Spherical Map Projections

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1 Introduction

This document presents a concise mathematical description of uninterrupted spherical map projections in common use, particularly with reference to their application in astronomy. It is currently incomplete in four ways,

- Two diagrams are missing.
- The inverse formulae for the conics are missing.
- Some important map projections are missing.
- Many projections are known by several names and a complete list of aliases will be produced.

These will be added in good time.

Spherical projections can be classified as either *perspective* or *non-perspective.* Perspective projections may be constructed by ray-tracing of points on a *generating sphere* from a single point of projection onto a surface which can subsequently be flattened without further distortion. In zenithal projections, the surface is a plane; in cylindrical projections it is a cylinder which is conceptually cut along its length and unrolled; the surface in conic projections is a semi-cone which may likewise be cut and flattened. All perspective projections have as free parameters the distance of the point of projection from the centre of the generating sphere, and the distance of the surface of projection from the generating sphere.

Non-perspective projections are constructed mathematically so as to have particular useful properties. Conceptually, they are also based on a generating sphere, but the relationship between sphere and surface of projection cannot be defined in Euclidean terms - that is, with straight edge and compass. However, the surface of projection may still be described as planar, cylindrical, or conic.

Although certain perspective projections are important in astronomy, for example the *orthographic* and *gnomonic projections*, the general distinction between perspective and non-perspective projections is not particularly important - the days of ruler and compass are long past! A more important distinction is the mathematical one based on the plane of projection, and suggests the division into the following classes: *zenithal*, *cylindrical*, *conic*, and *conventional.* These classes may be further subdivided, for example *conic* projections can be either *one-standard*, *two-standard*, or *poly-standard.*

Conventional projections are pure-mathematical. Certain of them are closely related to particular projections from other classes, for example *Aitov's projection* is developed from the equatorial case of the *zenithal equal area projection*, and the *Sanson-Flamsteed projection* is actually the equatorial case of *Bonne's projection*, a poly-standard conic. From a mathematical viewpoint the conventional projections are closely related to the cylindricals, but really they belong in a class of their own.

When constructing a projection a distinction must be drawn between mapping the earth and mapping the celestial sphere. The two cases are related by a simple inversion, the difference between looking at a sphere from the outside (the earth), or from the inside (the sky). In the remainder of this document we will be concerned only with mapping the sky. To map the earth, simply apply the relation

$$(x_\oplus, y_\oplus) = (-x, y), \tag{1}$$

where (x, y) henceforth refers to the sky projection.

The relationship between the sphere and the surface of projection is a geometrical one, independent of any coordinate system that may be ascribed to the sphere. This is clearly the case for perspective projections, which may be constructed with straight edge and compass, but is also true for non-perspective projections. However, the mathematical description of a projection may only be made in terms of a spherical coordinate system. In a sense, this coordinate system is imposed on the sphere by the projection itself, and it may not bear any relationship to the coordinate systems usually associated with the celestial sphere, be they *ecliptic*, *equatorial*, or *galactic*. For example, zenithal projections are naturally described by a coordinate system in which the "north" pole is located at the point of tangency of the plane of projection.

Throughout the remainder of this document the longitudinal and latitudinal components of the native coordinate system will be denoted by the ordered pair (ϕ, θ).

An *oblique projection* is one in which the coordinate system of interest does not coincide with the coordinate system used to define the projection mathematically. The term "oblique projection" is somewhat of a misnomer, since obliquity is a property of a coordinate system and not of the spherical projection itself. It is best treated in terms of a spherical coordinate rotation from the (ϕ', θ') system to the (ϕ, θ) system defined by three Euler angles, (Φ', Θ', Φ) via the transformation equations:

$$\phi = \Phi + \arg(\cos\theta' \cos(\phi' - \Phi'), \sin\theta' \sin\Theta' + \cos\theta' \cos\Theta' \sin(\phi' - \Phi')), \tag{2}$$
$$\theta = \sin^{-1}(\sin\theta' \cos\Theta' - \cos\theta' \sin\Theta' \sin(\phi' - \Phi')). \tag{3}$$

The following equations derived from these are often found to be useful:

$$\cos(\phi - \Phi)\cos\theta = \cos\theta' \cos(\phi' - \Phi'), \tag{4}$$
$$\sin(\phi - \Phi)\cos\theta = \sin\theta' \sin\Theta' + \cos\theta' \cos\Theta' \sin(\phi' - \Phi'). \tag{5}$$

Oblique coordinate projections arise in many places in astronomy even without being noticed. The ancient Arab astronomers faced with the task of constructing astrolabes, hit upon the expedient of using the *stereographic projection* which has the interesting and valuable property that great circles and small circles alike are projected as circles on the plane of projection. This makes oblique coordinate grids in stereographic projections easier to compute and draw than in most other projections. Any planisphere which contains an ecliptic or galactic grid in addition to the equatorial grid contains an oblique coordinate projection. The so called SIN projection used in radio astronomy is really an oblique *orthographic projection* whose natural pole is at the field centre of the observation, offset from the *north celestial pole*. On the other hand, the so called NCP projection is, in fact, a non-oblique *orthographic projection* with rescaled y-coordinate.

There are a number of important properties which a spherical projection can have:

- Equidistance: where angles are to be measured directly from a map ease of measurement may be an important criterion. This is facilitated in equidistant projections such as *Plate Carreé*, in which the meridians and parallels of the native coordinate system are uniformly divided.
- Equal area: an equal area projection has the valuable property that two regions on the generating sphere which enclose equal areas also enclose equal areas in the plane of projection. Such projections

are referred to as being *authalic*. This property is particularly useful when mapping the density of objects in different regions of the sky, or for unbiassed political maps of the earth.

All of the equal area projections depicted in this document enclose the same area (16π cm^2).

- Conformality (orthomorphism): It can be proved mathematically that no spherical projection exists which can represent a finite portion of the sphere without distortion. Conformality is a property which refers to the faithful representation of regions of vanishingly small size. Conformality exists at points on a projection where the meridians and parallels are orthogonal and equally scaled, so that a tiny circle at the corresponding point of the generating sphere is projected as a circle in the plane of projection. Most projections are conformal within a restricted set of points, but certain projections are conformal at all points.

  Although conformality is a *local* property, distortions are usually minimized within the immediate vicinity of the conformal points. However, the importance of conformality can be overemphasised. For example, while the *Sanson-Flamsteed projection* is conformal along the equator and central meridian, it doesn't minimize distortions across the whole of the sphere as well as does *Aitov's projection* which is conformal only at the centre.

- Boundaries: Some projections are incapable of representing the whole sphere and diverge at some latitude in their native coordinate system. Other projections, particularly the poly-standard conics and conventional projections, are particularly suited to representing the whole sphere.

- A number of projections have properties shared by no other:
  - *gnomonic*: great circle arcs are projected as straight line intervals, but with non-uniform scale.
  - *stereographic*: small circles are projected as circles.
  - *orthographic*: represents the visual appearance of a sphere when seen from infinity.
  - *Mercator's projection*: lines of constant bearing (*rhumb lines*) are projected as straight lines.

Certain projections are particularly suited for representing "continent" sized regions of the sphere. In particular, the two-standard and poly-standard conic projections are much favoured in atlases of the world.

The following sections present mathematical formulae which may be used to construct the spherical projections. It is usually easier to derive formulae for computing coordinates in the plane of projection, (x, y), from the spherical coordinates in the native coordinate system of the projection, (ϕ, θ), than vica versa. However, the inverse formulae are also very useful, and are provided for all projections.

In some circumstances it may be necessary to do a hybrid calculation, that is, compute x and θ from y and ϕ, compute x and ϕ from y and θ, compute y and θ from x and ϕ, or compute y and ϕ from x and θ. However, the hybrid formulae often cannot be specified analytically and iterative methods must be used. We will not attempt to develop any hybrid formulae here but instead rely on iterative solutions in all such cases. Formulae of the following forms will be presented for each projection (or class of projections):

$$x = \mathrm{x}(\phi, \theta), \tag{6}$$
$$y = \mathrm{y}(\phi, \theta), \tag{7}$$
$$\phi = \phi(x, y), \tag{8}$$
$$\theta = \theta(x, y). \tag{9}$$
$$\tag{10}$$

Then for any mixed pair of coordinates, say x and ϕ, each of the unknown coordinates, y and θ, can be obtained by iterative solution of a single formula, for example y can be obtained from the equation $\phi = \phi(x, y)$ and θ from $x = \mathrm{x}(\phi, \theta)$.

Throughout the rest of this paper the radius of the generating sphere is denoted by r_0, and latitudes of significance in defining the projection as θ_x. The angles ϕ and θ are always specified in degrees rather than radians, and the formulae explicitly include the $\pi/180$ conversion factor to radians whenever required. It is assumed that all trigonometric and inverse trigonometric functions accept and return angles in degrees. The arg() function referred to in some of the formulae is such that if

$$(x, y) = (r \cos \alpha, r \sin \alpha), \tag{11}$$

then

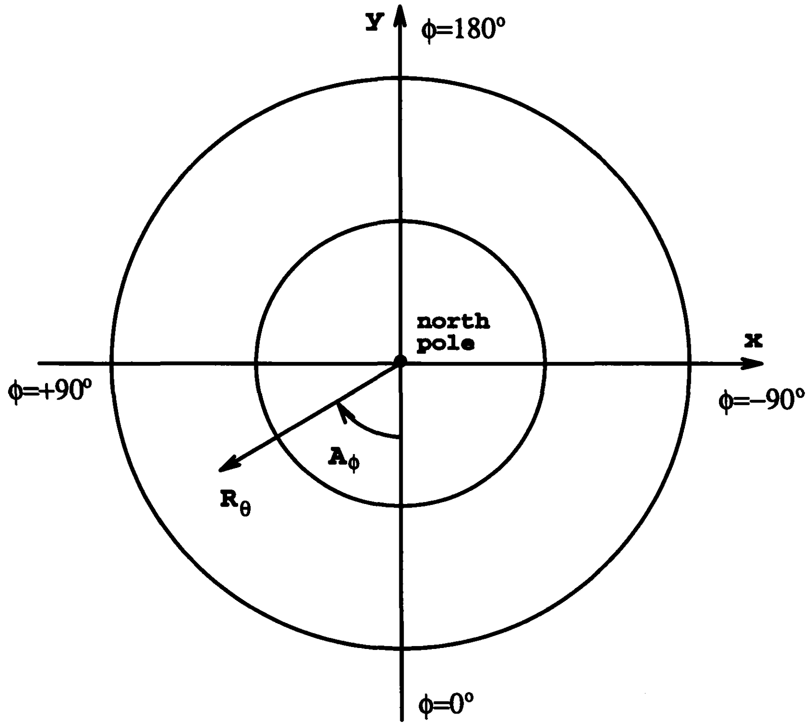
$$\arg(x, y) = \alpha. \tag{12}$$

In other words, $\arg(x, y)$ is effectively the same as $\tan^{-1}(y/x)$ except that it returns the angle in the correct quadrant.

Since the mathematical formulae which define the projections are hard to interpret by themselves, each projection is illustrated with a diagram of its native coordinate grid. The grid interval is 15°, and all projections are based on a generating sphere of radius r_0 = 2cm. The diagrams themselves were encoded directly in PostScript.

