EFFECT OF SAMPLER THRESHOLD ERRORS ON QUANTIZATION CORRECTIONS

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April 7, 1993

ABSTRACT

We consider effct of the sampler DC offset and threshold errors on the correlation coefficient measurements. To obtain the visibility values accuarate to 0.1%, we need to account for the effect of these errors on the quantization correction to be applied to the correlator measurements (for calculating the visibilities).

INTRODUCTION

In a digital correlator measured correlation values have to be corrected for the quantization scheme used, and other factors to obtain (true) visibility values. We have to know the correction factors which need to be applied to the correlator outputs to get the corrected visibilities. some of these corrections may be applied online in the correlator, and others may be applied offline during the image processing. In that case the correction factors, or the quantities which are required for generating the corrections, have to be passed on to the image processing system (along with the correlated data).

Schwab (1986, VLBA Correlator Memo. No. 75) has considered two bit correlator equation with idealized sampler having no errors. He has given linearized and polynomial approximations to the quantization corrections for the use in the FX (VLBA) correlator application. In this memo we consider effects of sampler DC offset and threshold errors which need to be accounted to get correct visibility values.

DERIVATIONS OF A TWO BIT CORRELATOR EQUATION

The output of a correlator for gaussian noise inputs x(t) and y(t) may be expressed as

$$R_{12} = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x)q(y)e^{-(\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)})}dxdy. \tag{1}$$

Here q(x) and q(y) are quantizing functions for x and y, given by

$$q(x) = \begin{cases} +n, & \text{for } x \geq a_1, \\ +1, & \text{for } 0 \leq x < a_1, \\ -1, & \text{for } b_1 \leq x < 0, \\ -n, & \text{for } x < b_1. \end{cases} \tag{2-1}$$

and

$$q(y) = \begin{cases} +n, & \text{for } y \geq a_2, \\ +1, & \text{for } 0 \leq y < a_2, \\ -1, & \text{for } b_2 \leq y < 0, \\ -n, & \text{for } y < b_2. \end{cases} \tag{2-2}$$

For a small ρ such that ρ^2 terms can be ignored, this may be simplified to give

$$R_{12} \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x)q(y)e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy. \tag{3}$$

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For the sampler threshold errors due to gain change of ϵ, and DC offset of δ, the output of a correlator (for a full 2-bit 4-levels by 2-bit 4-levels, with a high level weight $= n$, and sampler threshold value of v_0), for a small value of correlation coefficient ($\rho << 1$), can be written as (Appendix 1):

$$R_{12}(\epsilon_1, \epsilon_2, \delta_1, \delta_2) = \frac{2}{\pi}(n^2 - 2n + 4)e^{-v_0^2}\delta_1\delta_2 + \frac{2\rho}{\pi}\{(n-1)^2 e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2)$$
$$- \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2\epsilon_1\epsilon_2]$$
$$- (n-2)e^{-\frac{v_0^2}{2}}[1 - \frac{v_0}{2}(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2)] + 1\}. \quad (4)$$

Here n^2 dependent terms are due to high level products, n dependent terms are due to intermediate level products, and terms independent of n are due to low level products.

EFFECTS OF SAMPLER ERRORS ON CORRELATION MEASUREMENTS

If only high level products are considered, and intermediate and low level products are ignored, the output can be expressed as:

$$R_{12}(\epsilon_1, \epsilon_2, \delta_1, \delta_2) = \frac{2}{\pi}n^2 e^{-v_0^2}\delta_1\delta_2 + \frac{2\rho}{\pi}n^2 e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2\epsilon_1\epsilon_2]. \quad (5)$$

For only DC offsets (no gain changes) Eq.(5) simplifies to

$$R_{12}(\epsilon_1 = 0, \epsilon_2 = 0, \delta_1, \delta_2) = \frac{2}{\pi}n^2 e^{-v_0^2}\delta_1\delta_2 + \frac{2\rho}{\pi}n^2 e^{-v_0^2}[1 - \frac{1}{2}(1 - v_0^2)(\delta_1^2 + \delta_2^2)]. \quad (6)$$

For only gain variations (no DC offsets) Eq.(5) simplifies to

$$R_{12}(\epsilon_1, \epsilon_2, \delta_1 = 0, \delta_2 = 0) = \frac{2\rho}{\pi}n^2 e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2) + v_0^2\epsilon_1\epsilon_2]. \quad (7)$$

These are the same results as obtained by Thompson (1973, VLA Electronics Memo. No. 112, Eqs. A21 and A31).

