O'SULLIVAN'S ZERO-PADDING

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J. D. O'Sullivan [1982] suggests, among other things, that the desirable non-cyclic correlation of a sequence with itself or with another sequence can be obtained by augmenting the sequences with an equal number of zero-entries before using the FFT and a term-by-term product to obtain what would otherwise be the DFT of a cyclic correlation. The purpose of this note is to investigate such zero-padding as it may apply to VLBA processing.

Let g(t) be a real waveform and G(f) its spectrum. If N uniformly-spaced samples of g(t), taken at the sampling rate w, are used to describe the waveform and its spectrum, g and G are interrelated by the DFT pair

$$G\left(\frac{nw}{N}\right) = \frac{1}{w} \sum_{k=0}^{N-1} g\left(\frac{k}{w}\right) e^{-j2\pi\frac{nk}{N}} \quad , \; n = 0,\, 1,\, \ldots\, N-1$$

$$g\left(\frac{k}{w}\right) = \frac{w}{N} \sum_{n=0}^{N-1} G\left(\frac{nw}{N}\right) e^{j2\pi\frac{nk}{N}} \quad , \; k = 0,\, 1,\, \ldots\, N-1. \tag{1}$$

It is easy to show that this is a transform pair: If N values of G are computed from N samples of g using the first relation, and these N values are used in the second relation to compute N samples of g, these N samples agree with the N samples of g that were originally entered into the first relation. However, since the DFT pair makes G(f) periodic with period w and g(t) periodic with period N/w, aliasing in both time and frequency is possible: A spectral component may be shifted -- not folded -- up or down in frequency by an integer number of periods w from the position it would be given by the Fourier integral. Similarly, a section of waveform may be shifted ahead or behind by an integer number of periods N/w.

To examine the cyclic correlation that results when zero-padding is not employed, let g(t) and h(t) be real waveforms with spectra G(f) and H(f), respectively, and let the waveforms be represented by N coincident sample pairs, taken at the sampling rate w. The voltage spectra are then given by (1), and the values of the cross-power spectrum of g and h are proportional to

$$S\left(\frac{nw}{N}\right) = G^*\left(\frac{nw}{N}\right) H\left(\frac{nw}{N}\right) \quad , \; n = 0,\, 1,\, \ldots\, N-1. \tag{2}$$

Although their inverse DFT has not been properly scaled to make these numbers cross-power spectral values and their inverse DFT, itself, cross-correlation samples, I shall nevertheless refer to these numbers as cross-power spectral values and to their inverse DFT as cross-correlation samples. By this choice, I have opted to avoid consideration of the windowing of the cross-correlation function. I shall also omit consideration of the accumulation of successive sets of cross-power spectral values, which must be an essential piece of VLBA processing. With this understanding, the cross-correlation samples that would have to be DFT'd to produce the power spectrum values of (2) are

$$C(\tfrac{k}{w}) = \frac{w}{N} \sum_{n=0}^{N-1} S(\tfrac{nw}{N})\, e^{j2\pi\frac{nk}{N}} \quad , \quad k = 0,\ 1,\ \ldots\ N-1$$

$$= \frac{w}{N} \sum_{n=0}^{N-1} G^{*}(\tfrac{nw}{N})\, H(\tfrac{nw}{N})\, e^{j2\pi\frac{nk}{N}}$$

$$= \frac{1}{Nw} \sum_{n=0}^{N-1} e^{j2\pi\frac{nk}{N}} \sum_{i=0}^{N-1} g(\tfrac{i}{w})\, e^{j2\pi\frac{ni}{N}} \sum_{m=0}^{N-1} h(\tfrac{m}{w})\, e^{-j2\pi\frac{nm}{N}}$$

$$= \frac{1}{w} \sum_{i=0}^{N-1} g(\tfrac{i}{w}) \sum_{m=0}^{N-1} h(\tfrac{m}{w}) \cdot \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi\frac{n}{N}(i+k-m)} \ .$$

The last sum is

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi\frac{n}{N}(i+k-m)} = \begin{cases} 1 \ , & (i+k-m) = \text{integer} \times N \\ 0, & \text{otherwise.} \end{cases}$$

The only possible values of "integer x N" in the above condition are 0 -- if $i + k < N$ -- and N -- if $i + k \geq N$. In either case,

$$m = (i + k) \bmod N$$

when the condition is satisfied.