2 Zenithal (azimuthal) projections

Zenithal projections (also known as *azimuthal projections*) are a class of projections in which the surface of projection is a plane. The native coordinate system is such that the polar axis is orthogonal to the plane of projection, whence the meridians are projected as equispaced rays emanating from a central point, and the parallels are mapped as concentric circles centered on the same point. The projection is therefore defined by R_θ and A_ϕ as in the following diagram:



All zenithal projections have

$$A_\phi = \phi, \tag{13}$$

whence

$$\begin{aligned} x &= -R_\theta \sin\phi, & (14)\\ y &= -R_\theta \cos\phi. & (15) \end{aligned}$$

These equations may be inverted as follows

$$\begin{aligned} R_\theta &= \sqrt{x^2+y^2}, & (16)\\ \phi &= \arg(-y,-x). & (17) \end{aligned}$$

Since

$$\frac{\partial A_\phi}{\partial \phi} = 1, \tag{18}$$

the requirement for conformality of zenithal projections is

$$\left|\frac{\partial R_\theta}{\partial \theta}\right| = \frac{R_\theta}{\cos\theta}. \tag{19}$$

This differential equation has the general solution

$$R_\theta \propto \tan\left(\frac{90-\theta}{2}\right), \tag{20}$$

and this is the form of R_θ for the *stereographic projection*.

Let an oblique coordinate system be denoted by (ϕ', θ'), and let the coordinates of the pole of the native coordinate system in the oblique system be (ϕ'_0, θ'_0). The meridian of the oblique system defined by $\phi' = \phi'_0$ will be projected as a straight line segment; suppose it overlies the native meridian of $\phi = \phi_0$ in the same sense of increasing or decreasing latitude (to distinguish it from the native meridian on the opposite side of the pole), then the Euler angles for the transformation from (ϕ', θ') to (ϕ, θ) are

$$(\Phi', \Theta', \Phi) = (\phi'_0 + 90°, 90° - \theta'_0, \phi_0 + 90°). \tag{21}$$

2.1 Perspective zenithal projections

Let r_0 be the radius of the generating sphere and let the distance of the origin of the projection from the centre of the generating sphere be μr_0. If the plane of projection intersects the generating sphere at latitude θ_x in the native coordinate system of the projection as in the following diagram

...diagram...

then it is straightforward to show that

$$R_\theta = r_0 \cos\theta \left(\frac{\mu + \sin\theta_x}{\mu + \sin\theta}\right). \tag{22}$$

From this equation it can be seen that the effect of θ_x is to rescale r_0 by $(\mu + \sin\theta_x)/\mu$, thereby uniformly scaling the projection as a whole. Consequently the projections presented in this section only consider $\theta_x = 90°$.

The equation for R_θ is invertible as follows:

$$\theta = \arg(\rho, 1) - \sin^{-1}\left(\frac{\rho\mu}{\sqrt{\rho^2+1}}\right), \tag{23}$$

where

$$\rho = \frac{R_\theta}{r_0(\mu + \sin\theta_x)}. \tag{24}$$

For $|\mu| \le 1$ the perspective zenithal projections diverge at latitude $\theta = \sin^{-1}(-\mu)$, while for $|\mu| > 1$ the projection of the near and far sides of the generating sphere are superposed, with the overlap beginning at latitude $\theta = \sin^{-1}(-1/\mu)$.

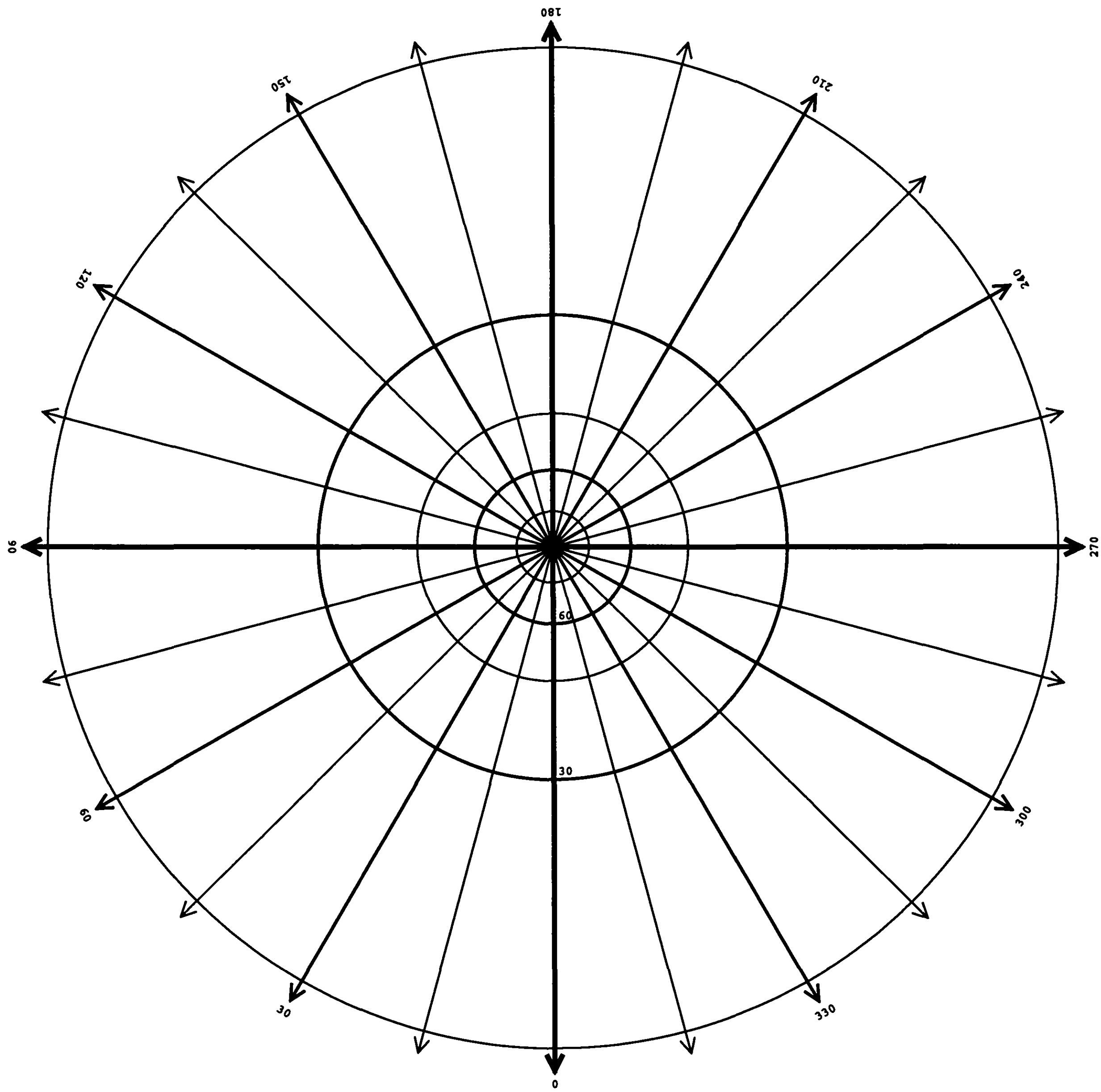
All perspective zenithal projections are conformal at latitude ($\theta = 90°$). The projection with $\mu = 1$ (*stereographic*) is conformal at all points.

2.1.1 The gnomonic projection

The perspective zenithal projection with $\mu = 0$ is known as the *gnomonic projection*. Since the projection is made from the centre of the generating sphere great circles are projected as straight lines. Thus the shortest distance between two points on the sphere is represented as a straight line interval which, however, is not uniformly divided.

$$R_\theta = r_0 \cot \theta, \tag{25}$$

$$\theta = \tan^{-1}\left(\frac{r_0}{R_\theta}\right). \tag{26}$$



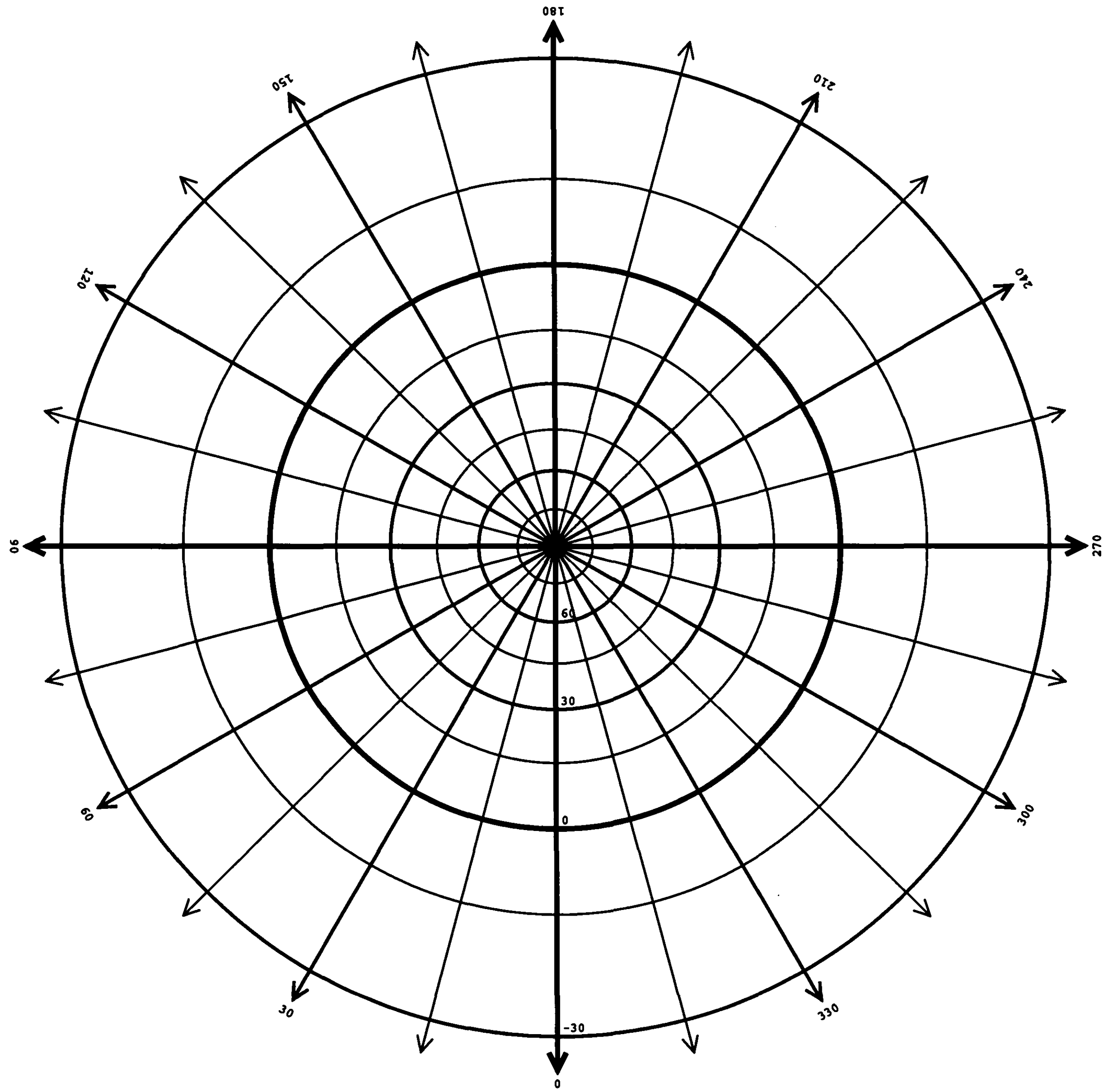
Limits: diverges at latitude $\theta = 0°$.
Conformal at latitude $\theta = 90°$.

