VLA Scientific Memorandum No. 161
Correction Schemes for Polarized Intensity

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ABSTRACT: We present calculations of the effects of applying various proposed correction schemes to polarized intensity images, in an attempt to statistically compensate for the intrinsic bias in this quantity. In particular we investigate the effects of masking the image with a signal-to-noise cutoff. We recommend some changes to the algorithms currently used by AIPS.

1 The problem

The polarized intensity,

$$p = \sqrt{Q^2 + U^2} \tag{1}$$

is a difficult quantity to handle at low signal-to-noise ratio (S/N), mainly because the value obtained is necessarily positive and therefore biassed by noise in the image. Nevertheless it is an important quantity in multi-frequency polarization work, since it is required to calculate physically-interesting quantities such as the degree of polarization and depolarization ratios. Because polarized signals are often weak, we are usually faced with the problem of getting the best out of low S/N data. In particular, when we cannot obtain an accurate value of p in each beam area, we would like to obtain a reasonable estimate of the integrated polarized intensity at a particular resolution over an area of the source. The phrase 'at a particular resolution' is the key problem: because p does not obey the convolution equation we cannot simply smooth the p image to obtain higher S/N. We can convolve Q and U separately, but the averaging over regions with different E-vector orientation reduces the net polarized intensity; essentially such a procedure answers a different question. We want to preserve the separation of differently polarized regions provided by high resolution but still gain some benefit from averaging over a large area.

In fact it is possible to do substantially better than the naive approach by applying a statistical correction scheme. Several such schemes have been advocated and discussed in the literature. For the integration problem mentioned above and for most other problems, we should choose a scheme which gives the least biassed average result when applied to an ensemble of data. In general, the level of residual bias is a function of S/N. Effective correction at S/N $>$ 4 is easy while all schemes give a relatively large residual at S/N $<$ 0.5. In this memorandum we calculate the residual bias as a function of S/N for several proposed schemes. A similar exercise has been done by Simmons & Stewart (1985; SS). We extend their approach by including also the simplest possible correction, which actually turns out to be extremely effective, and by considering differing S/N cutoffs in the algorithms. On the other hand they consider a number of points omitted here, notably the risk functions associated with the various schemes. This is the rms difference between true and corrected value as a function of signal-to-noise.

2 Calculations

We assume that the errors in Q and U are gaussian with the same standard deviation σ_p in both. This is a reasonable approximation for most VLA polarization data. The uncorrected error distribution of polarized intensity is then the Rice distribution,

$$F(p, p_0) = \frac{p}{\sigma_p^2} e^{-\frac{p^2+p_0^2}{2\sigma_p^2}} I_0\left(\frac{pp_0}{\sigma_p^2}\right) \tag{2}$$

Here p is the observed polarized intensity, p_0 is the true polarized intensity, and I_0 is a modified Bessel function of the first kind (see SS).

We considered four possible estimators of p_0 given p.

First, $\hat{p}_1 = p$. This shows how bad things are if no correction is applied.

Second, the simple first-order correction $\hat{p}_2 = \sqrt{p^2 - \sigma_p^2}$, which can be derived from a first-order expansion in $(\sigma_p/p)^2$:

$$p = (Q_0^2 + U_0^2 + 2(Q\,dQ + U\,dU) + dQ^2 + dU^2)^{1/2} \quad (3)$$

$$= p_0 \left(1 + \frac{2(Q\,dQ + U\,dU) + dQ^2 + dU^2}{2p_0^2} - \frac{4(Q^2dQ^2 + 2QU\,dQdU + U^2dU^2) + \ldots}{8p_0^4} \ldots \right) \quad (4)$$

$$\langle p \rangle = p_0 \left(1 + \frac{2\sigma_p^2}{2p_0^2} - \frac{4p_0^2\sigma_p^2}{8p_0^4} + \ldots \right) \quad (5)$$

$$p_0 \approx \langle p \rangle \left(1 - \frac{\sigma_p^2}{2\langle p \rangle^2}\right) \approx \sqrt{p^2 - \sigma_p^2} \quad (6)$$

We define $\hat{p}_2 = 0$ if $p < \sigma_p$; this is consistent since $\hat{p}_2 = 0$ when $p = \sigma_P$. It might be asked whether it is better to leave such points 'undefined'. It is not, because the existence of a substantial number of points with observed flux below the noise level is strong evidence that there is indeed very little polarized emission. To ignore them would lead to a substantial overestimate of the mean polarized intensity, as we shall show. This correction is available in AIPS as the POLC option in COMB.