It is believed that DC offset term in the output is not important for VLBI due to large fringe rate. Though this may be questionable, but for present we wiil ignore this term. However the error terms dependent on ρ will still contribute to the erroneous calculation of the correlation coefficient. For example if we assume that v_0=0.9, gain error terms $\epsilon_1 = \epsilon_2 = 0.02$, and DC offsets $\delta_1 = \delta_2 = 0.02$, then the calculated value of the correlation coefficient (using n=4 in Eq. 4) will be different by $\approx 3.25\%$ compared with the case when DC offsets and gain error terms were zero. This is much larger than the value ($\leq 0.1\%$) many people are hoping for.

Closure errors due to sampler gain errors: Consider a 3-antenna interferometer. For simplicity we will use only high level products, and ignore intermediate and low level products. From Eq.(4), ignoring $(1 - v_0^2)$ term as $v_0 \approx 1$, we can write

$$R_{ij} = G_iG_j\rho_{ij}[1 - v_0(\epsilon_i + \epsilon_j) + v_0^2\epsilon_i\epsilon_j]. \quad (8)$$

From Eq.(8) we can express antenna gain, G_i, for i^{th} antenna as

$$G_i = \{\frac{[1 - (\epsilon_i + \epsilon_j)v_0][1 - (\epsilon_i + \epsilon_k)v_0] + (\epsilon_i\epsilon_j + \epsilon_i\epsilon_k)v_0^2}{[1 - (\epsilon_i + \epsilon_j)v_0] + \epsilon_j\epsilon_k v_0^2}\}^{\frac{1}{2}} \quad (9)$$

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This may be simplified to:

$$G_i = [1 - 2v_0\epsilon_i + v_0^2(\epsilon_i^2 + \epsilon_j^2 + \epsilon_k^2 + 3(\epsilon_i\epsilon_j + \epsilon_j\epsilon_k + \epsilon_k\epsilon_i) - \epsilon_j\epsilon_k)]^{\frac{1}{2}}. \tag{10}$$

For $v_0 = 0.9$, and $\epsilon_1 = \epsilon_2 = 0.02$, and $\epsilon_3 = -0.02$, we get $G_1 = 0.982$, $G_2 = 0.982$, and $G_3 = 1.01764$; giving closure errors of 0.002, 0.002, and -0.0024 for the antennas 1, 2, and 3 respectively.

Effect of DC offset on closure errors due to only high level products is less serious, but as v_0 is not equal to one, $(1 - v_0^2)$ term will contribute to the closure errors. Also, intermediate and low level products, which are only a factor of a few less important(i.e. n= 3 to 4), will contribute to closure errors. In one bit case, similar analysis gives $G_i = \sqrt{1 - \delta_i^2}$. This means gain will be dependent on the DC offset, and may vary with time if DC offset is varying.

CONCLUSION

This suggests that if we are serious about the closure errors at 0.1% level, then we should measure the sampler quantizing threshold values using state counters at the outputs of the samplers (rather in Digital Switch modules in the Formatter), and pass-on this information for suitable quantization correction. This should be done continuously, as a sampler thresholds (in terms of rms input level) are going to vary with time due to the discrete nature of the automatic level control (ALC) circuitry in the Baseband Converters.