The cross-correlation samples are then

$$C(\tfrac{k}{w}) = \frac{1}{w} \sum_{i=0}^{N-1} g(\tfrac{i}{w})\, h(\tfrac{1}{w}[(i+k) \bmod N]) \ , \ k = 0,\ 1,\ \ldots\ N-1. \qquad (3)$$

Except for C(0), which is correctly calculated, these samples are all incorrect:

The first (N-k) products in the sum have the h-sample taken k/w seconds after the g-sample, but the last k products have the h-sample taken $\frac{N-k}{w}$ seconds <u>before</u> the g-sample! In simplified notation, the products in each sum are listed below for N = 4.

k = 0	k = 1	k = 2	k = 3
g_0h_0	g_0h_1	g_0h_2	g_0h_3
g_1h_1	g_1h_2	g_1h_3	$\underline{g_1h_0}$
g_2h_2	g_2h_3	$\underline{g_2h_0}$	$\underline{g_2h_1}$
g_3h_3	$\underline{g_3h_0}$	$\underline{g_3h_1}$	$\underline{g_3h_2}$

The underlined products are out of place. There are seven, rather than four, samples of the correctly-calculated cross-correlation function in the present case, and these are the sums of the products in the following columns:

k = -3	k = -2	k = -1	k = 0	k = 1	k = 2	k = 3
g_3h_0	g_2h_0	g_1h_0	g_0h_0	g_0h_1	g_0h_2	g_0h_3
	g_3h_1	g_2h_1	g_1h_1	g_1h_2	g_1h_3	
		g_3h_2	g_2h_2	g_2h_3		
			g_3h_3			

O'Sullivan's solution to this cyclic correlation difficulty is to augment the N-element sequences of g- and h-samples with zeros before they are FFT'd, so that they become 2N-element sequences. The spectral values G of g are then given by

$$G\left(\frac{nw}{2N}\right) = \frac{1}{w} \sum_{k=0}^{2N-1} g\left(\frac{k}{w}\right) e^{-j2\pi\frac{nk}{2N}} \quad , \quad n = 0, 1, \ldots 2N-1.$$

But since, for the zero-padded sequence

$$g\left(\frac{k}{w}\right) = \begin{cases} g\left(\frac{k}{w}\right) , & k < N \\ 0, & \text{otherwise,} \end{cases}$$

only the first half of the sum needs to be completed, which leaves

$$G\left(\frac{nw}{2N}\right) = \frac{1}{w} \sum_{k=0}^{N-1} g\left(\frac{k}{w}\right) e^{-j2\pi\frac{nk}{2N}} \quad , \quad n = 0, 1, \ldots 2N-1. \tag{4}$$

Values of the cross-power spectrum of g and h are

$$S\left(\frac{nw}{2N}\right) = G^*\left(\frac{nw}{2N}\right) H\left(\frac{nw}{2N}\right) \quad , \quad n = 0, 1, \ldots 2N-1, \tag{5}$$

and the cross-correlation samples that would have to be DFT'd to produce them are

$$C\left(\frac{k}{w}\right) = \frac{w}{2N} \sum_{n=0}^{2N-1} S\left(\frac{nw}{2N}\right) e^{j2\pi\frac{nk}{2N}} \quad , \quad k = 0, 1, \ldots 2N-1$$

$$= \frac{1}{2Nw} \sum_{n=0}^{2N-1} e^{j2\pi\frac{nk}{2N}} \sum_{i=0}^{N-1} g\left(\frac{i}{w}\right) e^{j2\pi\frac{ni}{2N}} \sum_{m=0}^{N-1} h\left(\frac{m}{w}\right) e^{-j2\pi\frac{nm}{2N}}$$

$$= \frac{1}{w} \sum_{i=0}^{N-1} g\left(\frac{i}{w}\right) \sum_{m=0}^{N-1} h\left(\frac{m}{w}\right) \cdot \frac{1}{2N} \sum_{n=0}^{2N-1} e^{j2\pi\frac{n}{2N}(i+k-m)} .$$