Third, the scheme proposed by Wardle & Kronberg (1974). Here the estimator is the value of p_0 for which the observed polarization p is the maximum of $F(p, p_0)$; i.e.

$$\partial F(p, \hat{p}_3)/\partial p = 0 \quad (7)$$

There is no solution for $p < 1$ so we define $\hat{p}_3 = 0$ in this case.

Finally we consider the maximum-likelihood scheme advocated by Killeen, Bicknell & Ekers (1986) and implemented as the AIPS task POLCO. Here $\hat{p}_4$ is the value of p_0 which maximises the probability of obtaining p:

$$\partial F(p, p_0)/\partial p_0|_{p_0 = \hat{p}_4} = 0 \quad (8)$$

Here there is no solution for $p < \sqrt{2}\sigma_p$; again we set $\hat{p}_4 = 0$ here.

SS considered all cases except the second and also two other schemes with distinctly worse performance. They derive the minimum values we cite above.

For each scheme we evaluated the mean corrected value of p, which we denote $\langle \hat{p} \rangle$, as a function of S/N, via numerical integration:

$$\langle \hat{p} \rangle = \int_0^\infty \hat{p}(p)\, F(p, p_0)\, dp \qquad (9)$$

For convenience we set the units so that $\sigma_p = 1$.

The results are given in Fig. 1, in the form of plots of the residual bias $\langle \hat{p} \rangle - p_0$ as a function of S/N (i.e. p_0). The cases overlapping with SS are in good agreement, thereby checking our programs (some 'droop' due to the relatively low upper bound we used in practice is visible for $p_0 > 5$; this has no effect on our discussion).

We made a second set of calculations in which $\hat{p}$ was set to zero for S/N < 2, simulating the common practice of 'masking' low S/N regions of the map. In fact AIPS enforces a minimum S/N cutoff of 2 in both the POLC option of COMB and in POLCO. The results are given in Fig 2. Finally we made another series of calculations in which the lower bound of the integration was 2, and the results were normalised by the integral of the Rice distribution over the same range. This is equivalent to ignoring the masked points, rather than counting them as zero. The results are given in Fig. 3.

3 Discussion

All schemes inevitably leave some residual bias at low signal-to-noise ratio. For signal-to-noise ratio of > 2, the scheme due to Wardle & Kronberg is clearly the best. This scheme is probably the best-behaved overall: it is the only one (apart from no correction) which never gives an underestimate, it allows valid (non-zero) estimates for points as low as 1σ, and it appears to approach the zero bias curve exponentially quickly, as opposed to the other schemes which tend to zero roughly as power laws. SS find that this scheme also has a low risk at high S/N.

In practice the simplest first-order correction is almost as good. Below 2σ the simple correction is actually superior, although not as good as the maximum-likelihood scheme for S/N < 1. However at these low signal-to-noise ratios, the residual bias depends mostly on the cutoff point below which pixels are set to zero. A higher cutoff results in a smaller bias at zero signal, but at the same time gives an underestimate for S/N between about 1 and 3. The maximum likelihood scheme has the highest such cutoff, which accounts for its 'good' performance at S/N < 1. In Fig. 4 we plot the maximum likelihood correction together with the simple correction, using the same cutoff ($\sqrt{2}$) for both. A somewhat higher cutoff for the simple correction would probably give even closer agreement.

We have considered the possibility of reducing the zero-signal residual bias by allowing negative corrected values. We do not think that this would be of any help. Experiments with free-form correction schemes suggest that *any* algorithm which reduces the bias at zero signal will introduce an underestimate at S/N of about 1.5, provided the algorithm is a function only of the observed polarized intensity at the point to be corrected. The reason is fairly clear: because the underlying noise distribution has a dispersion of unity by definition, corrections at values of p_0 separated by $O(1)$ in S/N are inevitably highly correlated. More complicated correction schemes based on the observed *distribution* of p values may be possible; the object would be to return an estimate of the underlying p_0 distribution.

