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PHASE SWITCHING, FRINGE ROTATION, AND THE ORTHOGONALITY OF WALSH FUNCTIONS AND SQUARE WAVES

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I. Interaction Between Phase Switching and Fringe Rotation

During the VLBA design review meeting held at Green Bank, September 10-11, 1985, the question of the interaction of phase switching and fringe rotation was briefly discussed. The phase switching is chiefly required to eliminate the effect of offsets in correlator outputs, which result mainly from offsets in the quantization thresholds in the samplers. With phase switching, these offsets become multiplied by the phase-switching waveforms applied to the two antennas, which are orthogonal. Thus the unwanted offsets are reduced to very small levels by the averaging of the visibility. However, if the fringe rotation is applied after the sampling and quantization of the data, the unwanted component of the correlator output is further multiplied by the fringe rotation waveform, which is a simulation of a sinewave at the natural fringe frequency. A spurious response can occur if the fringe rotation waveform is not orthogonal to the product of the phase switching waveforms.

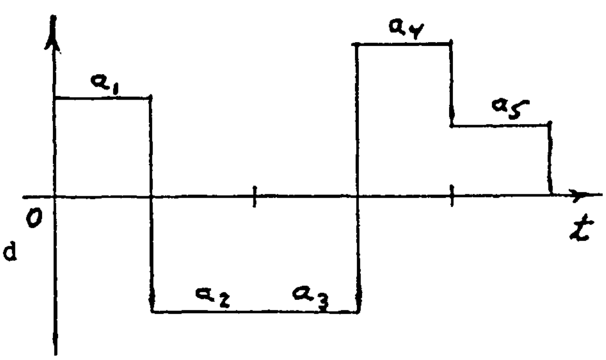
If the phase switching waveforms are Walsh functions, their product is another Walsh function. During the discussion of phase switching and fringe rotation at the review meeting, Dick Thompson noted that certain Walsh functions can be described as products of two or more harmonically-related square waves. Suppose that the product of the switching waveforms is such a Walsh function. Then when the fringe frequency becomes equal to that of one of the square wave components, it is still orthogonal to the product, and no spurious response will occur. Is it possible to choose Walsh functions that are orthogonal to all fringe-frequency waveforms? Further thought shows that it is not possible. This is shown by John Granlund's analysis in part II of this memorandum, which proves that no Walsh function is orthogonal to all square waves. (The orthogonality with square waves rather than sine-waves was considered here for mathematical convenience.) The same result can be visualized rather simply by considering that Walsh functions, being periodic, can be expressed as a summation of Fourier-series terms. If the fringe frequency becomes equal to one such term a spurious response can occur. Thus, as Barry Clark pointed out, to avoid the possibility of interaction between fringe rotation and phase switching the lowest Fourier component of the Walsh functions should be higher than the maximum fringe frequency. The maximum fringe frequency, for which we consider the Hawaii-St. Croix baseline at 86 GHz, is approximately 200 kHz. Phase switching at frequencies greater that 200 kHz would hardly be practicable, and would certainly complicate the operation of the array.

The question of interaction between phase switching and fringe rotation is largely avoided if fringe rotation is performed before the sampler, i.e., by phase shifts in a local oscillator. One then has only to consider the residual fringe frequencies produced by signals arriving from different directions within the antenna beam; these residual frequencies will not exceed 60 Hz. However, the simplification of the phase switching that occurs when the fringe rotation is performed before the sampling should not be a major factor in the choice of location of the fringe rotators. The fringe frequency in VLBI observations is usually high enough that phase switching is unnecessary. The time intervals during which the fringe frequency goes through zero for any antenna pair are short, and phase switching could be applied to individual antennas for short periods to cover such times. The control computer could select a switching frequency that would not cause problems by interaction with the fringe frequencies for any baselines involving the particular antenna. Thus by applying phase switching only when necessary, it should be possible to avoid unwanted interactions. The principal questions upon which the location of the fringe rotators should rest are, first, whether they can be implemented without degradation of astrometric measurements and, second, what will be the resulting cost saving in the correlator.

II. Orthogonality of Walsh Functions and Square Waves.

The set of sequency-ordered Walsh functions numbered* from 0 through 2^n-1, with n a positive integer, is an orthogonal and complete set of what I'll call clocked functions with 2^n intervals. Because this set is complete, no clocked function with 2^n intervals, and thus no square wave that is itself a clocked function with 2^n intervals, is orthogonal to each member of the set unless that function or square wave is identically zero. Much more to the point, the n-set of all square waves is itself a complete set of clocked functions with 2^n intervals. It follows that no non-trivial clocked function with 2^n intervals, and thus none of the first 2^n Walsh functions, is orthogonal to each member of the n-set of all square waves. Definitions and discussions of the underlined terms follow.

