Bending of a 4-Cornered Plate

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The question was raised whether or not a trapezoidal plate of our present design (as opposed to a triangular plate) could have the disadvantage of some unfavorable bending shape if its four corners are not adjusted in a plane; especially, whether or not its central parts could show "oil-canning", meaning two different stable configurations with a possible "snapping" from one to the other.

We have not found the bending of 4-cornered plates in textbooks. For approaching an understanding, we consider three extreme models: a thin rubber membrane in tension, a thin cardboard plate, and a very thick rubber plate, supported at 4 corners. Let us use coordinates and definitions of Fig. 1, with

$$z(B) = z(C) = z(D) = 0 \quad \text{and} \quad z(A) = \Delta. \tag{1}$$

The thin rubber membrane will follow Laplace's potential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{2}$$

and it cannot show oil-canning. At the center (if a $\approx$ c) we must have, from reasons of symmetry, the average of all four corners which is

$$z_o = z(0, 0) = \Delta/4. \tag{3}$$

The cardboard plate does have strong oil-canning: it is stable with either a straight diagonal BD and a curved diagonal AC and with $z_o = 0$, or with a straight AC and a curved BD and with $z_o = \Delta/2$:

$$z_o = \begin{cases} 0 \\ \Delta/2 \end{cases} \tag{4}$$

The thick rubber plate cannot have oil-canning. I do not know its exact shape, but I guess it will not be too different from (2), and it must again follow (3).

Our actual surface plate may behave somewhere in between the thin cardboard plate and the thick rubber plate, and whether or not it shows oil-canning depends on its thickness/size ratio. I feel pretty sure that our plates will behave properly without oil-canning because their rib structure is thick enough, and also because their paraboloidal double curvature should prevent any oil-canning.

In a discussion with Woon-Yin Wong we agreed that this question should best be decided by an experiment: Level all 4 corners of one of our plates and measure the surface shape at the points of some grid. Then raise or lower one or the other corner by Δ = 10 or 20 times our measuring error; measure the new shape, subtract the old one, and call $z(x,y)$ the deformation. I would call it "well behaved" if $z(x,y) < \Delta$ for all points except A, and if $\mathrm{rms}(z) = \left\{(1/n)\sum z^2\right\}^{1/2}$ is not much larger than $\Delta/3$ (see following paragraph), and especially if $z_o \approx \Delta/4$ as in (3), neither close to zero nor close to $\Delta/2$ as in (4).

I would like to venture the following guess. The deformation will show approximately the most simple non-planar shape

$$z(x,y) \quad = \quad \frac{\Delta}{4}\ (1 + x/a)\ \ (1 + y/b) \tag{5}$$

which fulfills (1), (2) and (3), and which yields (if a = c):

$$\mathrm{rms}(z) \quad = \quad \Delta/3. \tag{6}$$

If the measured deformation of our surface plates is not too different from (5), then another urgent question can be answered: what are the best four corner adjustments of a plate once the whole telescope surface has been measured and the best-fit paraboloid has been determined? First, let us call A, B, C, D the adjustment amounts by which the corner heights are changed. One can show that the plate surface then changes its height by

$$z(x,y) = \underbrace{\frac{1}{4}\left\{\underbrace{(A+B+C+D)}_{\text{lift}} + \underbrace{\left(\frac{A-B}{a} + \frac{D-C}{c}\right)x}_{\text{tilt, y-axis}} + \underbrace{\frac{A+B-C-D}{b}\,y}_{\text{tilt, x-axis}}\right.}_{\text{rigid-body translation}} + \left.\underbrace{\left(\frac{A-B}{ab} + \frac{C-D}{bc}\right)xy}_{\text{internal bending}}\right\} \tag{7}$$

We call Z(x,y) the deviation of the measured surface from the best-fit paraboloid, measured at many points on the plate, and we assume a symmetric grid such that

$$\overline{x} = \overline{y} = \overline{xy} = \overline{x^2y} = \overline{xy^2} = 0. \tag{8}$$

We want to choose A, B, C, D such that the resulting change z(x,y) is most similar to the deviation Z(x,y). We thus set Z(x,y) = z(x,y) of (7), average over all x and y, and obtain the lift: $\frac{1}{4}$ (A+B+C+D) = $\overline{Z(x,y)}$. Similarly, we multiply (7) by x and average, and also multiply by y and by xy. This gives 4 equations with the 4 unknowns, and after a bit of polishing the system can be written as, with d = a/c,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -d & d \\ 1 & 1 & -1 & -1 \\ 1 & -1 & d & -d \end{pmatrix} \times \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 4 \begin{pmatrix} \overline{Z} \\ a\ \overline{xZ} \,/\, \overline{x^2} \\ b\ \overline{yZ} \,/\, \overline{y^2} \\ ab\ \overline{xyZ} \,/\, \overline{x^2y^2} \end{pmatrix} = 4\vec{V} \tag{9}$$

This matrix is so regular that I inverted it right away, and the result is, with q = 1/d = c/a, and with vector $\vec{V}$ from (9)

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -q & -1 & q \\ 1 & q & -1 & -q \end{pmatrix} \times \vec{V} \qquad (10)$$

If, however, the suggested measurements of deformed plates should not be approximately described by (5), then some other easy approximation must be looked for, and equation (7) must be changed accordingly. In (7), it is only the last term, the "internal bending", which needs any change, not the "rigid-body translations"; this means that only the fourth line of the matrix and the fourth element of the vector in (9) need to be changed, while the first three lines and elements stay the same. If needed, all this can be worked out, but I think that approximation (5) will already be good enough.