2.1.2 The stereographic projection

The perspective zenithal projection with $\mu = 1$ is known as the *stereographic projection*. It is conformal at all points and also has the property that all circles on the generating sphere are projected as circles. The relative ease of constructing oblique stereographic projections led to its use by Arab astronomers for constructing astrolabes.

$$R_\theta = 2r_0 \tan\left(\frac{90-\theta}{2}\right), \tag{27}$$

$$\theta = 90^\circ - 2\tan^{-1}\left(\frac{R_\theta}{2r_0}\right). \tag{28}$$



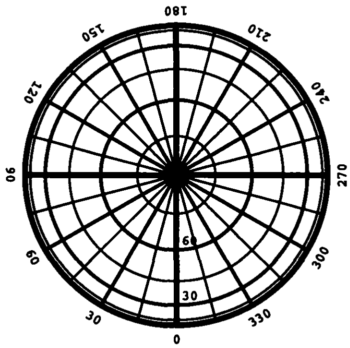
Limits: diverges at latitude $\theta = -90^\circ$.
Conformal at all points.

2.1.3 The orthographic projection

The perspective zenithal projection with $\mu = \infty$ is known as the *orthographic projection*. It gives a representation of the visual appearance of a sphere (for example a planet) when seen from a great distance.

$$R_\theta = r_0 \cos\theta, \tag{29}$$

$$\theta = \cos^{-1}\left(\frac{R_\theta}{r_0}\right). \tag{30}$$



Limits: the front and rear sides of the generating sphere begin to overlap at latitude $\theta = 0°$.
Conformal at latitude $\theta = 90°$.

2.1.4 Approximate equidistant perspective zenithal projection

The perspective zenithal projection with $\mu = (\pi/2 - 1)^{-1} = 1.7519$ has approximately equidistant parallels, the length of the meridian from $\theta = 90°$ to $\theta = 0°$ being true. See also ***zenithal equidistant projection.***

2.1.5 Approximate equal area perspective zenithal projection

The perspective zenithal projection with $\mu = \sqrt{2} + 1 = 2.4142$ is approximately equal area, the area of the hemisphere being true. See also ***zenithal equal area projection.***

2.2 Non-perspective zenithal projections

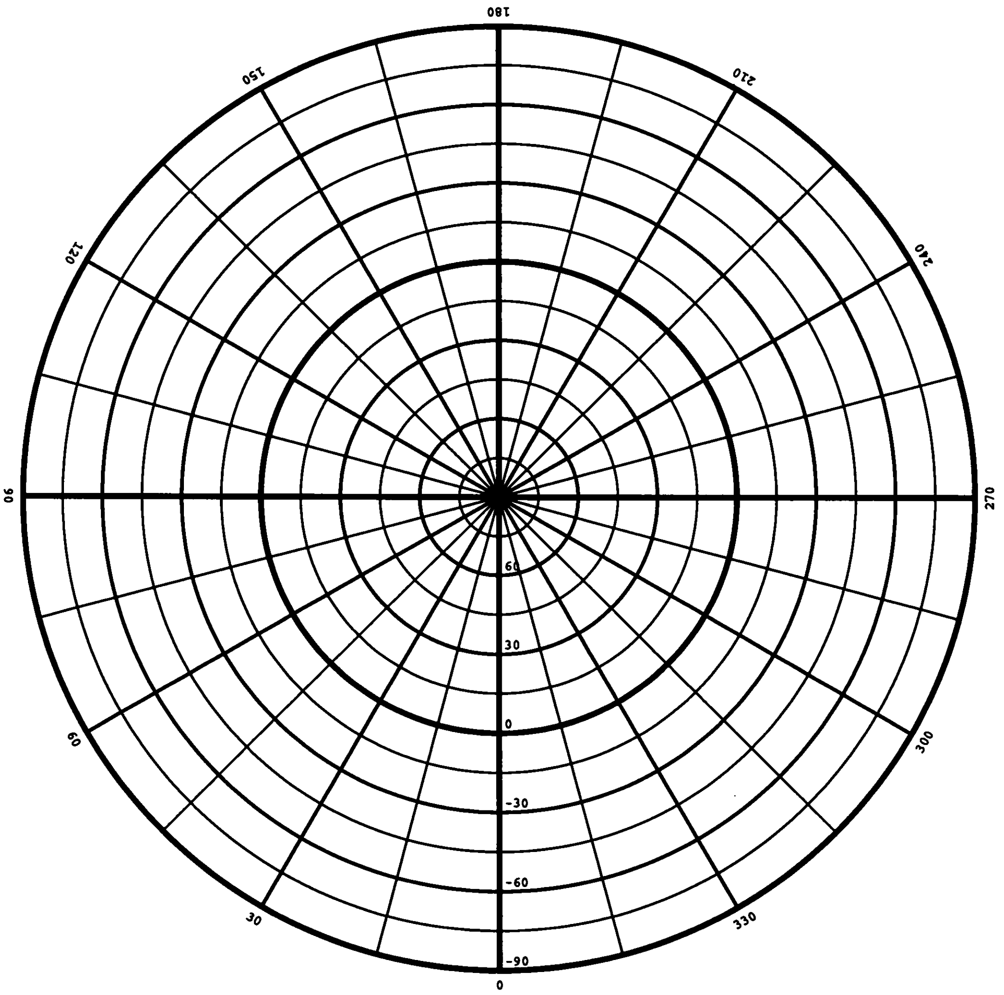
This group contains mathematically defined projections which are designed to have certain useful properties.

2.2.1 Zenithal equidistant projection

The meridians are uniformly divided so as to give uniformly spaced parallels. This projection is useful for mapping the polar regions of a spherical coordinate system when ease of construction and measurement are required.

$$R_\theta = r_0(90 - \theta)\frac{\pi}{180}, \tag{31}$$

$$\theta = 90^\circ - \frac{180 R_\theta}{\pi r_0}. \tag{32}$$



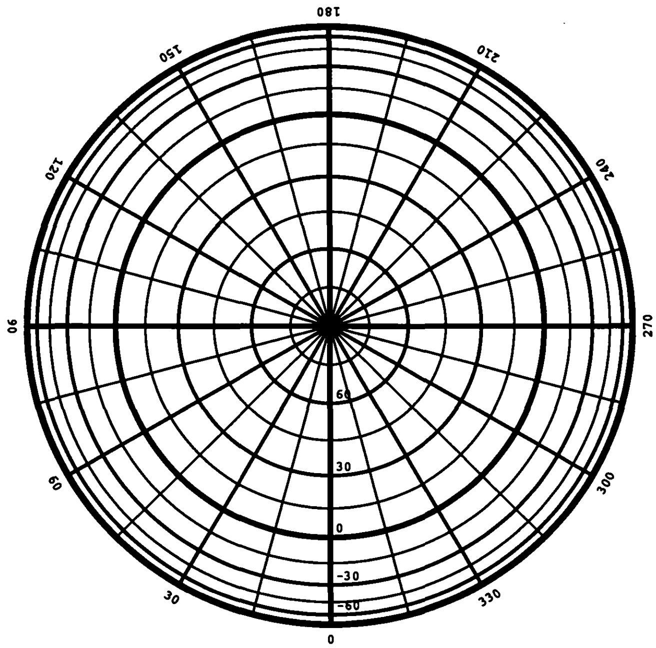
Limits: none.
Conformal at latitude $\theta = +90^\circ$.

2.2.2 Zenithal equal area projection

The spacing between parallels is computed to make this an equal area projection.

$$R_\theta = r_0\sqrt{2(1-\sin\theta)}, \tag{33}$$

$$\theta = \sin^{-1}\left(1 - \frac{R_\theta^2}{2r_0^2}\right). \tag{34}$$



Limits: none.
Conformal at latitude $\theta = +90°$.

3 Cylindrical projections

Cylindrical projections are a class of projections in which the surface of projection is a cylinder whose axis intercepts the centre of the generating sphere. The native coordinate system is such that the polar axis is coincident with the axis of the cylinder. Meridians and parallels are mapped onto a retangular grid. The projection is therefore described directly by formulae which return x, and y. All cylindricals have

$$x \propto -r_0 \phi \frac{\pi}{180}, \tag{35}$$

and this may be inverted as

$$\phi \propto -\frac{180x}{\pi r_0}. \tag{36}$$

The requirement for conformality of cylindrical projections is

$$\left|\frac{\partial y}{\partial \theta}\right| = \left|\frac{\partial x}{\cos\theta \partial \phi}\right|, \tag{37}$$

and since

$$\frac{\partial x}{\partial \phi} \propto r_0 \tag{38}$$

the general solution is

$$y \propto r_0 \ln \tan \left(\frac{90-\theta}{2}\right), \tag{39}$$

and this is the basis of *Mercator's projection.*

Let an oblique coordinate system be denoted by (ϕ', θ'), and let the coordinates of the pole [??? (0,0)] of the native coordinate system in the oblique system be (ϕ'_0, θ'_0). The meridian of the oblique system defined by $\phi' = \phi'_0$ will be projected as a straight line segment; suppose it overlies the native meridian of $\phi = \phi_0$ in the same sense of increasing or decreasing latitude (to distinguish it from the native meridian on the opposite side of the pole), then the Euler angles for the transformation from (ϕ', θ') to (ϕ, θ) are

$$(\Phi', \Theta', \Phi) = (\phi'_0 - 90°, \theta'_0, \phi_0 - 90°). \tag{40}$$

3.1 Perspective cylindrical projections

In the general case the radius of the cylinder is λr_0, where r_0 is the radius of the generating sphere, and the point of projection is distant μr_0 from the centre of the sphere. If $\mu \neq 0$ the point of projection moves around a circle of radius μr_0 in the equatorial plane of the generating sphere depending on the meridian being projected. The case $\mu < 0$ is allowable, however $\mu \neq -\lambda$.

...diagram...

It is straightforward to show that

$$x = -r_0 \lambda \phi \frac{\pi}{180}, \tag{41}$$

$$y = r_0 \sin\theta \left(\frac{\mu + \lambda}{\mu + \cos\theta}\right), \tag{42}$$

For $\mu = 0$, where the point of projection is at the centre of the generating sphere, the effect of λ is to scale the projection uniformly in x and y.

The equations for x and y are invertible as follows:

$$\phi = -\frac{180x}{\pi \lambda r_0}, \tag{43}$$

$$\theta = \arg(1, \eta) + \sin^{-1}\left(\frac{\eta\mu}{\sqrt{\eta^2 + 1}}\right), \tag{44}$$

where

$$\eta = \frac{y}{r_0(\mu + \lambda)}. \tag{45}$$

Limits for the perspective cylindrical projections are determined by the value of μ rather than λ which mainly affects the relative scaling of the x and y axes.

- For $\mu < -1$: the projection begins to overlap at latitudes $\theta = \pm\cos^{-1}(-1/\mu)$.
- For $-1 \leq \mu \leq 0$: diverges at latitudes $\theta = \pm\cos^{-1}(-\mu)$.
- For $\mu > 0$: the projection does not diverge.

