression 5-1, with $m = 1$). This might be a useful modification for the case of low S/N observations, especially at short observing wavelengths or whenever atmospheric effects are dominant. The $n - 1$ degrees of freedom in the solution for antenna phases would essentially be reduced as λ is increased; for $\lambda = 0$, one would have the case of ordinary self-calibration. However, there mightn't be very suitable *a priori* information for choosing the ‘right’ value of λ . This modification could, with little effort, be incorporated in the AIPS program for self-calibration.

2) Both of the techniques of §6, for incorporating an assumption of temporal correlation, are already employed in some fashion in self-calibration. A rectangular data window, whose width is specified by the program user, is used in the AIPS implementation; the window, however, does not ‘slide’—but rather it ‘jumps’, in time increments equal to its full width. Cornwell and Wilkinson in [3] describe the use of a penalty function method in order to constrain the moduli of the computed antenna gains (rather than to constrain the antenna phases, as in §6). This modification is incorporated in a version of the AIPS self-calibration program which is used primarily for VLBI data reduction.

3) The remarks in §§ 5–6 that cross-validation could be used to choose the parameters λ and λ' are a bit flippant. Cross-validation is probably too expensive a technique to be practical in this application, but in a sense it would be the ‘ideal’ method.

4) It may be useful to compare the computational expense of the solution for antenna phases (as outlined in § 2) in the anisoplanatic case, against the case of ordinary self-calibration. In the present implementations of self-calibration in the AIPS software package, two methods, both of them iterative, are used in the solution for antenna gains or antenna phases. The method used most often for VLA data reduction is a successive substitution algorithm [9], requiring only first-order information—the gradient, ∇S , of the right-hand side of Eq. 2-4. The other method, which is used both in the global fringe search algorithm for VLBI [11] and in a version of the self-calibration algorithm which incorporates a penalty function to constrain the moduli of the computed antenna gains, requires second-order information, $\nabla^2 S$; moreover, at each iteration it requires the numerical solution of a system of M simultaneous linear equations, where M denotes the total number of solution parameters.

In the anisoplanatic case, one would need to compute ∇S , and perhaps $\nabla^2 S$, where S now is given by Eq. 2-7 (derivatives of the penalty terms of §§ 5–6 also have to be added in—but see below). M is increased by a factor m , equal to the number of isoplanatic patches. S and ∇S still are comprised of $n(n-1)/2$ terms, where n is the number of array elements. There is of order m times as much expense in evaluating each term of ∇S , because of the $2(m-1)$ additional antenna phases, and their accompanying interpolation weights, in the argument of the exponential function in Eq. 2-7. Each observation contributes to $4m^2$ elements of $\nabla^2 S$, rather than only to four, so one would want to avoid evaluating $\nabla^2 S$; this means that one would use a successive substitution algorithm, requiring more iterations, but less work per iteration, than a Newton method.

Additionally, the larger number of unknowns requires a larger number of iterations—but the rate of increase in this number ought not to be proportional to m , but more gentle. Incorporating the penalty terms of §§ 5–6 entails little added expense, because the observational data are not included in the penalty sums. In fact the penalty constraints tend to stabilize the solutions, and so should produce better-conditioned minimization problems.

Overall, for fixed n , the computational expense in a careful implementation is increased by a factor equal to a few times or several times m , rather than a few times m^2 . With m in the range say, 5 to 20, and n smaller than 50 or so, the burden, though considerable, would seem manageable. In the present implementation of self-calibration in AIPS, the solutions for antenna gains or antenna phases are computed by the AIPS host computer, rather than by a high-speed array processor (AP). In the anisoplanatic case one would want to compute ∇S (at least) in an AP, as AP’s are fairly well-suited to the task (and should yield a speed-up factor perhaps in the range of 20 to 60).

5) In the isoplanatic case, one has $n(n-1)/2$ visibility measurements, and (given a source model) $n-1$ unknowns, the ψ_k , to be determined by the antenna phase solution algorithm. Hence, the ψ_k are overdetermined by a factor $q = n/2$; that is, there are $n/2$ times more equations than there are unknowns. When the source model is inaccurate or the data are noisy, this overdeterminacy is essential to the success of self-calibration.

In the anisoplanatic case, for a fixed number of antennas, and ignoring (for the moment) the possibility of incorporating the penalty functions of §§ 5–6, q is reduced by a factor which is nearly equal to the reciprocal of the number of isoplanatic patches;

i.e., $q = \frac{n(n-1)}{2(nm-1)}$. In order for the solutions for the f_{kl} to be overdetermined, m must be less than $r = \frac{n-1}{2} + \frac{1}{n}$. For a 27 element array, such as the VLA, $r = 13\frac{1}{27}$; and for a 50 element array, $r = 24\frac{13}{55}$. When m is close to r , both a very accurate source model and very high-quality data are required in order to obtain accurate solutions for the f_{kl} .

