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MILLIMETER ARRAY MEMO NO. 33

Factors Affecting Sensitivity for the Millimeter Arrays

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I. Introduction

Previous Millimeter Array memos have discussed various aspects of the sensitivity of the proposed millimeter arrays. Unfortunately, a completely correct formulation of the problem has not yet been made because: 1) the basis of the theoretical sensitivity formula was not discussed; 2) errors by factors of 2.5 and $10^{1/2}$ have occured; and the major effect the atmosphere can have on signal to noise has not been mentioned. This memo is intended to address these problems.

II. Fundamental Sensitivity Equations

If we define

$\Delta\nu$ = IF bandwidth

T_{rcvr} = receiver temperature (= 100 K in this memo)

T_{atmo} = atmospheric temperature (= 280 K in this memo)

T_{sys} = system temperature due to receiver, atmosphere, etc.

D = diameter of each antenna

ε_a = aperture efficiency (= 0.5 in this memo)

ε_c = correlator efficiency (= 0.82 for 3 level correlation)

then the theoretical rms noise flutuation in the amplitude for a single antenna-receiver interferometer pair is (cf. "Introduction to the NRAO VLA")

$$\sigma_o = 4(2^{1/2}) k T_{sys} / [\pi \varepsilon_a \varepsilon_c D^2 (\Delta\nu\ \Delta t)^{1/2}]$$

$$= 0.192\ (T_{sys}/100)/[(D_m/10)^2 (\Delta\nu_{GHz}\ \Delta t_{sec})^{1/2}]\ \ \text{Jy} \qquad (1)$$

as correctly stated in Equation (1) of Millimeter Array Memo 29, where Δt is the integration or observing time, k = Boltzmann constant, the antenna aperture efficiency is taken to have a value of 0.5, we assume a three level correlator, and the numerical coefficient corresponds to T_{sys} = 100 K, D = 10 meters, and

$\Delta\nu = 1$ GHz (appropriate for a continuum observing situation).

The system temperature is dominated by a combination of the receiver noise temperature and the radiative transfer through the atmosphere. For an isothermal atmosphere this can be expressed as

$$T_{sys} = T_{rcvr} + T_{atmo}[1 - \exp(-\tau_1 \sec \zeta)]$$
$$= 100 + 280[1 - \exp(-\tau_1 \sec \zeta)] \quad (2)$$

where τ_1 is the optical depth for unit air mass (at the zenith, ζ is the zenith angle, and for simplicity we approximate the air mass by sec ζ. While T_{rcvr} = 100 K and T_{atmo} = 280 K are reasonably assumptions for the planned millimeter array, the value of τ_1 varies with both frequency and atmospheric conditions. The major atmospheric parameters are the column densities (n) of precipitable water vapor (PWV) and molecular oxygen. A discussion of these dependencies can be found in Zammit and Ade (Nature, **293**, 550, 1981). As a rule of thumb $\tau_1 \approx 0.06$ n(mm^{-1}PWV) at 230 GHz. For the purposes of this memo we will treat τ_1 at 230 GHz as the most important atmospheric parameter.

Sensitivity to surface brightness can be determined from Equation (1) by using the relationship between brightness temperature and flux density (cf. MMA Memo 29), which is

$$\Delta T_b = (\sigma_o/2k)(\lambda^2/\Omega_b) = (\sigma_o/2k)[\lambda^2/(1.1331\theta_b^2)] = 1.36\ \sigma_o(\text{mJy})[\lambda/\theta_b('')]^2\ \text{K} \quad (3)$$

where Ω_b is the beam solid angle, λ is the observing wavelength, θ_b is the half-power width of a gaussian beam solid angle, and ΔT_b is the rms brightness temperature sensitivity. It is obvious that without aperture synthesis with a reasonable number of antennas, the beam referred to in Equation (3) will not be very well defined. Also, brightness temperature sensitivity is independent of wavelength because of the the wavelength dependence of the beam solid angle; therefore, the beamwidth (for a uniformly weighted u-v plane) is

$$\theta_{b,un} = 1.9'' \ (\lambda_{cm}/B_{km}) \quad (4)$$

where B_{km} is the maximum size of the array in km, and Equation (3) becomes

$$\Delta T_b = 0.377\ \sigma_o(\mathrm{mJy})\ B_{km}^2 \quad . \tag{5}$$

Adopting scaling parameters appropriate to a one minute (snapshot) observation with 21 antennas of 10 m diameter, Equations (1) and (5) become

$$\sigma = 1.71\ (T_{sys}/100)/\{(D_m/10)^2[\Delta\nu_{GHz}\ \Delta t_{min}(N_B/210)]^{1/2}\}\ \ \mathrm{mJy} \tag{6}$$