The requirement for conformality of perspective cylindrical projections is

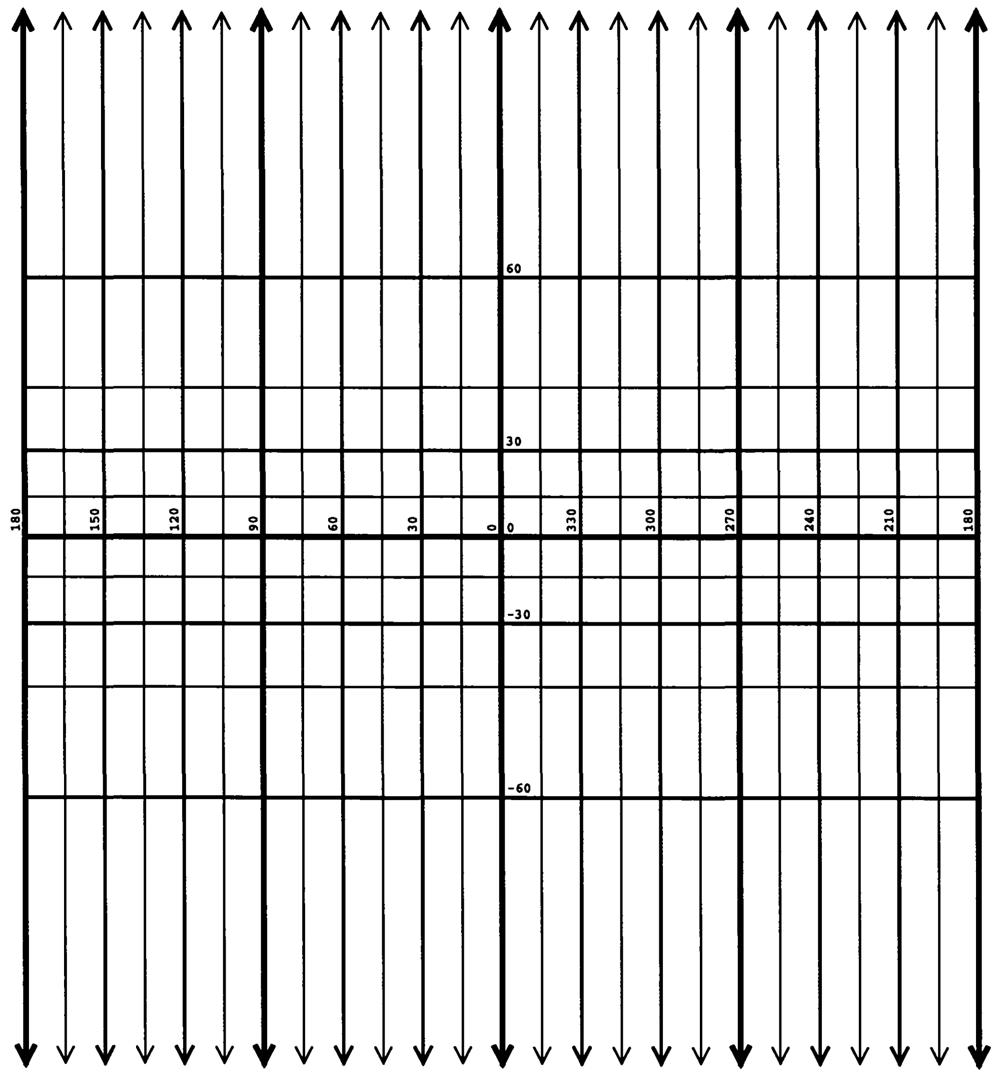
$$[\mu(\mu + \lambda) - \lambda]\cos^2\theta + (\mu + \lambda - 2\mu\lambda)\cos\theta - \mu^2\lambda = 0. \tag{46}$$

For $\mu = 0$ the projection is conformal at the equator for all values of λ. For $\mu = 1$ the projection is conformal at latitudes $\theta = \pm\cos^{-1}\lambda$, and this is the basis of *Gall's projection*.

3.1.1 Simple perspective cylindrical projection

In this simplest case of a perspective cylindrical projection the point of projection is at the centre of the generating sphere and the cylinder intersects the equator.

$$\mu = 0, \quad (47)$$
$$\lambda = 1, \quad (48)$$
$$x = -r_0\phi\frac{\pi}{180}, \quad (49)$$
$$y = r_0 \tan\theta, \quad (50)$$
$$\phi = -\frac{180x}{\pi r_0}, \quad (51)$$
$$\theta = \tan^{-1}\left(\frac{y}{r_0}\right). \quad (52)$$



Limits: diverges at latitudes $\theta = \pm 90°$.
Conformal at latitude $\theta = 0°$.

3.1.2 Gall's projection

This projection, which is conformal at latitudes $\theta = \pm 45°$, minimizes distortions in the equatorial regions.

$$\mu = 1 \tag{53}$$

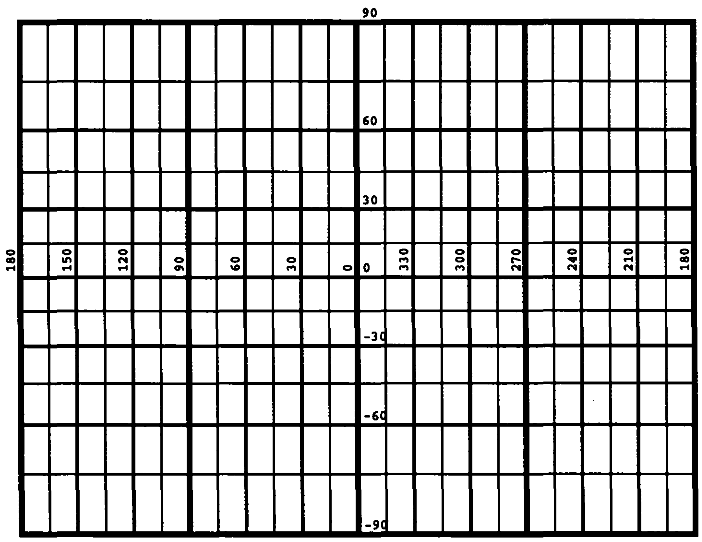
$$\lambda = \sqrt{2}/2 \tag{54}$$

$$x = -r_0 \phi \frac{\sqrt{2}\pi}{360}, \tag{55}$$

$$y = r_0 \left(1 + \frac{\sqrt{2}}{2}\right) \tan\left(\frac{\theta}{2}\right), \tag{56}$$

$$\phi = -\frac{360x}{\sqrt{2}\pi r_0}, \tag{57}$$

$$\theta = 2\tan^{-1}\left(\frac{y(2+\sqrt{2})}{r_0}\right). \tag{58}$$

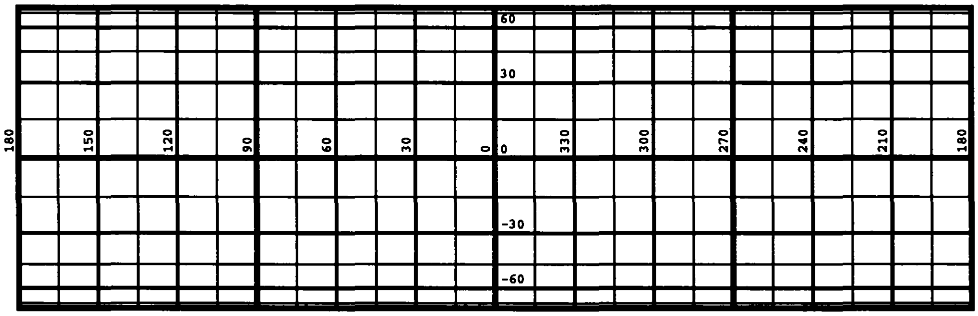


Limits: none.
Conformal at latitudes $\theta = \pm 45°$.

3.1.3 Lambert's equal area projection

This perspective projection is a specific instance of a subclass of non-perspective equal area cylindrical projections. See also ***generalized equal area cylindrical projections.***

$$\mu = \infty, \tag{59}$$
$$\lambda = 1, \tag{60}$$
$$x = -r_0\phi\frac{\pi}{180}, \tag{61}$$
$$y = r_0 \sin\theta, \tag{62}$$
$$\phi = -\frac{180x}{\pi r_0}, \tag{63}$$
$$\theta = \sin^{-1}\left(\frac{y}{r_0}\right). \tag{64}$$



Limits: none.
Conformal at latitude $\theta = 0°$.

3.2 Non-perspective cylindrical projections

Each of the non-perspective projections discussed here has

$$x = -r_0\phi\left(\frac{\pi}{180}\right), \tag{65}$$

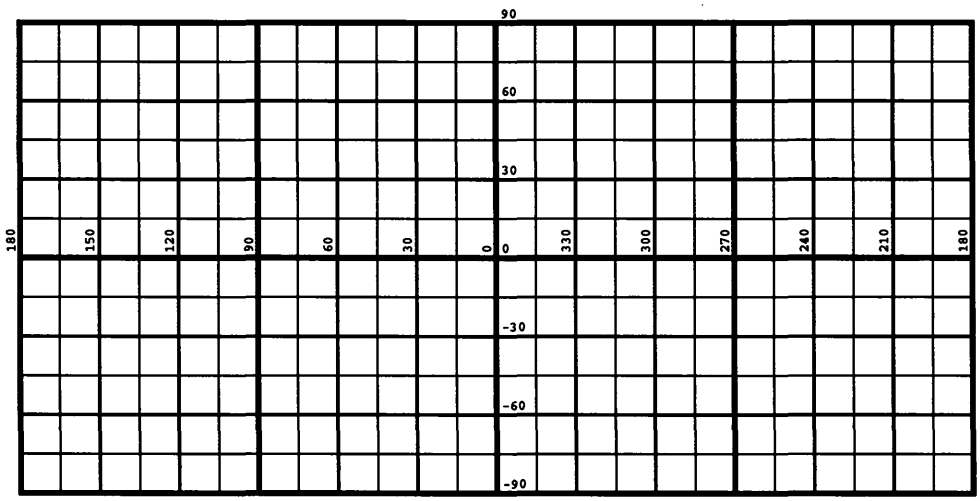
and inverse

$$\phi = -\frac{180x}{\pi r_0}. \tag{66}$$

3.2.1 Plate Carreé projection

Also known as *Cartesian projection*, this projection is easy to construct and take measurements from.

$$\begin{aligned} y &= r_0\theta\left(\frac{\pi}{180}\right), &(67)\\ \theta &= -\frac{180y}{\pi r_0}. &(68) \end{aligned}$$



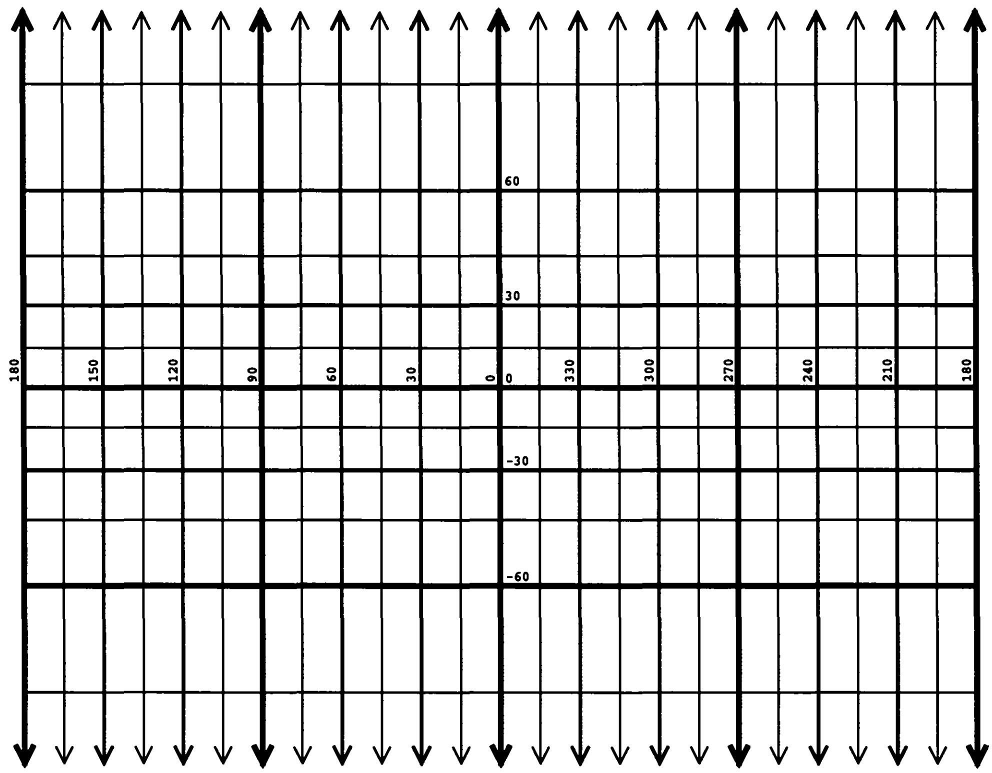
Limits: none
Conformal at latitude $\theta = 0°$.

