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# Sensitivity Criteria for Aperture Synthesis Arrays

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## I. INTRODUCTION

Previous memos (18, 20, 21) in the mm Array series have discussed various sensitivity criteria for aperture synthesis arrays in general, and the proposed NRAO mm Array in particular. The purpose of the present memo is to clarify a number of related issues: the statistical analysis of the way visibility errors (noise) propagate into the map plane under different conditions of weighting and tapering; various approaches to translating this error analysis into surface brightness sensitivity criteria; an error whereby brightness temperature sensitivities were underestimated by a factor of 2.5 in mm Array memo 21; and the distinctions between a "mapping" approach to surface brightness sensitivity and the "matched filter" approach discussed in mm Array memo 18.

Let M be the number of visibility data points obtained with a fundamental integration time $\Delta t$, a bandwidth $\Delta\nu$, and system temperature $T_{sys}$ when observing with m antennas (of diameter D), forming m(m-1)/2 antenna pairs. If $\sigma_o$ is the rms noise level for a single antenna pair,

$$\sigma_o = 0.194(T_{sys}/100^oK)/[(D/10m)^2(\Delta t_{sec}\ \Delta\nu_{GHz})^{1/2}] \ \text{Jy} \tag{1}$$

assuming antenna aperture efficiencies of 0.5. The quantity $\sigma_o$ can be considered equivalent to the statistical standard deviation for the amplitude of the visibility for one antenna pair and integration time $\Delta t$. With M of these data points, used in maps with specified weighting and tapering, how does one estimate the errors (noise) in these maps? Before addressing this question in the context of aperture synthesis, let us summarize the way errors propagate in some simple statistical situations.

## II. STATISTICS OF DATA ORGANIZED IN GROUPS OR CELLS

Let M data points be distributed in N groups (occupied cells), where x is the data variable, and each data point has a standard deviation $\sigma_o$. The sequence $x_1$, $x_2$, ···, $x_N$ enumerates the cell averages for x, the sequence $n_1$, $n_2$, ···, $n_N$ enumerates the number of data points in each group (cell), and the sequence $w_1$, $w_2$, ···, $w_N$ enumerates the arbitrarily assignable weights for each group. The standard deviation for the s-th cell is

$$\sigma_s = \sigma_o/n_s^{1/2} \quad . \tag{2}$$

The weighted mean of all x is

$$\langle x\rangle = \sum_{s=1}^{N} (w_s\ x_s) / \sum_{s=1}^{N} w_s \quad . \tag{3}$$

Note that $\Sigma\ w_s(x_s - x)^2$ is minimized when $x = \langle x\rangle$. The standard deviation of $\langle x\rangle$ is $\sigma$, where

$$\sigma^2 = \sum_{s=1}^{N} w_s\ \sigma_s^2/(N \sum_{s=1}^{N} w_s) \quad . \tag{4}$$

For so-called "natural" weighting of data, $w_s = n_s$, so, combining Equations (2) and (4), one obtains

$$\sigma_{na}^{\ 2} = \sigma_o^2/\sum_{s=1}^{N} n_s = \sigma_o^2/(N\langle n\rangle_M) \tag{5}$$

where $\langle n\rangle_M$ is the arithmetic mean of the number of data points in each group. For so-called "uniform" weighting of each group of data points (cell), $w_s = 1$, so

$$\sigma_{un}^{\ 2} = \sigma_o^2[\sum_{s=1}^{N} (1/n_s)]/N^2 = \sigma_o^2/(N\langle n\rangle_{HM}) \tag{6}$$

where

$$1/\langle n\rangle_{HM} = (1/N) \sum_{s=1}^{N} (1/n_s) \tag{7}$$

defines the "harmonic" mean number of data points in each group. From the results of Equations (5)-(6) it is obvious that the formula

$$\sigma_s = \sigma_o/(N\langle n\rangle)^{1/2} \tag{8}$$

is valid as long as long as one is careful about using the appropriate arithmetic mean or harmonic mean for $\langle n\rangle$.

The concept of taper, as used in aperture synthesis, corresponds to applying another "weighting" function, $T_s$, to each group (cell). The above results are then generalized, and $\sigma$ is still given by Equation (8), if one defines the "tapered" aritmetic mean, $\langle n\rangle_{TM}$, by

$$\langle n\rangle_{TM} = (\sum_{s=1}^{N} T_s)/N \quad , \tag{9}$$

and the "tapered" harmonic mean, $\langle n\rangle_{THM}$, by

$$\langle n\rangle_{THM} = (\sum_{s=1}^{N} T_s)/(\sum_{s=1}^{N} T_s/n_s) \quad . \tag{10}$$