The most obvious implication of our results is that masking at high S/N seriously corrupts mean values (and hence integral values) derived from the maps. The AIPS minimum cutoff at S/N of 2 creates a gross underestimate of the polarized flux for S/N between 0.6 and 3.5, or an even worse overestimate if masked points are ignored rather than counted as zero.

4 Recommendations

- AIPS should allow masking with any S/N cutoff down to 1.
- Observers should be aware that there are inevitable trade-offs in applying any correction scheme. The most important tool for manipulating the residual bias is the S/N cutoff. Examination of the distri-

bution function should suggest whether to optimise for low- or high-signal-to-noise. It is always a good idea to use auxiliary information to reduce the area of 'blank sky' in a way that does not bias against regions of low polarization, and especially of strong depolarization. The best such way is to mask the p maps on the basis of an I map.

- **When integrating polarized intensity over a region, blanked points should generally be treated as zero rather than ignored.**

- An implementation of the Wardle & Kronberg correction scheme within AIPS would be worthwhile.

5 References

Killeen, N., Bicknell, G. V. & Ekers, R. D., 1986. *Ap. J.*, **302**, 306.
Simmons, J. F. L. & Stewart, B. G., 1985. *Astron. Ap.*, **142**, 100.
Wardle, J. F. C. & Kronberg, P. P., 1974. *Ap. J.*, **194**, 249.