3.2.2 Mercator's projection

This is a conformal projection with the useful property that lines of constant bearing (*rhumb lines*) are projected as straight lines. Consequently, it has been widely used for navigational purposes.

$$y = r_0 \ln \tan\left(\frac{90+\theta}{2}\right), \tag{69}$$

$$\theta = 2\tan^{-1}\left(e^{\left(\frac{y}{r_0}\right)}\right). \tag{70}$$



Limits: diverges at latitudes $\theta = \pm 90°$.
Conformal at all points.

3.2.3 Generalized equal area cylindrical projection

Parallels are spaced so as to make the projection equal area, and scaled so as to make it conformal at latitudes $\theta = \pm\theta_x$. The area of a region on the projection is $1/\cos^2\theta_x$ times larger than that of the corresponding region on the generating sphere. The case with $\lambda = 1$ is *Lambert's projection.*

$$y = r_0 \frac{\sin\theta}{\cos\theta_x^2}, \tag{71}$$

$$\theta = \sin^{-1}\left(\frac{y\cos^2\theta_x}{r_0}\right). \tag{72}$$

Limits: none.
Conformal at latitudes $\theta = \pm\theta_x$.

4 Conic projections

Parallels are projected as arcs of circles. Projections for which the parallels are concentric may be described by R_θ and A_ϕ, where R_θ is the radius of the arc for latitude θ. Then,

$$x = -R_\theta \sin A_\phi, \tag{73}$$
$$y = -R_\theta \cos A_\phi. \tag{74}$$

The requirement for conformality is

$$\left|\frac{\partial R_\theta}{\partial \theta}\right| = \frac{R_\theta}{\cos\theta}\left|\frac{\partial A_\phi}{\partial \phi}\right|. \tag{75}$$

In the special case satisfied by one-, and two-standard conic projections

$$\frac{\partial A_\phi}{\partial \phi} = C, \tag{76}$$

a constant known as the constant of the cone, this differential equation has the general solution

$$R_\theta = \kappa\left[\tan\left(\frac{90-\theta}{2}\right)\right]^C \tag{77}$$

where κ is a constant. This solution is used to construct orthomorphic conic projections. The apical angle of the projected cone is $2\pi C$ where

$$C = \frac{\partial A_\phi}{\partial \phi} \tag{78}$$
$$= \frac{r_0 \cos\theta_1}{R_{\theta_1}} \tag{79}$$
$$= \frac{r_0 \cos\theta_2}{R_{\theta_2}} \tag{80}$$

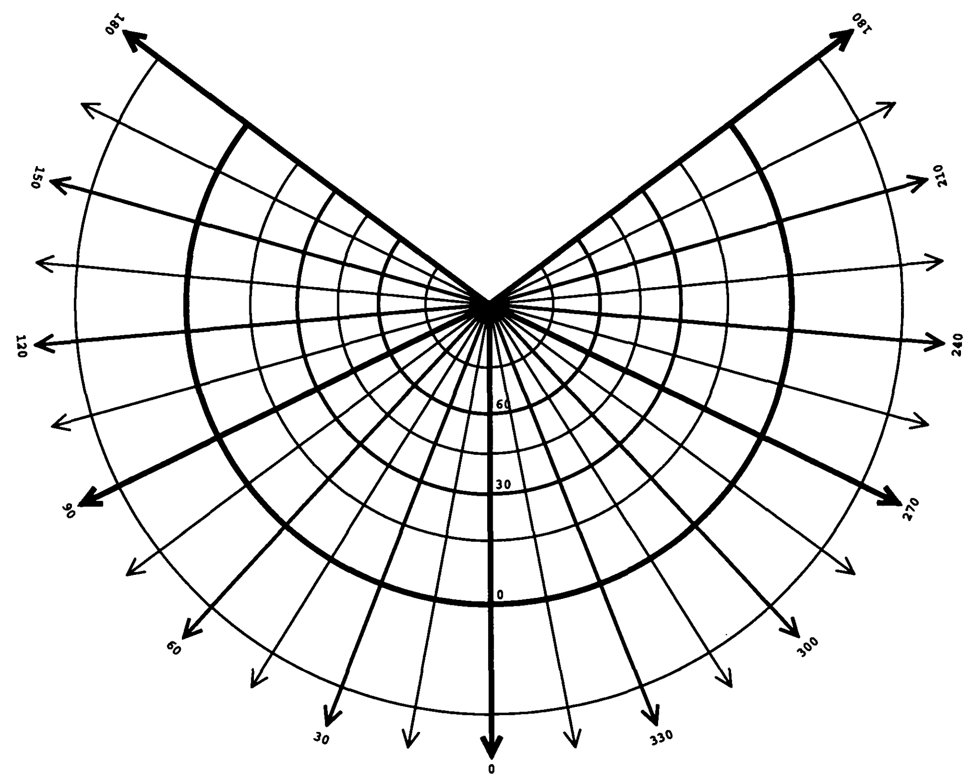
where θ_1 (and θ_2) is the latitude of the standard parallel(s).

4.1 One-standard conic projections

The standard parallel is at latitude θ_1. All of the diagrams presented here have been computed for $\theta_1 = 45°$.

4.1.1 One-standard perspective conic projection

$$R_\theta = r_0[\cot\theta_1 - \tan(\theta - \theta_1)] \tag{81}$$
$$A_\phi = \phi\sin\theta_1 \tag{82}$$

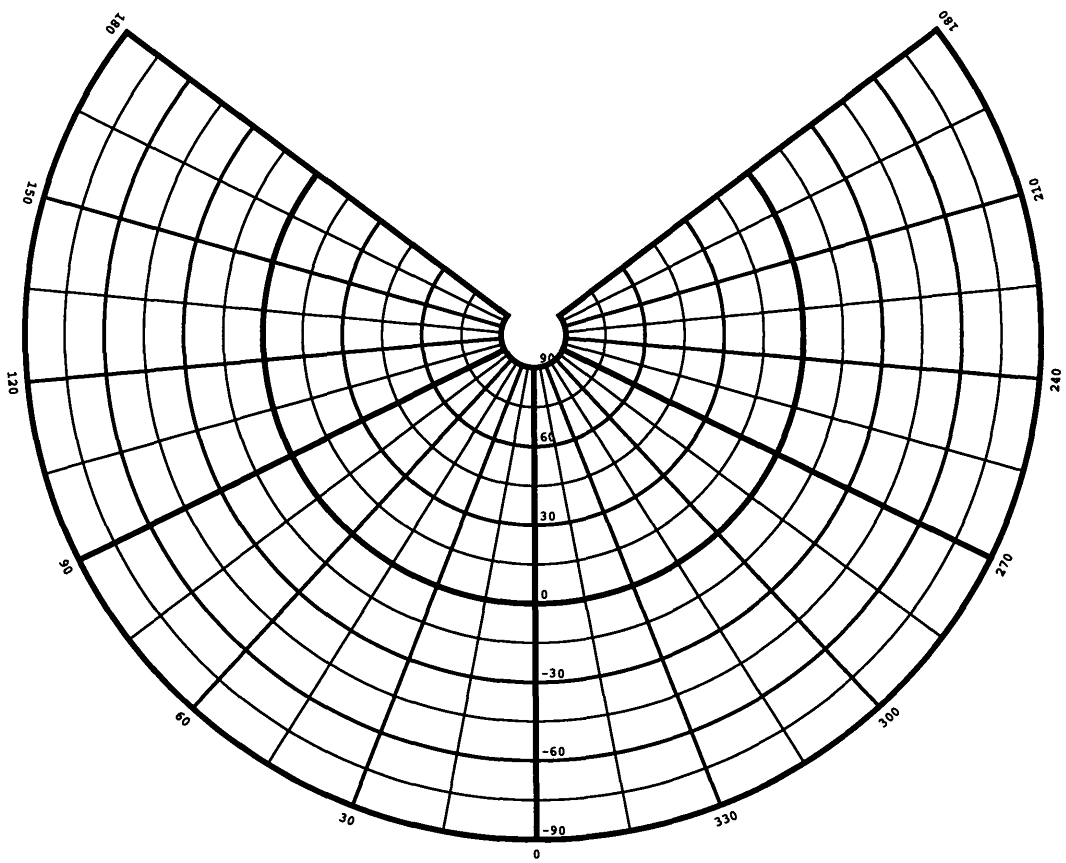


Limits: diverges at latitude $\theta = \theta_1 - 90°$.
Conformal at latitude θ_1.
Constant of the cone: $C = \sin\theta_1$.

4.1.2 One-standard equidistant conic projection

The spacing between parallels is true.

$$R_\theta = r_0[\cot\theta_1 - (\theta - \theta_1)\frac{\pi}{180}] \quad (83)$$
$$A_\phi = \phi\sin\theta_1 \quad (84)$$



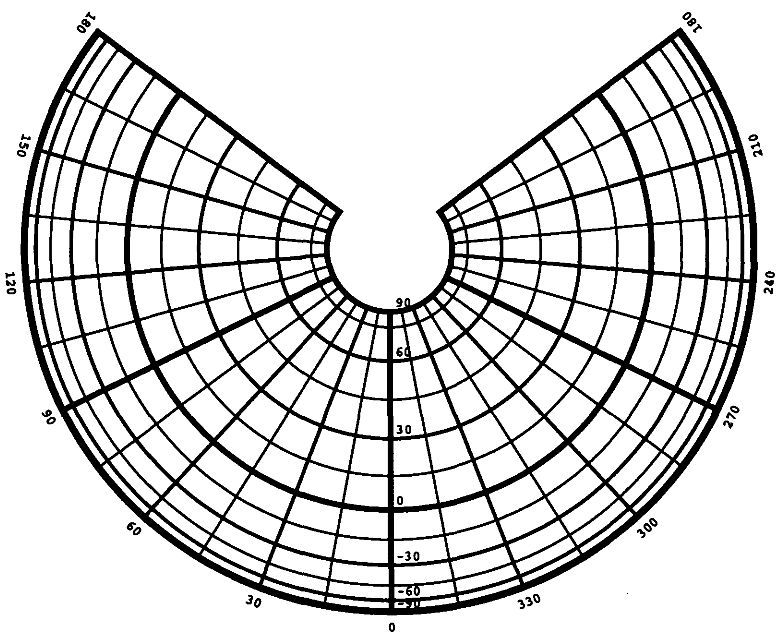
Limits: none.
Conformal at latitude θ_1.
Constant of the cone: $C = \sin\theta_1$.