ACKNOWLEDGEMENTS

APPENDIX 1

DERIVATIONS OF A TWO BIT CORRELATOR EQUATIONS

The output of a correlator for gaussian noise inputs x(t) and y(t) may be expressed as

$$R_{12} = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x)q(y)e^{-(\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)})}dxdy. \qquad (A1)$$

Here q(x) and q(y) are quantizing functions for x and y, given by

$$q(x) = \begin{cases} +n, & \text{for } x \geq a_1, \\ +1, & \text{for } 0 \leq x < a_1, \\ -1, & \text{for } b_1 \leq x < 0, \\ -n, & \text{for } x < b_1. \end{cases} \qquad (A2-1)$$

and

$$q(y) = \begin{cases} +n, & \text{for } y \geq a_2, \\ +1, & \text{for } 0 \leq y < a_2, \\ -1, & \text{for } b_2 \leq y < 0, \\ -n, & \text{for } y < b_2. \end{cases} \qquad (A2-2)$$

For a small ρ such that ρ^2 terms can be ignored, this may be simplified to give

$$R_{12} \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x)q(y)e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy. \qquad (A3)$$

Using piecewise constant weights this can be expressed as a sum of 16 integrals with appropriate weights for q(x) and q(y) from Eqs. (A2). The sixteen integrals are:

$$P_{22} = \frac{1}{2\pi} \int_{a_1}^{\infty} \int_{a_2}^{\infty} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-01)$$

$$P_{21} = \frac{1}{2\pi} \int_{a_1}^{\infty} \int_{0}^{a_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-02)$$

$$P_{2\bar{1}} = \frac{1}{2\pi} \int_{a_1}^{\infty} \int_{b_2}^{0} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-03)$$

$$P_{2\bar{2}} = \frac{1}{2\pi} \int_{a_1}^{\infty} \int_{-\infty}^{b_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-04)$$

$$P_{12} = \frac{1}{2\pi} \int_{0}^{a_1} \int_{a_2}^{\infty} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-05)$$

$$P_{11} = \frac{1}{2\pi} \int_{0}^{a_1} \int_{0}^{a_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-06)$$

$$P_{1\bar{1}} = \frac{1}{2\pi} \int_{0}^{a_1} \int_{b_2}^{0} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-07)$$

$$P_{1\bar{2}} = \frac{1}{2\pi} \int_{0}^{a_1} \int_{-\infty}^{b_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-08)$$

$$P_{\bar{1}2} = \frac{1}{2\pi} \int_{b_1}^{0} \int_{a_2}^{\infty} e^{-(\frac{x^2+y^2-2\rho xy}{2})}dxdy, \qquad (A4-09)$$

$$P_{\bar{1}1} = \frac{1}{2\pi} \int_{b_1}^0 \int_0^{a_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy, \qquad (A4-10)$$

$$P_{\bar{1}\bar{1}} = \frac{1}{2\pi} \int_{b_1}^0 \int_{b_2}^0 e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy, \qquad (A4-11)$$

$$P_{\bar{1}2} = \frac{1}{2\pi} \int_{b_1}^0 \int_{-\infty}^{b_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy, \qquad (A4-12)$$

$$P_{\bar{2}2} = -\frac{1}{2\pi} \int_{-\infty}^{b_1} \int_{a_2}^{\infty} e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy, \qquad (A4-13)$$

$$P_{\bar{2}1} = \frac{1}{2\pi} \int_{-\infty}^{b_1} \int_0^{a_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy, \qquad (A4-14)$$

$$P_{\bar{2}\bar{1}} = \frac{1}{2\pi} \int_{-\infty}^{b_1} \int_{b_2}^0 e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy, \qquad (A4-15)$$

$$P_{\bar{2}\bar{2}} = \frac{1}{2\pi} \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} e^{-(\frac{x^2+y^2-2\rho xy}{2})} dx\, dy. \qquad (A4-16)$$