An easy way to understand the effect of the penalty function method of §5 (to incorporate the assumption of spatial correlation) is to consider the case of a low-frequency array in which the array elements are placed down in pairs that are so closely spaced that each element may be considered to be looking through the same part of the ionosphere as one nearest neighbor. Then, as λ is increased, the number of unknowns is essentially halved, and the overdeterminacy is doubled—i.e., now $q = \frac{n(n-1)}{nm-2}$, and $r = n - 1 + \frac{2}{n}$.

An advantage of the technique of §5 is that the penalty can be made continuously variable as a function of antenna separation. For very closely spaced antennas, the antenna phase solutions may be constrained to be very close, whereas for proximate antennas which are not so closely spaced, the solutions still may be constrained to be close, but need not be as close.

6) As an alternative to use of computed antenna phase surfaces given by the interpolation formula Eq. 2-5, one might consider use of a bilinear or biquadratic polynomial form, such as

$$\hat{\psi}_k(x, y) = a_k x^2 + b_k xy + c_k y^2 + d_k x + e_k y + f_k. \quad (7-1)$$

Then Eq. 2-7 would be replaced by

$$S(\mathbf{a}, \dots, \mathbf{f}) = \sum_{1 \leq j < k \leq n} w_{jk} \left| \hat{V}_{jk} - \sum_{q=1}^N p_q e^{i(2\pi(u_{jk}l_q + v_{jk}m_q + w_{jk}n_q) + \hat{\psi}_j(l_q, m_q) - \hat{\psi}_k(l_q, m_q))} \right|^2, \quad (7-2)$$

where $\mathbf{a} \equiv (a_1, \dots, a_n)$, etc., denote the new solution parameters (six unknowns per antenna, in this example). One can go a step further by assuming that ψ_k is, say, locally quadratic in the neighborhood of each of a family of interpolation nodes—that the phase surface for antenna k , in the neighborhood of (x_l, y_l) , is given by

$$\begin{aligned} \phi_{kl}(x, y) = & a_{kl}(x - x_l)^2 + b_{kl}(x - x_l)(y - y_l) + c_{kl}(y - y_l)^2 \\ & + d_{kl}(x - x_l) + e_{kl}(y - y_l) + f_{kl}. \end{aligned} \quad (7-3)$$

These surfaces can be pieced together by means of a 'generalized' Shepard interpolation formula,

$$\hat{\psi}_k(x, y) = \frac{\sum_{l=1}^m \phi_{kl}(x, y)/r_l^{\alpha/2}}{\sum_{l=1}^m 1/r_l^{\alpha/2}}, \quad (7-4)$$

where, as before, r_l denotes the Euclidean distance between (x, y) and (x_l, y_l) . Then, for m interpolation nodes, there are $6m$ unknown parameters per antenna. Probably, however, this technique would be most useful when the anisoplanatism is not too severe (e.g., in 327 MHz VLA observations), so that just one 'interpolation node' would suffice (i.e., $m = 1$, with $x_1 = y_1 = 0$).

In practice, the numerical stability of the solution algorithm would be improved by using, in place of Eq. 7-3, a linear combination of two-dimensional polynomials orthogonal over an appropriate domain. One could, for example, choose polynomials

orthogonal over a square region of the sky. However, since the array element primary beam patterns often are nearly circularly symmetric, a better choice (in the case $m = 1$) might be polynomials orthogonal over a disk—or polynomials defined over the entire x - y plane, but orthogonal with respect to a radially symmetric weight function. Suitable candidate families of bivariate orthogonal polynomials are described in [5, p. 264 ff.].

7) Armstrong and Sramek [1] and Sramek [12] characterize the tropospheric phase scintillations affecting interferometric observations at microwave frequencies by means of a so-called structure function, and by means of Allan variances. The structure function represents the spatial statistics of that component of the interferometer phase variations which is due to the troposphere. The Allan variances of observed phase variations bear a close relation to the structure function. In observations at 5 GHz, Armstrong and Sramek find the structure function to be well approximated by a power law relation—the variance of the observed variation is proportional to some power γ of the baseline length, and the value $\gamma \approx 1.4$ is typical for interferometer baseline lengths of 1–35 km. At 22 GHz Sramek finds the Allan variance of observed phase variations, computed over a time scale of 16^m, to be well modeled, again, by a power law relation—over baselines of 100 m to 3 km—with an exponent $\gamma \approx .72$

These results suggest that an appropriate choice for the weight function appearing in Expressions 5-1-5-3 might be the choice $1/r_{jk} = (u_{jk}^2 + v_{jk}^2)^{\gamma/2}$, for some empirically derived value of γ , identical to the γ used above. A further refinement to the technique described in § 5, for incorporating an assumption of spatial correlation, would be to constrain the mean square variation of each of the computed antenna phase surfaces to be in accord with some empirical determination of the structure function. Again, one might employ a penalty function, say one of the form

$$\lambda'' \sum_{k=1}^n \sum_{j \neq l} \frac{(f_{kj} - f_{kl})^2}{h(\gamma; \theta_{kjl})}, \quad (7-5)$$

where θ_{kjl} is the angular separation between the ray paths by which antenna k sees interpolation nodes j and l , and where h is an appropriately chosen monotone increasing function of this separation.

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