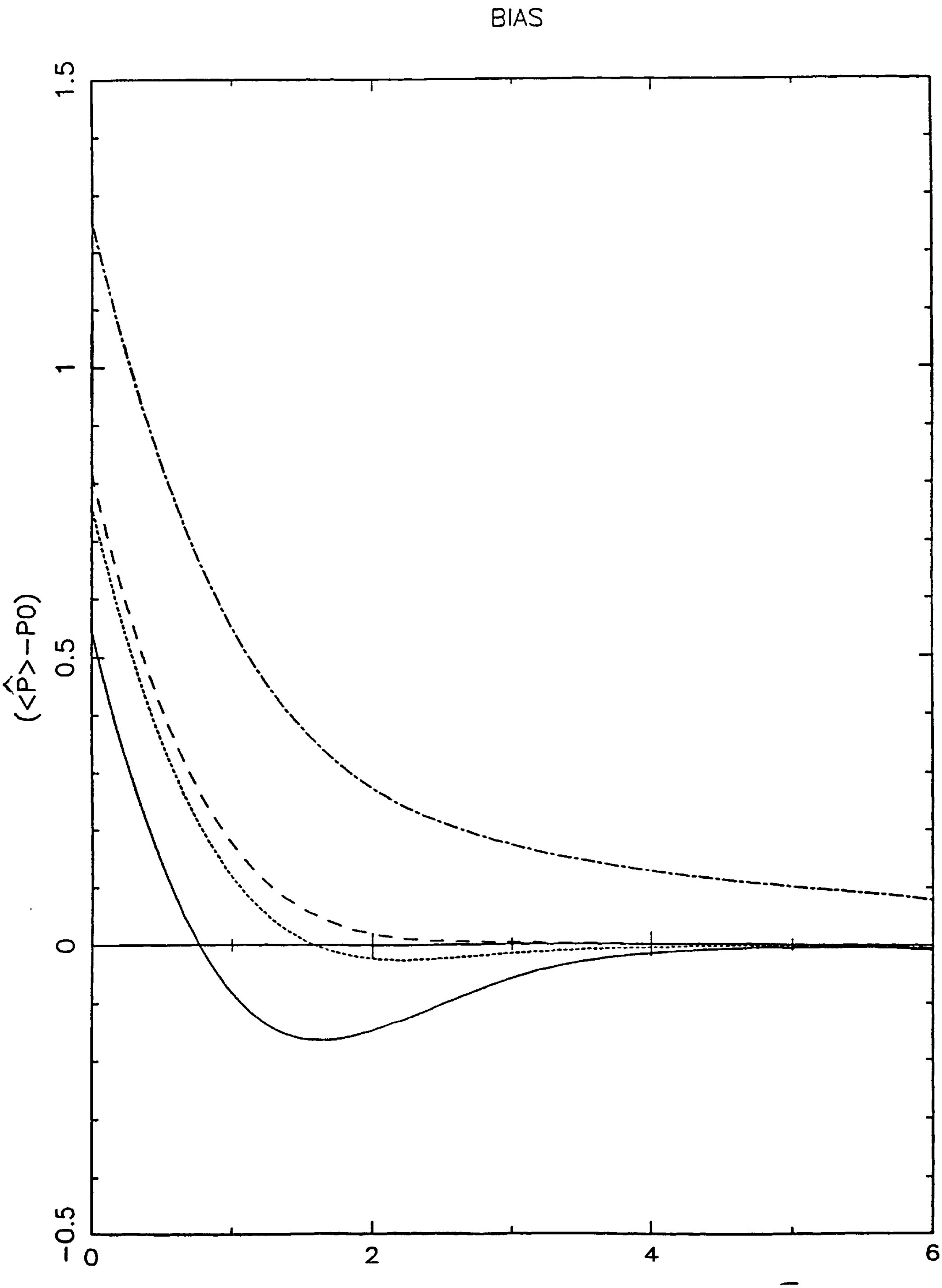

Fig 1

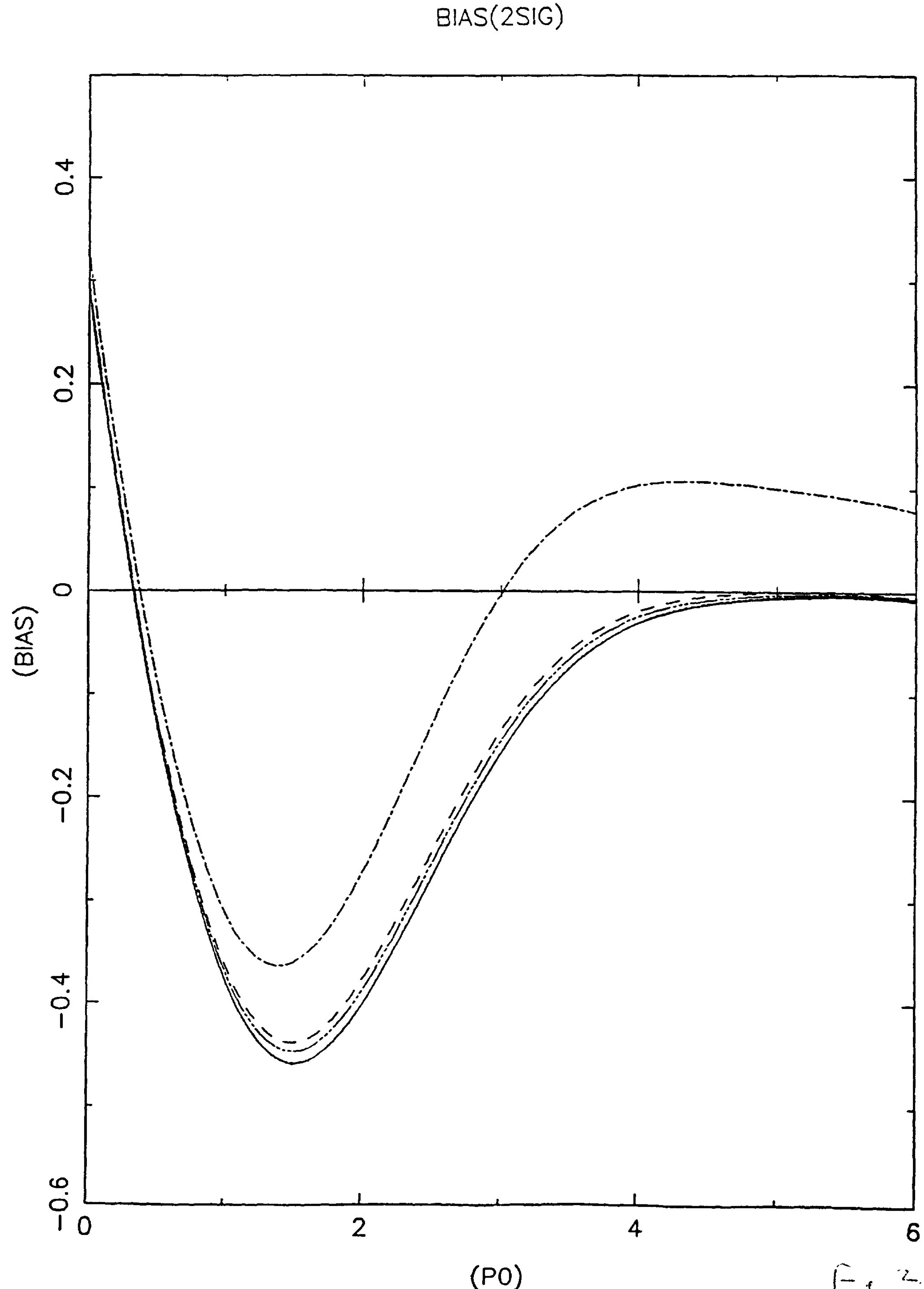