Using above integrals and appropriate weights from Eq.(A2) we can write

$$R_{12} = n^2(P_{22}-P_{2\bar{2}}-P_{\bar{2}2}+P_{\bar{2}\bar{2}}) + n(P_{21}-P_{2\bar{1}}+P_{12}-P_{1\bar{2}}-P_{\bar{1}2}+P_{\bar{1}\bar{2}}-P_{\bar{2}1}+P_{\bar{2}\bar{1}}) + (P_{11}-P_{1\bar{1}}-P_{\bar{1}1}+P_{\bar{1}\bar{1}}). \quad (A5)$$

Now we simplify Eq.(A4-01) by substituting $e^{\rho xy} = 1 + \rho xy$, as $\rho << 1$.

$$P_{22} = \frac{1}{2\pi} \int_{a_1}^{\infty} \int_{a_2}^{\infty} e^{-(\frac{x^2}{2})}e^{-(\frac{y^2}{2})}(1+\rho xy) dx\, dy$$

$$= [\frac{1}{\sqrt{2\pi}} \int_{a_1}^{\infty} e^{-(\frac{x^2}{2})}dx][\frac{1}{\sqrt{2\pi}} \int_{a_2}^{\infty} e^{-(\frac{y^2}{2})}dy] + \frac{\rho}{2\pi}[\int_{a_1}^{\infty} xe^{-(\frac{x^2}{2})}dx + \int_{a_2}^{\infty} ye^{-(\frac{y^2}{2})}dy]$$

$$= (\frac{1-\phi_{a1}}{2})(\frac{1-\phi_{a2}}{2}) + \frac{\rho}{2\pi}E_{a1}E_{a2}, \qquad (A6)$$

where $\phi_x = erf(x/\sqrt{2}) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt = 1 - \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt$, and $E_x = e^{-\frac{x^2}{2}}$. Therefore we can write

$$P_{22} = \frac{1}{4}(1-\phi_{a1})(1-\phi_{a2}) + \frac{\rho}{2\pi}E_{a1}E_{a2}. \qquad (A7-01)$$

Similarly we can simplify other integrals in Eqs.(A4) to give

$$P_{21} = \frac{1}{4}(1-\phi_{a1})\phi_{a2} + \frac{\rho}{2\pi}E_{a1}(1-E_{a2}). \qquad (A7-02)$$

$$P_{2\bar{1}} = \frac{1}{4}(1-\phi_{a1})\phi_{b2} - \frac{\rho}{2\pi}E_{a1}(1-E_{b2}). \qquad (A7-03)$$

$$P_{2\bar{2}} = \frac{1}{4}(1-\phi_{a1})(1-\phi_{b2}) - \frac{\rho}{2\pi}E_{a1}E_{b2}. \qquad (A7-04)$$

$$P_{12} = \frac{1}{4}\phi_{a1}(1-\phi_{a2}) + \frac{\rho}{2\pi}(1-E_{a1})E_{a2}. \qquad (A7-05)$$

$$P_{11} = \frac{1}{4}\phi_{a1}\phi_{a2} + \frac{\rho}{2\pi}(1-E_{a1})(1-E_{a2}). \qquad (A7-06)$$

$$P_{1\bar{1}} = \frac{1}{4}\phi_{a1}\phi_{b2} - \frac{\rho}{2\pi}(1-E_{a1})(1-E_{b2}). \qquad (A7-07)$$

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$$P_{1\bar{2}} = \frac{1}{4}\phi_{a1}(1 - \phi_{b2}) - \frac{\rho}{2\pi}(1 - E_{a1})E_{b2}. \qquad (A7 - 08)$$

$$P_{\overline{1}2} = \frac{1}{4}\phi_{b1}(1 - \phi_{a2}) - \frac{\rho}{2\pi}(1 - E_{b1})E_{a2}. \qquad (A7 - 09)$$

$$P_{\overline{1}1} = \frac{1}{4}\phi_{b1}\phi_{a2} - \frac{\rho}{2\pi}(1 - E_{b1})(1 - E_{a2}). \qquad (A7 - 10)$$