Although the concepts of arithmetic mean, harmonic mean, tapered arithmetic mean, and tapered harmonic mean are very useful understanding the effects of various weighting and taperings, these concepts (in connection with Equation (8)) are misleading when there are groups (cells) with no data points, as is common in aperture synthesis. However, the following formulations for natural and uniform weighting cases will always be valid:

$$\sigma_{na}^{2} = \sigma_o^2/(\sum_{s=1}^{N} T_s\ n_s) \tag{11}$$

and

$$\sigma_{un}^{2} = \sigma_o^2[\sum_{s=1}^{N} (T_s/n_s)]/[N(\sum_{s=1}^{N} T_s)^2\} \quad . \tag{12}$$

All of the above considerations apply to the aperture synthesis case where x is a complex visibility, $V_{jk}$, for antenna pairs j and k. The generation of a map is the equivalent of Equation (2), as can be seen from the discrete Fourier inversion formula

$$I(x,y) = \sum_{j,k} w_{jk}\{V_{jk}e^{2\pi i(u_{jk}x + v_{jk}y)} + V_{jk}^{*}e^{-2\pi i(u_{jk}x + v_{jk}y)}\}/(2 \sum_{j,k} w_{jk}) \quad . \tag{14}$$

The only conceptual change from the averaging formula (Equation (3)) is counting each visibility point both as itself and its complex conjugate (so I(x,y) is real), and adding each visibility with a phasing given by the appropriate Fourier transform factors.

## III. SURFACE BRIGHTNESS SENSITIVITY CRITERIA

The basic connection between sensitivity criteria as assumed above, and sensitivity criteria for surface brightness is the equation

$$S = (2k/\lambda^2)\int T_b \, d\Omega \tag{15}$$

where the integral is over the source, beam, etc. depending upon circumstances. If the integral is over a beam area where we can assume (or cannot avoid assuming) that $T_b$ is constant, then

$$S = (2k/\lambda^2)\, T_b \, \Omega_B \tag{16}$$

where $\Omega_B$ is the beam solid angle. If the beam can be assumed to be a Gaussian, $\Omega_B$ is related to the Gaussian HPBW by $\Omega_B = 1.1331\ \theta_{HPBW}^2$. However, if a gaussian is not a good approximation, it is better to think of $\theta_B$ as defining the width of the Gaussian that gives the correct beam solid angle, i.e.

$$\theta_B = (\Omega_B/1.1331)^{1/2} \quad . \tag{17}$$

Because of the wide range of beam shapes, and the errors of up to a factor of two that can ensue from assuming Gaussian beam solid angles, in subsequent mm Array design studies we will compute the beam solid angle from numerical integration over the synthesized beam (from the center out to a cut-off where the beam intensity reaches zero) computed from Equation (14) or its equivalent. Thus, in general, we will derive surface brightness sensitivity criteria from $\sigma$ computed as discussed above using

$$T_b = (\sigma/2k)(\lambda^2/\Omega_B) = (\sigma/2k)[\lambda^2/(1.1331\theta_B^2)] \quad . \tag{18}$$

Note that since $\theta_B$ is proportional to $\lambda$, this is a result which is independent of wavelength.

## IV. "MAPPING" AND "MATCHED FILTER" APPROACHES TO SURFACE BRIGHTNESS

The approach implied by the above discussion is one whereby we compute $\sigma$'s and $T_b$'s from the specified equations, and in obtaining surface brightness the beam solid angle is computed from the core of the synthesized beam of a simulated observing situation. There is no ambiguity about the results, and one obtains realistic estimates of the differences for the simple extremes of natural or uniform weighting, with no tapering. One can extend this to evaluating sensitivity for various taper functions and taper parameters.

How does this relate to the suggested surface brightness criteria discussed the T.J. Cornwell in mm Array memo 18? In terms of the above discussion, Cornwell argues that there is a special significance to the $\sigma$'s and $T_b$'s computed for taper functions which are squared gaussians, in that it is the best prescription for estimated sensitivity to structures that have size scales corresponding to the gaussian HPBW. However, another way of expressing this conclusion is that, if the basic taper function is a gaussian, then the squared gaussian "tapering" amounts to a normal gaussian taper, but the taper width is interpreted differently. The "width" in the u-v plane corresponds to a width in the x-y plane that is twice the "normal" size scale .

In applying this criterion for surface brightness sensitivity in mm Array memos 20 and 21, which is what we are calling a "matched filter" approach, the author of this memo was concerned that this approach resulted in sensitivities a factor of 2-4 lower than expected from simple considerations for cases where the HPBW of the "taper" applied corresponded to the size of the u-v plane including all the data. The reason was that the effect of the Gaussian squared was to effectively exclude much of the higher resolution data in the u-v plane.

The best current view is that both approaches should be applied in the evaluation of various designs for the proposed NRAO mm array. The matched filter approach gives a well defined result for larger scale structures where the higher resolution data play no significant role. The "mapping" approach with no taper is necessary to evaluate the high resolution characteristics of any array.