Fig 2

ALL ESTIM.:(2SIG),NO WEAK PTS

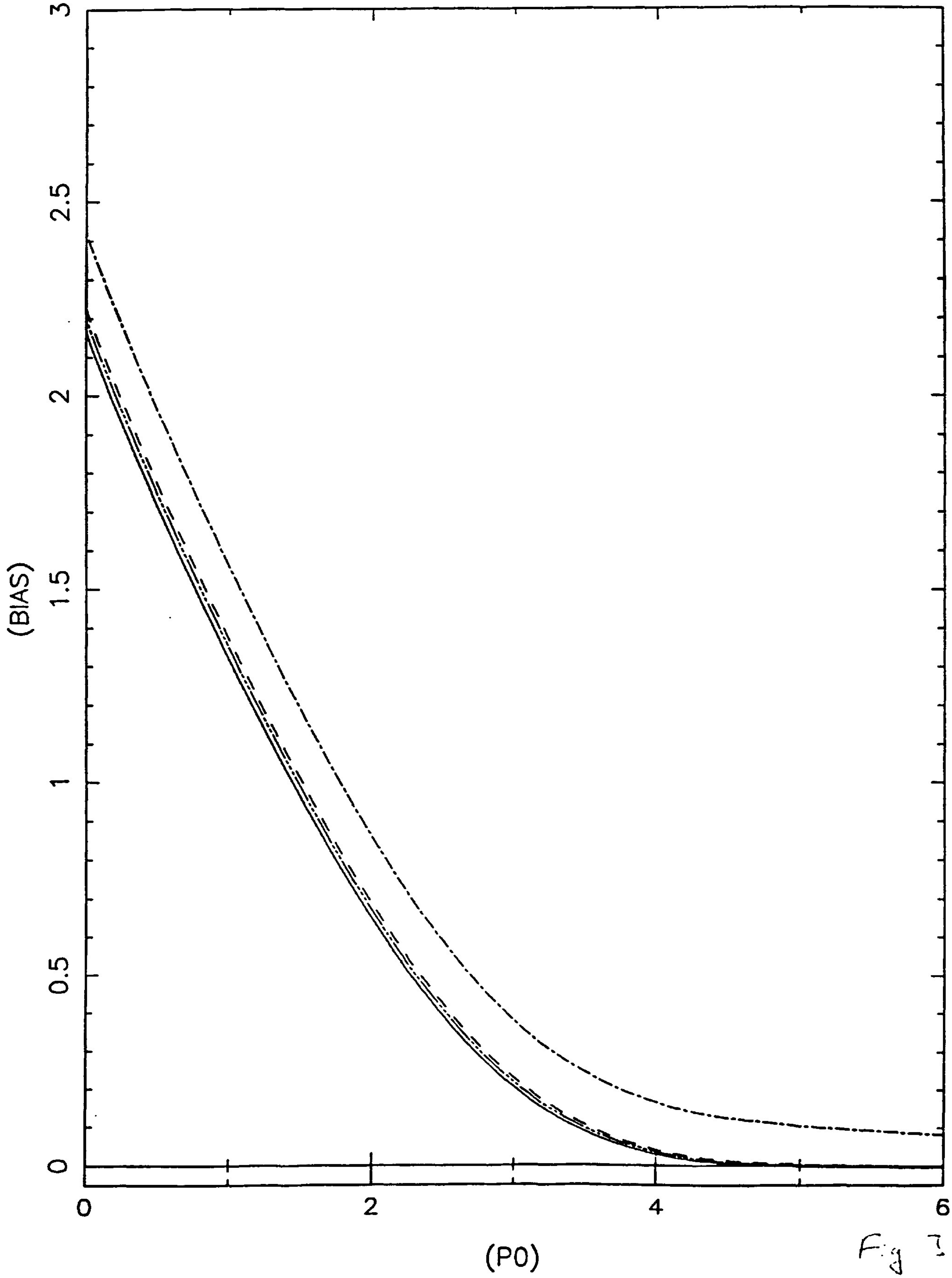

Fig 3.