$$P_{\overline{1}\overline{1}} = \frac{1}{4}\phi_{b1}\phi_{b2} + \frac{\rho}{2\pi}(1 - E_{b1})(1 - E_{b2}). \qquad (A7 - 11)$$

$$P_{\overline{1}\bar{2}} = \frac{1}{4}\phi_{b1}(1 - \phi_{b2}) + \frac{\rho}{2\pi}(1 - E_{b1})E_{b2}. \qquad (A7 - 12)$$

$$P_{\bar{2}2} = \frac{1}{4}(1 - \phi_{b1})(1 - \phi_{a2}) - \frac{\rho}{2\pi}E_{b1}E_{a2}. \qquad (A7 - 13)$$

$$P_{\bar{2}1} = \frac{1}{4}(1 - \phi_{b1})\phi_{a2} - \frac{\rho}{2\pi}E_{b1}(1 - E_{a2}). \qquad (A7 - 14)$$

$$P_{\bar{2}\overline{1}} = \frac{1}{4}(1 - \phi_{b1})\phi_{b2} + \frac{\rho}{2\pi}E_{b1}(1 - E_{b2}). \qquad (A7 - 15)$$

$$P_{\bar{2}\bar{2}} = \frac{1}{4}(1 - \phi_{b1})(1 - \phi_{b2}) + \frac{\rho}{2\pi}E_{b1}E_{b2}. \qquad (A7 - 16)$$

Substituting from Eqs.(A7) in Eq.(A5) we get

$$
\begin{aligned}
R_{12} = {} & \frac{1}{4}(n^2 - 2n + 4)(\phi_{a1}\phi_{a2} - \phi_{a1}\phi_{b2} - \phi_{b1}\phi_{a2} + \phi_{b1}\phi_{b2}) \\
& + \frac{\rho}{2\pi}\{n^2[E_{a1}E_{a2} + E_{a1}E_{b2} + E_{b1}E_{a2} + E_{b1}E_{b2}] \\
& + n[(E_{a1} + E_{a2} + E_{b1} + E_{b2}) - 2(E_{a1}E_{a2} + E_{a1}E_{b2} + E_{b1}E_{a2} + E_{b1}E_{b2})] \\
& + [4 - 2(E_{a1} + E_{a2} + E_{b1} + E_{b2}) + (E_{a1}E_{a2} + E_{a1}E_{b2} + E_{b1}E_{a2} + E_{b1}E_{b2})]\}.
\end{aligned}
\qquad (A8)
$$

To simplify notations in Eq.(A8) we use following symbols

$$\Phi = \phi_{a1}\phi_{a2} - \phi_{a1}\phi_{b2} - \phi_{b1}\phi_{a2} + \phi_{b1}\phi_{b2}, \qquad (A9 - 1)$$

$$\Psi = E_{a1}E_{a2} + E_{a1}E_{b2} + E_{b1}E_{a2} + E_{b1}E_{b2}, \qquad (A9 - 2)$$

$$\Theta = E_{a1} + E_{a2} + E_{b1} + E_{b2}. \qquad (A9 - 3)$$

Using notations given in Eqs.(A9), we can write Eq.(A8) as

$$R_{12} = \frac{1}{4}(n^2 - 2n + 4)\Phi - + \frac{\rho}{2\pi}\{n^2\Psi + n(\Theta - 2\Psi) + (4 - 2\Theta + \Psi)\}. \qquad (A10 - 1)$$

We can rewrite this as

$$R_{12} = \frac{1}{4}(n^2 - 2n + 4)\Phi + \frac{\rho}{2\pi}\{(n - 1)^2\Psi + (n - 2)\Theta + 4\}. \qquad (A10 - 2)$$

Now consider DC offsets of δ_1 and δ_2, and input power level (sampler gain) change of ϵ_1 and ϵ_2. Then assuming ideal threshold of v_0, we can write

$$a_1 = v_0 + \epsilon_1 + \delta_1, \qquad (A11 - 1)$$

$$b_1 = -v_0 - \epsilon_1 + \delta_1 = -v_0 - (\epsilon_1 - \delta_1), \qquad (A11 - 2)$$

