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An error analysis of calibration

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Introduction :

This document attempts to describe the effect of noise on calibration and selfcalibration . The algorithms used will be assumed to be ANTSOL for calibration and Schwab's algorithm for selfcalibration . Minimisation of L2 is also implicit .

The basic problem in both ordinary and self calibration is to calculate or improve estimates of the complex gains of the antennae used in an observation . In ordinary calibration (OC) an unresolved source is used to constrain the antenna gains whoreas in selfcalibration (SC) an estimate of the true shape of the source is used to make the source of interest look unresolved so that OC can be used . Hence in one sense OC and SC are similar but in the latter the effect of noise is much more complicated . In particular , the effect of noise on the selfcalibration of extended sources presents an intractable analytical problem since , in general , it is necessary to solve a set of simultaneous linear equations in the gain variances . It is possible , however , to perform this solution in a computer as the selfcalibration is performed but I doubt that this is really useful .

The influence of closure errors is also briefly discussed .

The algorithm :

Both OC and SC act to minimise the residuals between a model of the source and the observed data . The degrees of freedom available are the complex antenna gains g(i) . The quantity minimised is of the form :

$$S = \mathrm{Sum}(i,j)([V(i,j) - g(i)*(\text{complex conj}(g(j))*T(i,j)]**2*w(i,j))$$

- equation 1

where V(i,j) = observed visibility between telescopes i and j
T(i,j) = "true" visibility between telescopes i and j
w(i,j) = weighting factor
the square brackets [...] denote the modulus
and Sum(i,j) represents a sum over all i,j where i<j .
Some averaging over time may also be done .

This can be rewritten in a more illuminating fashion :

$$S = \mathrm{Sum}(i,j)([X(i,j)-g(i)*(\text{complex conj}(g(j)))]**2*w(i,j))$$

- equation 2

where X(i,j) = correlator-based gain between telescopes i and j :

$$X(i,j) = V(i,j)/T(i,j) \qquad \text{- equation 3}$$

and the weighting factors have been changed .

One may arbitrarily constrain the complex gains to have amplitude unity so that only the phases are corrected , this is usually only done in SC .

In OC the true visibility is estimated from the previously known position and flux of the calibrator whereas in SC it is estimated from a model of the source e.g. a set of clean components .

The effect of noise :

Thompson and D'Addario (preprint , 1981) give an algorithm for solving the above maximisation . This involves solving , by iterative substitution , the non-linear equation for the g(i) obtained by differentiating Equation 2 with respect to the gains :

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g(i) = [Sum(j, not=i)( w(i,j)*X(i,j)*g(j) )]
        /  Sum(j, not=i)( w(i,j)*[g(j)]**2 ) -equation 4
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Using the law of propagation of errors we find that the variance in the estimate of g(i) , var(g(i)) is :

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var(g(i)) = [Sum(j, not=i )( w(i,j)**2 *
      ( [2*g(i)*(complex conj(g(j))) - X(i,j)]**2 * var(X(i,j))
        + 2*[g(i)]**2 * var(g(j)) )]
      / (Sum(j, not=i)(w(i,j)*[g(j)]**2))**2 - equation 5
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All variances quoted are those appropiate for the given interval between gain solutions .

In principle this set of simultaneous equations in the variance of the gains can be solved by matrix inversion to yield an answer in terms of the driving terms var(X(i,j)) . In order to investigate the structure of these equations we can replace the gains by their expected value of unity . In all but some pathological cases this should not alter our conclusions too much . We then have :

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var(g(i)) = [Sum(j, not=i)( w(i,j)**2 *
     ( var(X(i,j)) + 2*var(g(j)) ) )]
      / (Sum(j, not=i)w(i,j))**2          - equation 6
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As an example of the use of this equation consider calibration on a point source . The weights should be equal if all telescopes have uniform levels of sensitivity . Furthermore the variances should be independent of both i and j . Thus we have , after some rearrangement :

$$\mathrm{var}(g) = \mathrm{var}(X)/(N-3) \qquad \text{- equation 7}$$

where N is the number of telescopes .

Note that the errors become infinite for a three telescope array in which the number of constraints (the X(i,j)s) is equal to the number of degrees of freedom (the g(i)s) . If we only allow correction of the phases i.e. [g(i)] = 1 . then the variance becomes :

$$\mathrm{var}(g) = \mathrm{var}(X)/(N-2) \qquad \text{- equation 8}$$

Further constraints on the gains similarly reduce the variance .