Clocked function. A clocked function with N intervals assumes a fixed value throughout each of N consecutive and equal time intervals; elsewhere it is identically zero. It is completely specified by these N values and its duration. For Walsh functions and square waves, the values are restricted to +1 and -1. In these cases, it will be convenient to represent a clocked function with 5 intervals as +--++, for example.



Orthogonal. Two clocked functions with N intervals and the sets of values $\{a_k\}$ and $\{b_k\}$ are orthogonal if

*K. G. Beauchamp, Walsh Functions and Their Applications. London: Academic Press, 1975.

$$\sum_{k=1}^{N} a_k b_k = 0.$$

Because the values of Walsh functions and square waves are restricted to +1 and −1, N must be even if a pair of such functions is to be orthogonal.

Complete. A set of clocked functions with N intervals is complete if an arbitrary clocked function with N intervals can be expressed as a weighted sum of the members of the set. Clearly the set must contain at least N members, and it would be redundant if it contained more; we shall limit to N members sets of clocked functions with N intervals. Let the values of the k^{th} clocked function in the set be contained in left-to-right order in the row vector $\underline{a_k}$, and from these vectors form the N × N matrix

$$[A] = \begin{bmatrix} \underline{a_1} \\ \underline{a_2} \\ \vdots \\ \underline{a_N} \end{bmatrix}.$$

Let $\underline{x}$ contain the ordered values of the arbitrary clocked function to be fitted, and let $\underline{w}$ contain the required weights for the weighted sum, also in order. Then the linear equations to be solved for the weights, in matrix form, read

$$\underline{w} \times [A] = \underline{x}$$

and have the solution

$$\underline{w} = \underline{x} \times [A]^{-1} .$$

This unique solution will fail to exist if and only if $|[A]|$, the determinant of [A], vanishes.

The case in which each member of the set of N clocked functions with N intervals is orthogonal to every other member is easy to check. Consider the matrix product

$$[B] \equiv [A] \times [A]_t \ ,$$

in which $[A]_t$ is the transpose of [A]. [B] is diagonal in this case, with each diagonal element consisting of the sum of N squares. The determinant $|[B]|$ cannot vanish if none of the clocked functions of the orthogonal set is identically zero. But

$$|[B]| = |[A]|^2 \ ,$$

so $|[A]|$ cannot then vanish either. This shows that every set of N mutually orthogonal -- and non-trivial -- clocked functions with N intervals is complete. A complete set, however, need not have mutually orthogonal members.

Beauchamp (op. cit.) introduces the clocked function set consisting of 8 mutually orthogonal block pulses and proceeds to argue that this is not a complete set! There are two reasons for this discrepancy: First, Beauchamp insists that a set of functions -- clocked functions, perhaps -- is not complete unless an arbitrary function can be fitted arbitrarily well by a weighted sum of members of the set. It is clear, then, that each of his complete sets must contain infinitely many members. Second, once the 8 mutually orthogonal block pulses have been chosen as the first 8 members of the set of block pulses, these original 8 cannot be removed or reshaped, and evidently all additions to the set must have the same pulse width and duration as the first 8. We are not fettered by these difficulties. Even with N not a power of 2, the set of N mutually orthogonal block pulses is a complete set in the sense of this memo. Herein, a complete set of clocked functions with N intervals is one that spans the space of values of an arbitrary clocked function with N intervals.

It has been noted that, for two Walsh functions or square waves to be orthogonal, the number of intervals N must be even. For such clocked functions with values restricted to +1 and -1, the largest number of mutually orthogonal functions that can be found is the highest power of 2 contained as a factor in N. If a complete and mutually orthogonal set of such functions is to exist, N must then itself be a power of 2.