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$$a_2 = v_0 + \epsilon_2 + \delta_2, \tag{$A11-3$}$$

$$b_2 = -v_0 - \epsilon_2 + \delta_2 = -v_0 - (\epsilon_2 - \delta_2). \tag{$A11-4$}$$

Substituting following in Eqs. (A11-1) to (A11-4)

$$\alpha_1 = \epsilon_1 + \delta_1, \tag{$A11-5$}$$

$$\alpha_2 = \epsilon_2 + \delta_2, \tag{$A11-6$}$$

$$\beta_1 = \epsilon_1 - \delta_1, \tag{$A11-7$}$$

$$\beta_2 = \epsilon_2 - \delta_2. \tag{$A11-8$}$$

we get

$$a_1 = v_0 + \alpha_1, \tag{$A12-1$}$$

$$b_1 = -v_0 - \beta_1, \tag{$A12-2$}$$

$$a_2 = v_0 + \alpha_2, \tag{$A12-3$}$$

$$b_2 = -v_0 - \beta_2. \tag{$A12-4$}$$

If we write $x = v_0 + \Delta$, we can simplify $E_x = e^{-\frac{x^2}{2}} = e^{-\frac{(v_0+\Delta)^2}{2}} = e^{-(\frac{v_0^2}{2} + \frac{\Delta^2}{2} + \Delta v_0)}$. This may be further simplified by substituting $e^{-\Delta v_0} = 1 - \Delta v_0$, and $e^{-\frac{\Delta^2}{2}} = 1 - \frac{\Delta^2}{2}$, and ignoring terms with order higher than Δ^2, to give

$$E_x = e^{-\frac{v_0^2}{2}}[1 - (\Delta v_0 + \frac{\Delta^2}{2}) + \frac{(\Delta v_0)^2}{2}]. \tag{13}$$

Substituting from Eqs.(A12) in Eq.(A13), we get

$$E_{a1} = e^{-\frac{v_0^2}{2}}[1 - (\alpha_1 v_o + \frac{\alpha_1^2}{2}) + \frac{(\alpha_1 v_0)^2}{2}], \tag{$A14-1$}$$

$$E_{b1} = e^{-\frac{v_0^2}{2}}[1 - (\beta_1 v_o + \frac{\beta_1^2}{2}) + \frac{(\beta_1 v_0)^2}{2}], \tag{$A14-2$}$$

$$E_{a2} = e^{-\frac{v_0^2}{2}}[1 - (\alpha_2 v_o + \frac{\alpha_2^2}{2}) + \frac{(\alpha_2 v_0)^2}{2}], \tag{$A14-3$}$$

$$E_{b2} = e^{-\frac{v_0^2}{2}}[1 - (\beta_2 v_o + \frac{\beta_2^2}{2}) + \frac{(\beta_2 v_0)^2}{2}]. \tag{$A14-4$}$$

Now we express $x = v_0 + \Delta$, and use Taylor series expansion to give

$$\phi_x = \phi_{v_0} + \Delta\phi'_{v_0} + \frac{\Delta^2}{2!}\phi''_{v_0} + \frac{\Delta^3}{3!}\phi'''_{v_0} + ... \tag{$A15$}$$

Here $\phi'_{v_0} = \frac{1}{\sqrt{2\pi}}e^{-\frac{v_0^2}{2}}$, $\phi''_{v_0} = -\frac{2}{\sqrt{2\pi}}v_0 e^{-\frac{v_0^2}{2}}$, and $\phi'''_{v_0} = -\frac{2}{\sqrt{2\pi}}(1 - v_0^2)e^{-\frac{v_0^2}{2}}$. Substituting these in Eq.(A15), we get

$$\phi_x = \phi_{v_0} + \frac{1}{\sqrt{2\pi}}e^{-\frac{v_0^2}{2}}[2\Delta - v_0\Delta^2 - \frac{1}{3}(1 - v_0^2)\Delta^3 + ...] \tag{$A16$}$$