4.1.3 One-standard equal area conic projection

R_θ is defined so that the area between any two parallels on the projection is true.

$$R_\theta = r_0 \left(\cot^2 \theta_1 + 2 - 2\frac{\sin\theta}{\sin\theta_1} \right)^{\frac{1}{2}} \tag{85}$$

$$A_\phi = \phi \sin\theta_1 \tag{86}$$

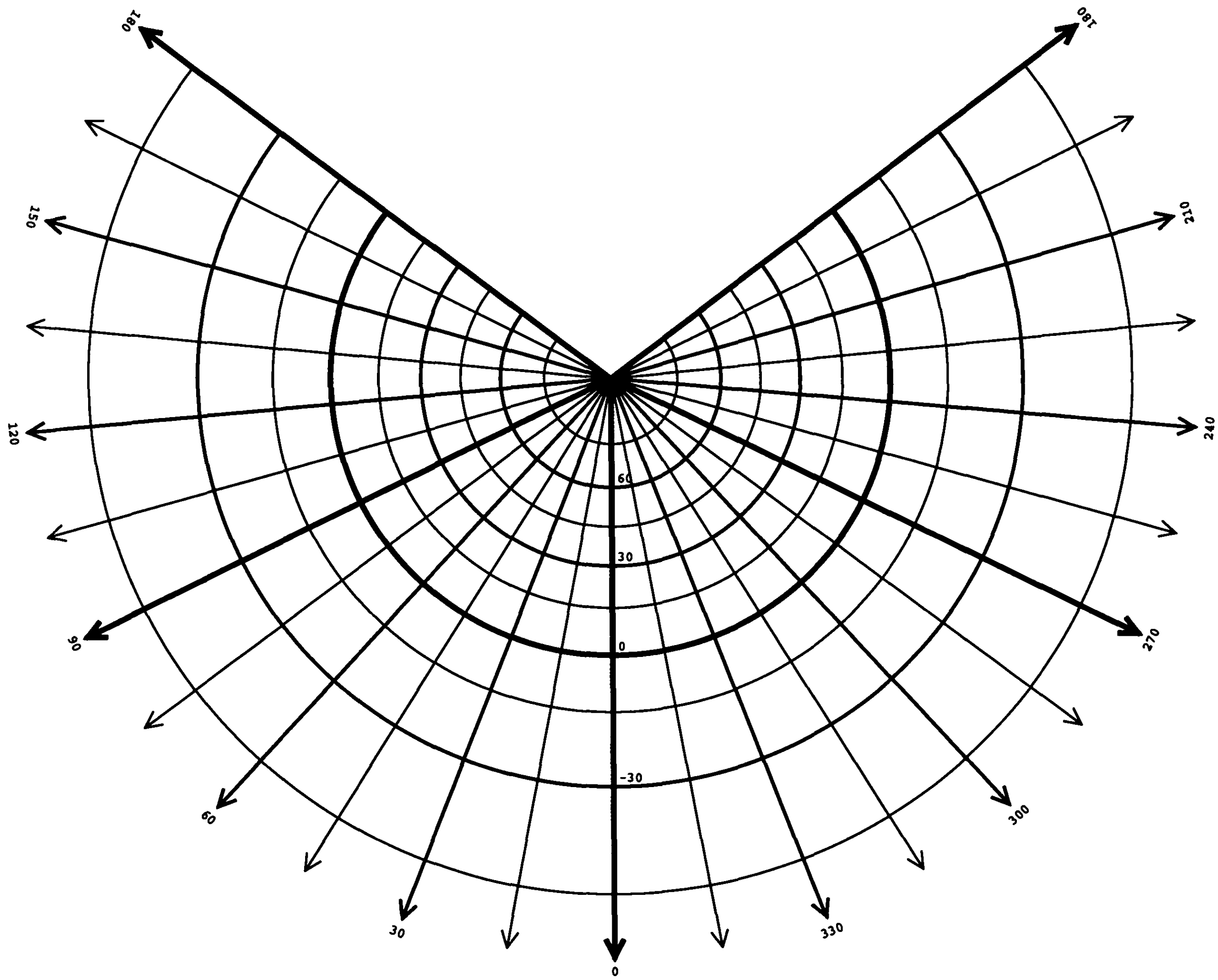


Limits: none.
Conformal at latitude θ_1.
Constant of the cone: $C = \sin\theta_1$.

4.1.4 One-standard orthomorphic conic projection

$$R_\theta = r_0 \cot\theta_1 \left[\frac{\tan\left(\frac{90-\theta}{2}\right)}{\tan\left(\frac{90-\theta_1}{2}\right)}\right]^{\sin\theta_1} \quad (87)$$

$$A_\phi = \phi\sin\theta_1 \quad (88)$$



Limits: diverges at latitude $\theta = -90°$.
Conformal at all points.
Constant of the cone: $C = \sin\theta_1$.

4.2 Two-standard conic projections

The standard parallels are at latitudes θ_1 and θ_2, $\theta_1 < \theta_2$. All of the diagrams presented here have been computed for $\theta_1 = 30°$ and $\theta_2 = 60°$.

4.2.1 Two-standard perspective conic projection

$$R_\theta = r_0\sqrt{1+\alpha^2}\frac{\sin\theta_1 + \alpha\cos\theta_1}{\alpha + \tan\theta} \tag{89}$$

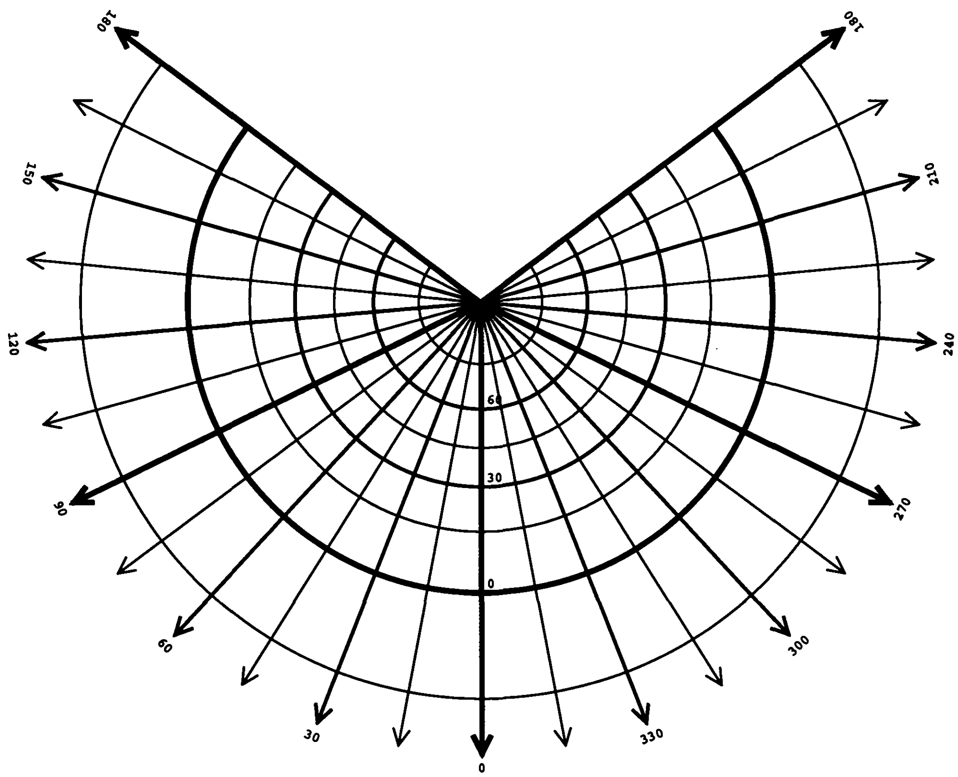
$$A_\phi = \phi(1+\alpha^2)^{-\frac{1}{2}} \tag{90}$$

where

$$\alpha = \frac{\sin\theta_2 - \sin\theta_1}{\cos\theta_1 - \cos\theta_2} \tag{91}$$

Note that

$$\begin{aligned} \sin\theta_1 + \alpha\cos\theta_1 &= \sin\theta_2 + \alpha\cos\theta_2 & (92) \\ &= \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1 - \cos\theta_2} & (93) \end{aligned}$$



Limits: diverges at latitude $\theta = \tan^{-1}(-\alpha)$.
Conformal at latitude $\theta = \tan^{-1}(1/\alpha)$.
Constant of the cone: $C = (1 + \alpha^2)^{-\frac{1}{2}}$.

4.2.2 Two-standard equidistant conic projection

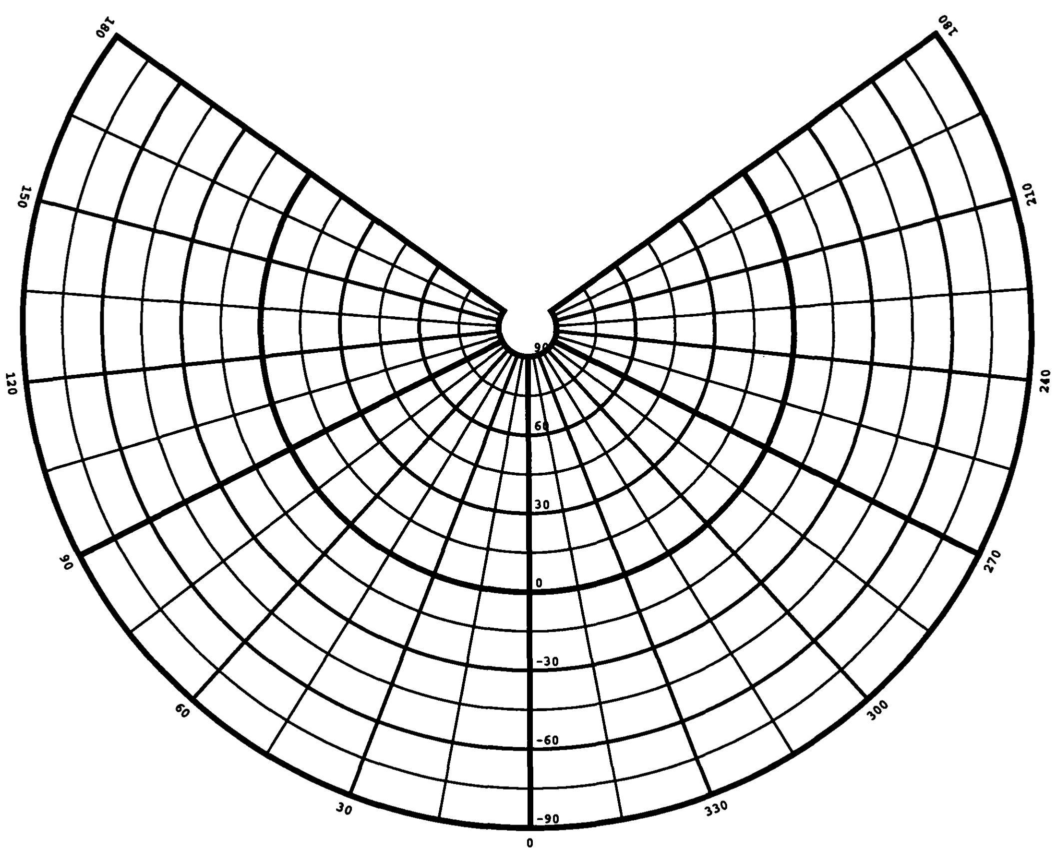
The spacing between parallels is true.

$$R_\theta = r_0 \frac{\pi}{180} \left[\frac{1}{\alpha} (\theta_2 \cos\theta_1 - \theta_1 \cos\theta_2) - \theta \right] \tag{94}$$

$$A_\phi = \phi\alpha / [\frac{\pi}{180}(\theta_2 - \theta_1)] \tag{95}$$

where

$$\alpha = \cos\theta_1 - \cos\theta_2 \tag{96}$$



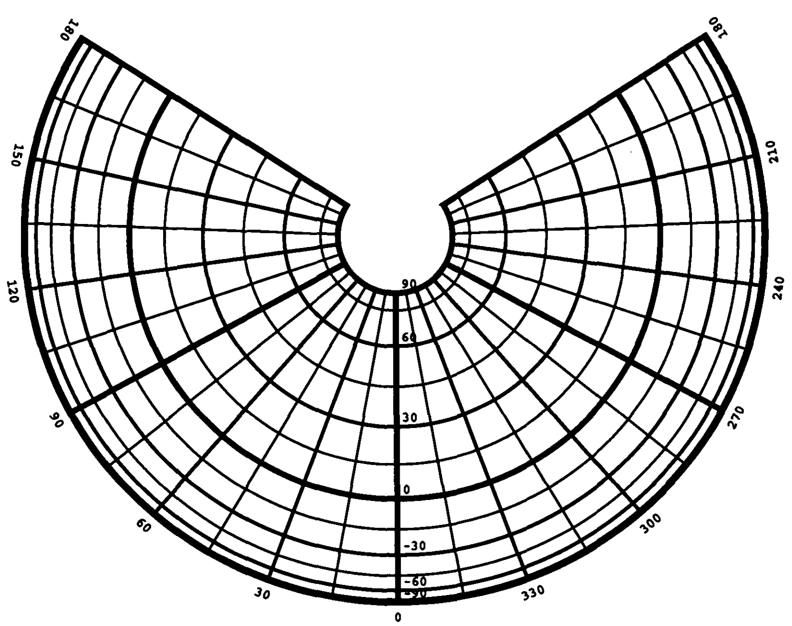
Limits: none.
Conformal at latitudes θ_1 and θ_2.
Constant of the cone: $\alpha/[\frac{\pi}{180}(\theta_2 - \theta_1)]$.