A proof of this bound on the number of orthogonal functions uses two facts about the orthogonality, displayed by the matrix [B] defined above, of a set of mutually orthogonal clocked functions with N intervals. First, changing the sign of the k^{th} value of each member of the set does not change [B] and thus does not affect the orthogonality of the set. Second, modifying, in the same way, the order in which the values of <u>all</u> functions appear changes neither [B] nor the orthogonality of the set. In seeking mutually orthogonal functions for the set of clocked functions with N intervals, the first fact offers the option that the first function may contain all +'s. If it is to be orthogonal to this first function, the second function must then have N/2 values of +1 and N/2 of -1. The second fact allows these values to be ordered so that the +'s appear first. If additional functions orthogonal to the first two exist, they can be constructed by seeking a new clocked function with N/2 intervals that is orthogonal to the clocked function with N/2 intervals and having the -- all + 1 -- values of the first half of the first function. The search will succeed if and only if N/2 is even. The new function will have N/4 values of +1 and N/4 values of -1, which will be ordered, using the second fact, with the +'s first. If the search succeeds, the number of clocked functions with N intervals will be doubled again by adding a twin to each function already found. In the twin, the all + 1 values of the original function in its first and last N/2 intervals are replaced by the values of the new function, and the all -1 values are replaced by those values with signs changed. The extended set of clocked functions with N intervals is again seen to be mutually orthogonal.

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+++ +++ +++ +++

+++ +++ --- ---

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Starting with the first member of the set of mutually orthogonal functions and repeating the process outlined above until it fails, produces $\log_2$ (highest power of 2 contained as a factor in N) doublings of the set size. Thus the size of the final set is the highest power of 2 contained as a factor in N. The constructive proof that has been used has generated only one of the possible sets of mutually orthogonal functions. The others can be generated from this set by using the two facts noted above.

<u>n-set of all square waves</u>. For $N = 2^n$, the n-set of all square waves will be <u>chosen</u> to be a complete set of clocked functions with 2^n intervals, in such a way that each function is a square wave. For n = 1, 2, and 3 the chosen function values are arranged in the rows listed below.

<u>n = 1</u>	<u>n = 2</u>		<u>n = 3</u>
+ +	+ + + +	(1)	+ + + + + + + +
+ -	+ + - -	(2)	+ + + + - - - -
	+ - - +	(3)	+ + + - - - - +
	+ - + -	(4)	+ + - - - - + +
		(5)	+ - - - - + + +
		(6)	+ + - - + + - -
		(7)	+ - - + + - - +
		(8)	+ - + - + - + -

The members of the 1-set of chosen functions are seen to be mutually orthogonal, as are the members of the 2-set, so these two sets are complete. The members of the 3-set are not mutually orthogonal, but the following set of weighted sums of its rows is orthogonal:

(1)	+ + + + + + + +
1/2[(2)-(3)]	0 0 0 + 0 0 0 -
1/2[(3)-(4)]	0 0 + 0 0 0 - 0
1/2[(4)-(5)]	0 + 0 0 0 - 0 0
1/2[(5)+(2)]	+ 0 0 0 - 0 0 0
(6)	+ + - - + + - -
(7)	+ - - + + - - +
(8)	+ - + - + - + -

The values resulting from these weighted sums can be +1, 0, or -1, which explains the entries 0 above. The set of weighted sums is seen to be mutually orthogonal and thus complete. A complete set has been produced from weighted sums of the 3-set members, so an arbitrary clocked function with 2^3 intervals can be expressed as a weighted sum of the members of the original 3-set: The 3-set of all square waves is therefore complete.

For n = 4, each function of the 3-set is extended to 2^4 intervals by copying it a second time as, for example

(3) + + + - - - - + + + + - - - - +.

Then the number of members of the set is doubled -- to 2^4 -- by adding the following "phase-shifted" versions of the square wave with period 2^4 intervals:

+ + + + + + + + - - - - - - - -
+ + + + + + + - - - - - - - - +
+ + + + + + - - - - - - - - + +
+ + + + + - - - - - - - - + + +
+ + + + - - - - - - - - + + + +
+ + + - - - - - - - - + + + + +
+ + - - - - - - - - + + + + + +
+ - - - - - - - - + + + + + + +

For consistency, these additions should appear between rows (1) and (2) of the 3-set. In this 4-set, the 4 square waves with a period of 8 intervals are not mutually orthogonal, and neither are the 8 square waves with a period of 16 intervals. Subsets of square waves with periods of 8 intervals or more are not mutually orthogonal. If the arguments of the previous paragraph are to be used to prove that the 4-set of square waves is complete, each such subset must be treated with the weighted-sum technique of the previous paragraph. But these arguments apply again to the 4-set of all square waves and show that it, too, is a complete set.

Proceeding inductively in this manner, it must be concluded that for all n > 0, the n-set of all square waves is complete. Then <u>no</u> non-trivial clocked function -- or Walsh function -- with 2^n intervals is orthogonal to every member of the n-set of all square waves.