To test this calculation I used the AIPS ASCAL program to selfcalibrate point sources of various strengths : 1ØØ to Ø.1 mJy in quasilogarithmic intervals . The noise added was 25 mJy 1Ø seconds and about 7 minutes of data was used .
In Table I I show the r.m.s. of the difrerence between the selfcalibrated map and the initial clean map . This subtraction was performed to eliminate , to first order , the effects of aliasing .

Table I :

Flux of point source , mJy	R.m.s. diff microJy	Signal to noise on map
1ØØ	41	5Ø8
5Ø	41	254
25	42	127
1Ø	49	51
5	67	25
2.5	134	13
1.Ø	266	5
Ø.5	279	3
Ø.25Ø	286	1
Ø.1ØØ	286	Ø.5

In the regime of good signal to noise we expect that , since only the phases were corrected , the noise will increase by SQRT(N/(N-2)) = 1.Ø5 for the twenty antennae used here . Before selfcalibration the r.m.s. noise was 197 microJy per beam hence the test indicates that the (incoherent) increase in noise is about 4.5% which is in reasonable agreement with the expected value .

In the regime of poor signal to noise the derived values of the gains will be completely wrong and therefore equation 5 should be used in place of equation 6 . Consequently the noise behaviour becomes much more complicated . I hope that nobody wanted to selfcalibrate a map with S/N about unity !

The application of equations 6,7 or 8 to the selfcalibration of resolved sources is made rather more difficult by two factors . First , the weights in equation 6 are not identical and so a true matrix inversion is required to calculate the variances . Secondly , since we are forcing the data to look like our model of the source any errors in that model will propagate into the final map . The element of feedback introduced by continued iteration exacerbates this problem . In particular , since we use a mapping technique such as CLEAN to tell us what the sky really looks like any treatment of the noise behaviour of selfcalibration must include the noise behaviour of CLEAN . Hence the only case that we can plausibly treat analytically is single pass selfcalibration .

As an example , consider the selfcalibration of a point or near-point source using as a model a point source . From equation 6 we know the variance of the gains . We then have that the variance in the corrected visibility C is :

var(C) = 2*var(g)*[V]**2 + var(V) - equation 9

where the values of the gains have been replaced by unity .

Substituting equation 7 and approximating [T]**2 by [V]**2 we have :

$$var(C) = (N-1)*var(V)/(N-3) + 2*var(T)/(N-3) \quad \text{- equation 1Ø}$$

Two interesting points arise from this result :

1. The noise is increased by a factor SQRT((N-1)/(N-3)) which for the full VLA is 1.Ø4 . For MERLIN the amplification is 1.29 and for a ten station VLBA it is 1.13 .
2. Errors in the model (var(T)) are much less important than errors in the original data . For the VLA they contribute , incoherently , Ø.Ø8 of the errors in the final map . This explains the success of selfcalibration a la Perley . For smaller arrays such as MERLIN or the various VLBAs the effect of errors in T is much greater : For MERLIN Ø.4Ø and for a VLBA Ø.22 . However , it is still important to minimise var(T) by choosing the model (T) sensibly . In particular , in CLEAN a small number of clean components is advisable to avoid the inclusion in T of the calibration errors we are trying to eliminate .

Clearly this type of calculation can be made for the more general case of a well resolved source and so we can , given some estimate for the errors in T , find the noise level on the output map .

The effect of closure errors :

We may now also investigate the effect of closure errors on calibration . Considering the case of a point source we have that the variance of the correlator based gain X is :

$$var(X) = (var(V) + var(T))/[T]**2 + var(D) \quad \text{- equation 11}$$

where var(D) is the variance of the closure errors . This then provides another driving term in equation 1Ø . We then have :

$$var(C) = (N-1)*var(V)/(N-3) + 2*(var(T) + var(D)*[V]**2)/(N-3) \quad \text{- equation 12}$$

where again [T]**2 has been replaced by [V]**2 .

We can now see that closure errors play an analogous role to model errors and are of similar importance in producing errors in the final map . One important difference remains , namely that the closure errors may not average down as will var(V) and var(T) .