4.2.3 Two-standard equal area conic projection

$$
\begin{aligned}
R_\theta &= 2r_0\sqrt{1 + \sin\theta_1 \sin\theta_2 - \alpha \sin\theta}/\alpha && (97)\\
A_\phi &= \phi\alpha/2 && (98)
\end{aligned}
$$

where

$$\alpha = \sin\theta_1 + \sin\theta_2 \qquad (99)$$



Limits: none.
Conformal at latitudes θ_1 and θ_2.
Constant of the cone: $\alpha/2$.

4.2.4 Two-standard orthomorphic conic projection

$$R_\theta = r_0\alpha\left[\tan\left(\frac{90-\theta}{2}\right)\right]^C \qquad (100)$$

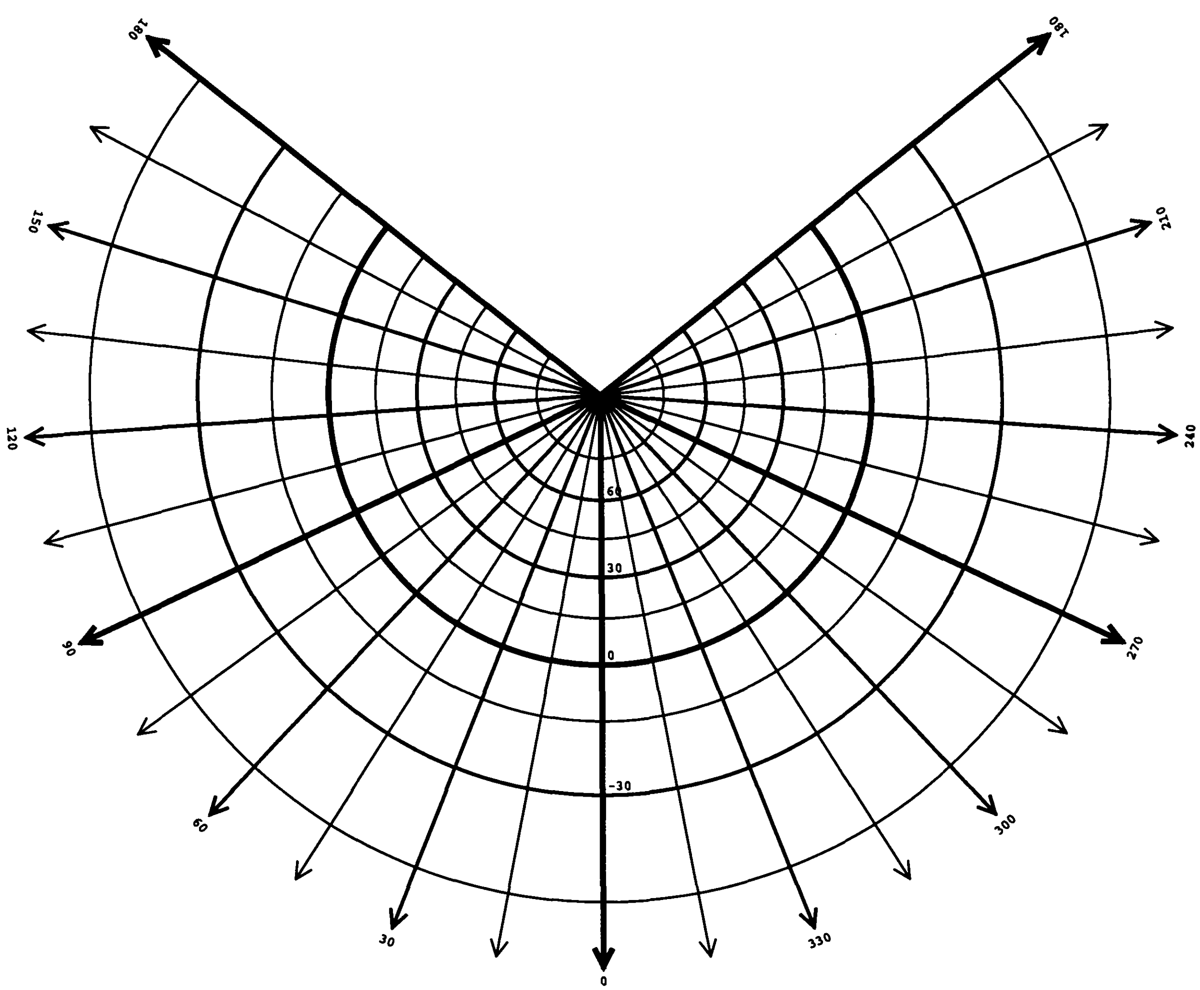
$$A_\phi = \phi C \qquad (101)$$

where

$$\alpha = \frac{\cos\theta_1}{C\left[\tan\left(\frac{90-\theta_1}{2}\right)\right]^C} \qquad (102)$$

$$= \frac{\cos\theta_2}{C\left[\tan\left(\frac{90-\theta_2}{2}\right)\right]^C} \qquad (103)$$

$$C = \frac{\ln\left(\frac{\cos\theta_2}{\cos\theta_1}\right)}{\ln\left[\frac{\tan\left(\frac{90-\theta_2}{2}\right)}{\tan\left(\frac{90-\theta_1}{2}\right)}\right]} \qquad (104)$$



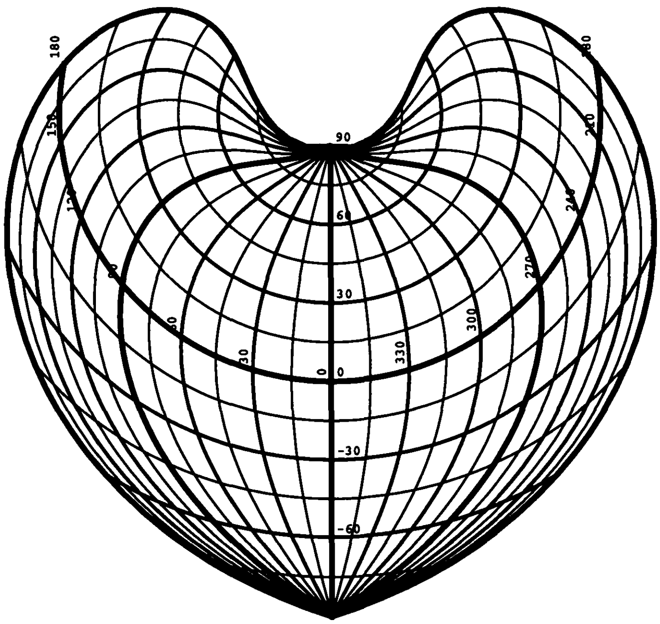
Limits: diverges at latitude $\theta = -90°$.
Conformal at all points.

4.3 Poly-standard conic projections

4.3.1 Bonne's equal area projection

Parallels are concentric equidistant arcs of circles of true length. The diagram presented here has been computed for $\theta_1 = 45°$.

$$R_\theta = r_0(\cot\theta_1 - (\theta - \theta_1)\pi/180) \tag{105}$$
$$A_\phi = r_0\phi\cos\theta/R_\theta \tag{106}$$



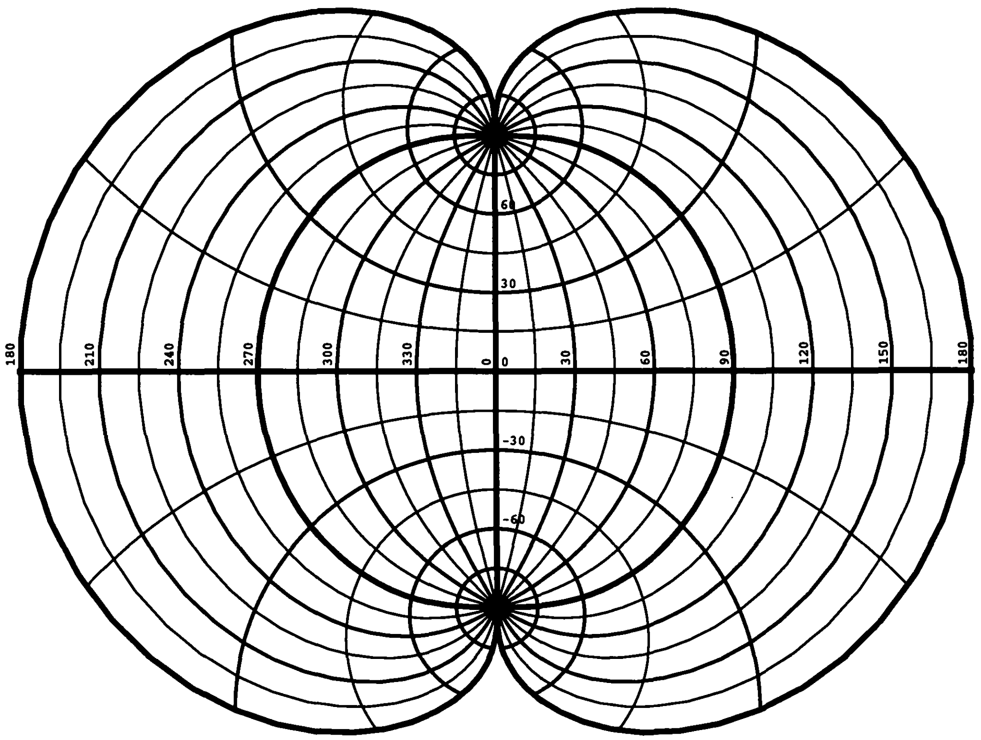
Limits: none.
Conformal at latitude θ_1 and along the central meridian.

4.3.2 Polyconic projection

Every parallel is projected as standard, i.e. as arcs of circles of radius $r\cot\theta$ at their true length, $2\pi r\cos\theta$, and correctly divided. The scale along the central meridian is true, consequently the parallels are not concentric.

$$x = r_0\cot\theta\sin(\phi\sin\theta) \tag{107}$$

$$y = r_0\{\cot\theta[1-\cos(\phi\sin\theta)]+\theta\left(\frac{\pi}{180}\right)\} \tag{108}$$



Limits: none.
Conformal along the central meridian.

5 Conventional projections

Conventional projections are pure-mathematical constructions designed to map the entire sphere with minimal distortion.