Thus, ignoring terms in Δ with powers higher than two, we can write

$$\phi_{a1} = \phi_{v_0} + \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \alpha_1 (1 - \frac{v_0 \alpha_1}{2})$$

$$= \phi_{v_0} + \chi \alpha_1 (1 - \frac{v_0 \alpha_1}{2}), \tag{17-1}$$

where $\chi = \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}}$. Similarly,

$$\phi_{b1} = \phi_{v_0} + \chi \beta_1 (1 - \frac{v_0 \beta_1}{2}), \tag{A17-2}$$

$$\phi_{a2} = \phi_{v_0} + \chi \alpha_2 (1 - \frac{v_0 \alpha_2}{2}), \tag{A17-3}$$

$$\phi_{b2} = \phi_{v_0} + \chi \beta_2 (1 - \frac{v_0 \beta_2}{2}). \tag{A17-4}$$

Substituting in Eq.(A9-1) from Eqs.(A17), we get

$$\Phi = \chi^2 (\alpha_1 \alpha_2 - \alpha_1 \beta_2 - \alpha_2 \beta_1 + \beta_1 \beta_2), \tag{A18}$$

and replacing α and β with ϵ and δ from Eqs.(A11-5) to (A11-8), and substituting for χ, we get

$$\Phi = \frac{2}{\pi} e^{-v_0^2} (4\delta_1 \delta_2). \tag{A19-1}$$

Similarly we solve Eqs.(A9-2) and (A9-3) by using relations given in Eqs.(A14) and (A17), and then substituting for α, and β from Eqs.(A11-5) to (A11-8). In solving these equations we ignore terms containing α, β, ϵ, and δ with powers higher than two. We get

$$\Psi = 4e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2 \epsilon_1 \epsilon_2], \tag{A19-2}$$

$$\Theta = 4e^{-\frac{v_0^2}{2}}[1 - \frac{1}{2}v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2)]. \tag{A19-3}$$

Substituting from Eqs.(A19) in Eq.(A10-1) we get

$$R_{12} = \frac{2}{\pi}(n^2 - 2n + 4)e^{-v_0^2}\delta_1\delta_2 + \frac{2\rho}{\pi}\{n^2 e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2 \epsilon_1 \epsilon_2]$$

$$+ n[e^{-\frac{v_0^2}{2}}[1 - v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2)]$$

$$- 2e^{-v_0^2}[1 - \frac{1}{2}v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2 \epsilon_1 \epsilon_2]]$$

$$+ [e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2 \epsilon_1 \epsilon_2]$$

$$- 2e^{-\frac{v_0^2}{2}}[1 - \frac{1}{2}v_0(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2)] + 1]\}. \tag{A20-1}$$

We can simplify this by substituting Eqs.(A19) in Eq.(A10-2), which gives

$$R_{12}(\epsilon_1, \epsilon_2, \delta_1, \delta_2) = \frac{2}{\pi}(n^2 - 2n + 4)e^{-v_0^2}\delta_1\delta_2 + \frac{2\rho}{\pi}\{(n - 1)^2 e^{-v_0^2}[1 - v_0(\epsilon_1 + \epsilon_2)$$

$$- \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2) + v_0^2 \epsilon_1 \epsilon_2] \tag{A20-2}$$

$$- (n - 2)e^{-\frac{v_0^2}{2}}[1 - \frac{v_0}{2}(\epsilon_1 + \epsilon_2) - \frac{1}{2}(1 - v_0^2)(\epsilon_1^2 + \epsilon_2^2 + \delta_1^2 + \delta_2^2)] + 1\}.$$

Here n^2 dependent terms are due to high level products, n dependent terms are due to intermediate level products, and n independent terms are due to low level products.

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