Finally, let us note that the initial formulas for $\sigma_o$ in mm Array memos contained an error of a factor of 2.5. The correct starting formula is given in Equation (1) of this memo. The correct value of $\sigma_{na}$ for a general observing situation is

$$\sigma_{na} = 5.5\ (T_{sys}/100)/[(D/10m)^2(\Delta t_{minutes}\ \Delta\nu_{GHz}\ (N_B/210))^{1/2}] \quad \text{mJy} \qquad (19)$$

where $N_B = m(m-1)/2$ is the number of baselines and D is the antenna diameter in meters. The value of $\sigma_{un}$ will always be less than or equal to that given by Equation (19) because of our general result that

$$\sigma_{un} = \sigma_{na}\ [\langle n\rangle_{HM}/\langle n\rangle_M]^{1/2} \qquad (20)$$

which cannot be evaluated without considering the geometric effects that determine the distribution of data points in cells.

## V. SUMMARY OF SENSITIVITY RESULTS FOR VARIOUS PROPOSED ARRAYS

Let us now summarize the correct results for the sensitivity of various proposed arrays. These results are excerpted from other memos (with corrections for the 300 m. configuration results), and represent only the "best" of the configurations discussed in these memos. Although we are now taking N = 21 for the number of the large antennas, there was not enough time to re-do all the 300 meter configurations for N = 21. In the following table all observations were for band-widths of 1 GHz, system temperatures of 100°, 60° declination, and integration time per data point of 300 seconds. All of the parameters in Table 1 have been defined earlier in this memo or are self-explanatory, except for the $N_{occ}$ parameter which is the number of occupied cell in u-v planes gridded for two points per synthesized beam (for uniform weighting).

Aside from the necessary background information, the main points of interest in Table 1 are: (1) the differences (or lack thereof) of beam-widths and map sensitivity for the different configurations and weighting (uniform or natural); (2) the numbers of occupied cells; and (3) the differences between the arithmetic and harmonic mean number of data points per cell which is a direct reflection of relative sensitivity for natural and uniform weighting. No tapering was applied for any of these results, so they reflect the raw beam-width and sensitivity characteristics of these arrays - with all the collected data used with natural or uniform weighting.

In Table 1 there are drastic differences between the 27 antenna arrays with a VLA-like Y configuration (Y27) and the randomized circular array (R2CIR27). These are due to the great differences in the radial distribution of data points in radial rings in the u-v plane, resulting in both the large differences in mean, and harmonic mean, numbers of data points per cell and the different synthesized beam widths. Each array has its virtues and reflects optimization for different types of observing problems. The VLA-line Y has high sensitivity to broad structures compared with high resolution structures, whereas the randomized circular array is optimized to put a higher proportion of sensitivity (for a single array observation) into the high resolution structures.

Table 1

Summary of Beam Width and Sensitivity Parameters of Various Arrays

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| No. Antennas | 27 | | 27 | | 21 | | 21 | |
| Antenna Diam. | 10 m. | | 10 m. | | 10 m. | | 4 m. | |
| Config. | R2CIR27 | | Y27 | | FCIR90M | | TRACKM21 | |
| Config. Diam. | 300 m. | | 300 m. | | 90 m. | | 25 m. | |
| Gr. u-v Plane | 71 X 71 | | 71 X 71 | | 17 X 17 | | 15 X 15 | |
| Obs. Time | $12^h$ | $10^m$ | $12^h$ | $10^m$ | $12^h$ | $10^m$ | $12^h$ | $10^m$ |
| $N_{occ}$ | 3400 | 478 | 2400 | 354 | 304 | 162 | 169 | 113 |
| $N_M$ | 15 | 1.5 | 21 | 2.0 | 99 | 2.6 | 179 | 3.7 |
| $N_{HM}$ | 10 | 1.3 | 4.4 | 1.5 | 19 | 2.0 | 45 | 2.4 |
| $\theta_{b,nat}/\lambda_{mm}$ | 0.61" | 0.63" | 1.5" | 1.5" | 2.2" | 2.0" | 7.1" | 7.1" |
| $\theta_{b,un}/\lambda_{mm}$ | 0.50" | 0.47" | 0.58" | 0.96" | 1.6" | 1.8" | 5.0" | 5.8" |
| $\sigma_{nat}$(mJy) | 0.050 | 0.43 | 0.05 | 0.43 | 0.065 | 0.55 | 0.40 | 3.4 |
| $\sigma_{un}$(mJy) | 0.062 | 0.45 | 0.11 | 0.49 | 0.15 | 0.63 | 0.81 | 4.3 |
| $T_{b,nat}$(mK) | 1.8 | 21 | 0.32 | 2.8 | 0.18 | 1.8 | 0.11 | 0.93 |
| $T_{b,un}$(mk) | 3.4 | 27 | 4.3 | 7.3 | 0.82 | 2.6 | 0.44 | 1.8 |