The last sum is

$$\frac{1}{2N} \sum_{n=0}^{2N-1} e^{j2\pi\frac{n}{2N}(i+k-m)} = \begin{cases} 1, & m = (i + k) \bmod 2N \\ 0, & \text{otherwise,} \end{cases}$$

so the cross-correlation samples are

$$C\left(\frac{k}{w}\right) = \frac{1}{w} \sum_{i=0}^{N-1} g\left(\frac{i}{w}\right) h\left(\frac{1}{w}[(i+k) \bmod 2N]\right) \quad , \quad k = 0, 1, \ldots 2N-1. \tag{6}$$

With this revision, unwanted products are not aliased to fall on top of wanted products. For N = 4 -- and 2N = 8 -- the cross-correlation samples produced are the sums of the products in the following columns:

k = 0	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
g_0h_0	g_0h_1	g_0h_2	g_0h_3	0	g_3h_0	g_2h_0	g_1h_0
g_1h_1	g_1h_2	g_1h_3				g_3h_1	g_2h_1
g_2h_2	g_2h_3						g_3h_2
g_3h_3							

The desired cross-correlation function has been aliased in time, but its values are all there. As O'Sullivan puts it, without zero-padding, the power density spectrum was not sampled densely enough to make the first calculation work.

The foregoing calculations have demonstrated conclusively that without zero-padding, the FX correlator produces erroneous correlations. Yet Chikada [1981] has announced plans for an FX correlator that does not employ zero-padding, and Ed Fomalont, in a 1986 visit to Nobeyama, did not hear of revisions to these plans. It is indeed disturbing to find such a fundamental paradox blocking the road to a successful VLBA processing design! Having resolved the paradox, it was my pleasure to discover that both O'Sullivan and Chikada have found viable means for achieving their goals. It is the goals, then, that are different.

Consider the cross-power spectral values that are produced with and without zero-padding. Without zero-padding, from (1) and (2), these are

$$S(\frac{nw}{N}) = \frac{1}{w^2} \sum_{i=0}^{N-1} g(\frac{i}{w})\ e^{j2\pi\frac{ni}{N}} \sum_{k=0}^{N-1} h(\frac{k}{w})\ e^{-j2\pi\frac{nk}{N}} \quad , \quad n = 0, 1, \ldots N-1, \qquad (7)$$

and with zero-padding, from (4) and (5),

$$S(\frac{nw}{2N}) = \frac{1}{w^2} \sum_{i=0}^{N-1} g(\frac{i}{w})\ e^{j2\pi\frac{ni}{2N}} \sum_{k=0}^{N-1} h(\frac{k}{w})\ e^{-j2\pi\frac{nk}{2N}} \quad , \quad n = 0, 1, \ldots 2N-1. \qquad (8)$$

Evidently the spectral values calculated without zero-padding agree with the even-numbered spectral values calculated with zero-padding; the former are a subset of the values calculated with zero-padding. If Chikada finds it satisfactory to have his results limited to the calculation of half as many spectral values as would be provided by zero-padding, then his FX correlator need not employ zero-padding. But if O'Sullivan wishes to inverse DFT his accumulated cross-power spectral values to obtain correctly-calculated cross-correlation samples (6), then his correlator must employ zero-padding.

Let it be noted that correctly-calculated cross-correlation samples are required for post-accumulation quantization (Van Vleck) correction and windowing adjustments. If these are to be a part of VLBA processing, then zero-padding must be employed. But several other conditions must also be satisfied to make correctly-calculated cross-correlation samples accessible: (1) Ray Escoffier's plan to accumulate sums of 2, 4, 8, and 16 adjacent cross-power spectral values must not be used; all cross-power spectral values must be separately accumulated. (2) I expect to show, in a future note, that fringe rotation and fractional bit-shift correction must be done before quantization at the antennas. A decision to achieve the major station time-keeping improvements required by these conditions must certainly not be lightly made.

Although there is more to come, I now favor a VLBA decision to omit post-accumulation quantization correction and windowing adjustments. Zero-padding would then not be required, Ray Escoffier could accumulate his sums of adjacent cross-power spectral values, and fringe rotation and fractional bit-shift correction could be done at the central station as the tapes are processed.

References:

J. D. O'Sullivan, "Efficient Digital Spectrometers - A Survey of Possibilities," Netherlands Foundation for Radio Astronomy, Note 375, September 1982.

Yoshihiro Chikada, "Techniques for Spectral Measurements," Nobeyama Radio Observatory, Technical Report No. 8, 1981.