ALL ESTIM. : BIAS (1.41)

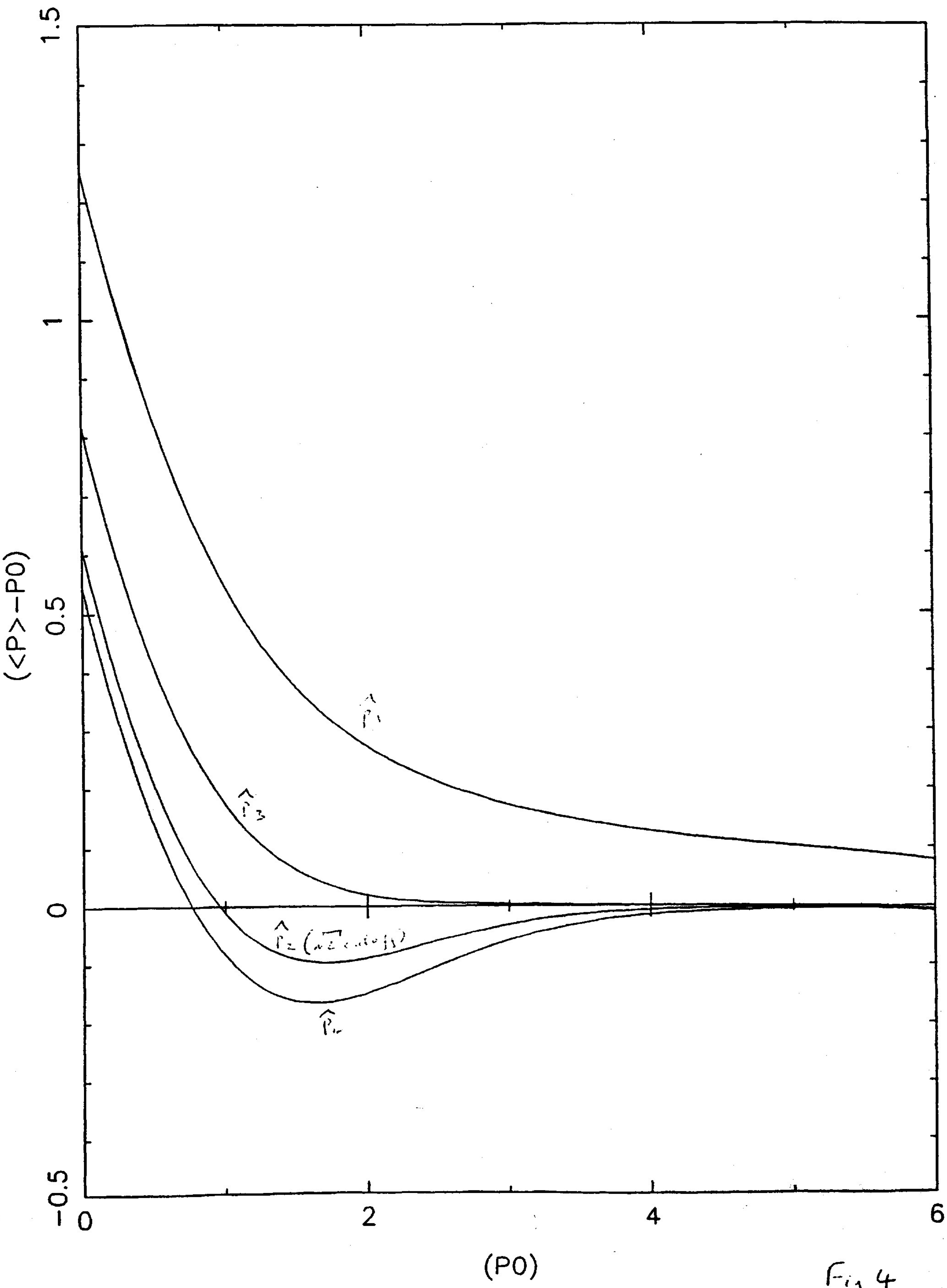

Fig 4