The Euler angles for constructing oblique forms of these projections are specified in the same way as for the cylindrical projections, i.e. as

$$(\Phi', \Theta', \Phi) = (\phi'_0 - 90^\circ, \theta'_0, \phi_0 - 90^\circ). \tag{109}$$

5.1 Sanson-Flamsteed (sinusoidal) projection

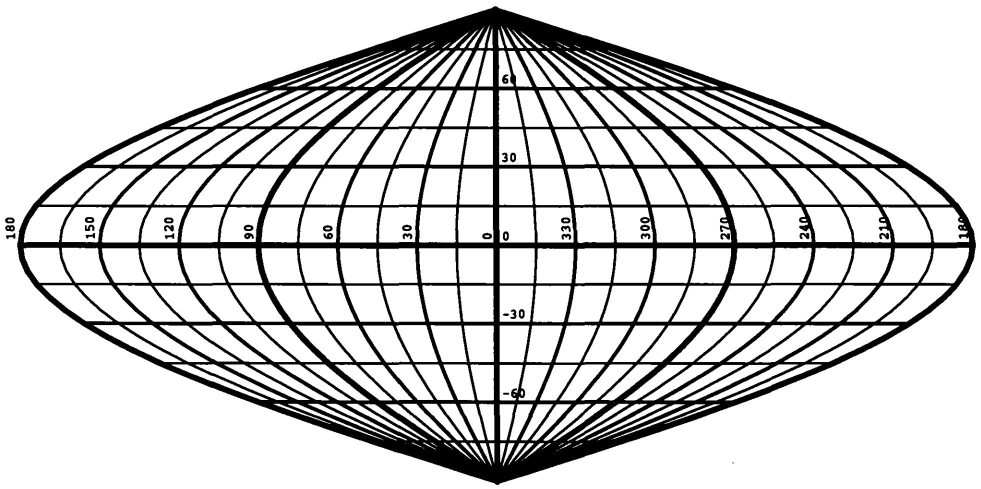
Parallels are equispaced and their length is chosen to make an equal area projection. It is, in fact, the equatorial case of *Bonne's projection.*

$$x = -r_0 \phi \cos\theta \left(\frac{\pi}{180}\right), \tag{110}$$

$$y = r_0 \theta \left(\frac{\pi}{180}\right), \tag{111}$$

$$\phi = \frac{-180x}{\pi r_0 \cos\left(\frac{180y}{\pi r_0}\right)}, \tag{112}$$

$$\theta = \frac{180y}{\pi r_0}. \tag{113}$$



Limits: none.
Conformal at latitude $\theta = 0$ and at longitude $\phi = 0$.

5.2 Aitov's projection

This projection is developed from the equatorial case of the *zenithal equal area projection* for a hemisphere. The equatorial scale and the longitude coverage are both doubled, thereby mapping the whole sphere while preserving the equal area property.

We will derive the equations for the projection here as an illustration of the construction of oblique projections, and also because the inverse equations are most easily derived by reversing the process.

The equations for the polar case of the zenithal equal area projection are (§2.2.2)

$$
\begin{aligned}
x'' &= -r_0\sqrt{2(1-\sin\theta'')}\sin\phi'', & (114)\\
y'' &= -r_0\sqrt{2(1-\sin\theta'')}\cos\phi'', & (115)
\end{aligned}
$$

where double primes denote the native polar system and single primes will denote the equatorial system. Following the prescription given in §2 for constructing oblique projections, the coordinates of the pole of the native system in the oblique system are $(\phi'_0, \theta'_0) = (0, 0)$, with $(\phi''_0 = 0)$. The Euler angles for the transformation are therefore $(\Phi', \Theta', \Phi'') = (90°, 90°, 90°)$. Thus

$$
\begin{aligned}
\phi'' &= 90° + \arg(\cos\theta'\sin\phi', \sin\theta'), & (116)\\
\theta'' &= \sin^{-1}(\cos\theta'\cos\phi'), & (117)
\end{aligned}
$$

whence

$$
\begin{aligned}
\cos\phi'' &= \frac{-\sin\theta'}{\sqrt{\cos^2\theta'\sin^2\phi' + \sin^2\theta'}}, & (118)\\
\sin\phi'' &= \frac{\cos\theta'\sin\phi'}{\sqrt{\cos^2\theta'\sin^2\phi' + \sin^2\theta'}}, & (119)\\
\sin\theta'' &= \cos\theta'\cos\phi'. & (120)
\end{aligned}
$$

Setting $(x', y') = (x'', y'')$ we have the equations for constructing the equatorial case of the *zenithal equal area projection*

$$
\begin{aligned}
x' &= -r_0\alpha'\cos\theta'\sin\phi', & (121)\\
y' &= r_0\alpha'\sin\theta', & (122)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha' &= \sqrt{\frac{2(1-\cos\theta'\cos\phi')}{\cos^2\theta'\sin^2\phi' + \sin^2\theta'}} & (123)\\
&= \sqrt{\frac{2}{1+\cos\theta'\cos\phi'}}. & (124)
\end{aligned}
$$

The equations for Aitov's projection may be deduced by making the substitutions

$$
\begin{aligned}
x &= 2x', && (125)\\
y &= y', && (126)\\
\phi &= 2\phi', && (127)\\
\theta &= \theta', && (128)
\end{aligned}
$$

whence

$$
\begin{aligned}
x &= -2r_0\alpha\cos\theta\sin\frac{\phi}{2}, && (129)\\
y &= r_0\alpha\sin\theta, && (130)
\end{aligned}
$$

where

$$\alpha = \sqrt{\frac{2}{1+\cos\theta\cos(\phi/2)}}. \qquad (131)$$

These equations would be difficult to invert without prior knowledge of how they were obtained. However, it is a fairly simple matter to apply the construction process to the inverse equations for the *zenithal equal area projection*. From §2.2.2 we have

$$
\begin{aligned}
\phi'' &= arg(-y'', -x''), && (132)\\
\theta'' &= \sin^{-1}\left(1 - \frac{x''^2+y''^2}{2r_0^2}\right). && (133)
\end{aligned}
$$

so that

$$
\begin{aligned}
\cos\phi'' &= \frac{-y''}{R''}, && (134)\\
\sin\phi'' &= \frac{-x''}{R''}, && (135)\\
\cos\theta'' &= \frac{R''}{r_0}\sqrt{1-\frac{R''^2}{4r_0^2}}, && (136)\\
\sin\theta'' &= 1 - \frac{R''^2}{2r_0^2}, && (137)
\end{aligned}
$$

where

$$R'' = \sqrt{x''^2 + y''^2}. \qquad (138)$$

The Euler angles for the transformation from the (ϕ'', θ'') system to the (ϕ', θ') system are the inverse of those presented previously, i.e. $(\Phi'', \Theta'', \Phi') = (\Phi', \Theta', \Phi'')^{-1} = (90°, -90°, 90°)$, so

$$
\begin{aligned}
\phi' &= 90° + \arg(\cos\theta''\sin\phi'', -\sin\theta''), && (139)\\
\theta' &= \sin^{-1}(-\cos\theta''\cos\phi''), && (140)
\end{aligned}
$$

and substituting with $(x'', y'') = (x', y')$ we have

$$\phi' = 90^\circ + \arg\left(\frac{-x'}{r_0}\sqrt{1-\frac{R'^2}{4r_0^2}}, \frac{R'^2}{2r_0^2}-1\right), \tag{141}$$

$$\theta' = \sin^{-1}\left(\frac{y'}{r_0}\sqrt{1-\frac{R'^2}{4r_0^2}}\right), \tag{142}$$

where

$$R' = \sqrt{x'^2+y'^2}. \tag{143}$$

The inverse equations for *Aitov's projection* are then readily obtained by substituting

$$x' = x/2, \tag{144}$$
$$y' = y, \tag{145}$$
$$\phi' = \phi/2, \tag{146}$$
$$\theta' = \theta, \tag{147}$$

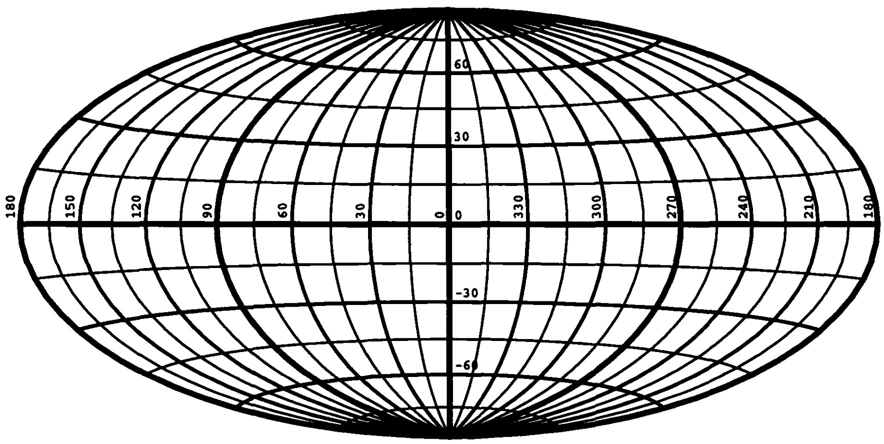
whence

$$\phi = 180^\circ + 2\arg\left(\frac{-x}{2r_0}\sqrt{1-\frac{R^2}{4r_0^2}}, \frac{R^2}{2r_0^2}-1\right), \tag{148}$$

$$\theta = \sin^{-1}\left(\frac{y}{r_0}\sqrt{1-\frac{R^2}{4r_0^2}}\right), \tag{149}$$

where

$$R = \sqrt{x^2/4+y^2}. \tag{150}$$



Limits: none.
Conformal at the centre only.

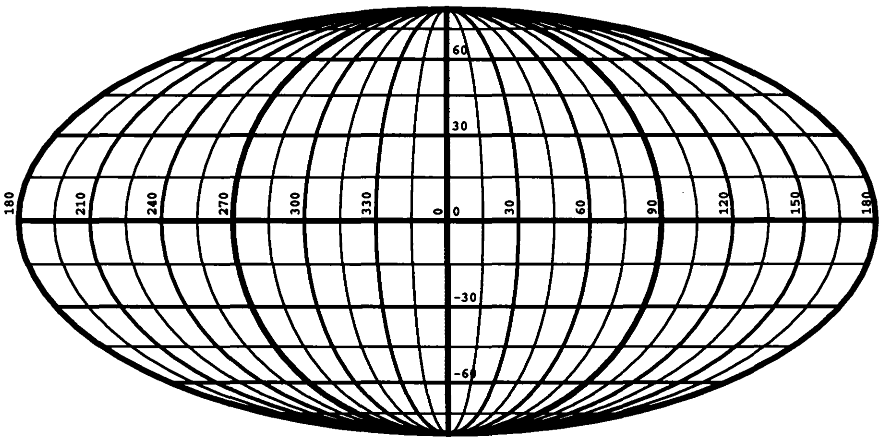
5.3 Mollweide's projection

Meridians are projected as ellipses, parallels as straight lines spaced so as to make the projection equal area.

$$
\begin{aligned}
x &= r_0\left(\frac{\phi}{90}\right)\sqrt{2}\cos\alpha, && (151)\\
y &= r_0\sqrt{2}\sin\alpha, && (152)\\
\phi &= \frac{90x}{r_0\sqrt{2-\frac{y^2}{r_0^2}}}, && (153)\\
\theta &= \sin^{-1}\left(\frac{1}{90}\sin^{-1}\left(\frac{y}{\sqrt{2}r_0}\right)+\frac{y}{r_0}\sqrt{2-\frac{y^2}{r_0^2}}\right). && (154)
\end{aligned}
$$

where α is given as the solution of the transcendental equation

$$\sin\theta = \frac{\alpha}{90}+\frac{\sin 2\alpha}{\pi}. \qquad (155)$$



Limits: none.