and

$$\Delta T_b = 0.64\ (T_{sys}/100)[B_{km}/(D_m/10)]^2/\{[\Delta\nu_{GHz}\ \Delta t_{min}(N_B/210)]^{1/2}\}\ \ \mathrm{K} \tag{7}$$

where N_B is the number of antenna pairs (= N(N-1)/2 if N is the number of antennas). Equation (6) applies to the situation where each measured data point is given equal weight, independent of location in the u-v plane. Unfortunately, Equations (4)-(5) are for uniform weighting in the u-v plane, so Equation (7) is a "compromise" between the uniform and "natural" weighting. As discussing in MMA memo 29 the uniform weight value of σ can be obtained by multiplying the right side of Equation (7) by the square root of the ratio of the harmonic mean (N_{HM})and mean (N_M) number of data points per occupied "cell" in a u-v plane gridded acording to the sampling theorem; however, there is no simple analytic way to express the natural weight beam width for an arbitrary distribution of data in the u-v plane. Table 1 is an up-dated version of the sensitivity parameters listed in the table in MMA Memo 29, but with 21 antenna configurations for the arrays of 10 m antennas (Y21, R5CIR21, and FCIR90M) and Multi-Telescope array (TRACKM21). In addition to quantities already defined, Table 1 contains entries for N_{occ} (the number of occupied cells in the gridded u-v plane), N_{occ}/N_{theo} (the fraction of theoretically occupiable cells that are occupied), and σ_{sid} (a fractional estimate of the beam sidelobe level as defined by Cornwell in MMA memo 18). The numbers in Table 1 and the coefficients in the above equations reflect system temperatures of 100 K which are due entirely to receiver noise. Unfortunately, one of the principle purposes of the present memo is to point out the difficulty in attaining this ideal system temperature when atmospheric effects are taken into account, therefore the sensitivity parameters in the last four lines of the table should be increased by at least a factor of two for some observations, particularly in the 1.3 mm band.

Table 1

Summary of Parameters for $\delta = 60°$ Obs. with Various Arrays and $T_{sys} = 100^{\circ}$

Config.	Y21		R5CIR21		FCIR90M		TRACKM21	
Config. Diam.	300 m.		300 m.		90 m.		25 m.	
Antenna Diam.	10 m.		10 m.		10 m.		4 m.	
Gr. u-v Plane	71 X 71		71 X 71		17 X 17		15 X 15	
Obs. Time	8^h	2^m	8^h	2^m	8^h	2^m	8^h	2^m
N_{occ}	2804	362	3418	410	308	204	176	136
N_{occ}/N_{theo}	.708	.091	.863	.104	.890	.589	.996	.770
N_M	7.10	0.57	5.90	0.51	65.3	1.03	114.6	1.54
N_{HM}	1.88	0.53	3.94	0.51	11.7	0.79	14.7	0.92
$\sigma_{sid,nat}/\lambda_{mm}$	.0325	.0565	.0200	.0499	.0717	.0798	.1006	.1064
$\sigma_{sid,un}/\lambda_{mm}$	.0189	.0526	.0171	.0494	.0570	.0700	.0754	.0857
$\theta_{b,nat}/\lambda_{mm}$	1.30"	1.20"	0.51"	0.49"	2.09"	1.99"	6.95"	6.90"
$\theta_{b,un}/\lambda_{mm}$	0.54"	0.82"	0.48"	0.49"	1.53"	1.75"	4.88"	5.44"
σ_{nat}(mJy)	0.079	1.22	0.079	1.22	0.079	1.22	0.49	7.6
σ_{un}(mJy)	0.154	1.28	0.096	1.23	0.187	1.40	1.38	9.9
$\Delta T_{b,nat}$(mK)	0.64	11.5	4.07	69.1	0.25	4.2	0.14	2.2
$\Delta T_{b,un}$(mk)	7.08	25.8	5.62	68.8	0.46	6.2	0.79	4.5

III. Effective System Temperature

Millimeter wavelength observations of astronomical sources are affected by both atmospheric absorption and atmospheric emission, which strongly limit results at the higher frequencies in the millimeter "window". In Equation (2) we expressed the system temperature as a composite of receiver noise and the effects of observing emission from a relatively "hot" atmosphere. However, another obvious effect is absortion, so that observation of a source with brightness temperature T_b through an atmosphere with temperature T_{atmo}, zenith optical depth of τ_1, and zenith angle ζ will give an observed brightness temperature which is

$$T_{b,obs} = T_b \exp(-\tau_1 \sec \zeta) \quad . \tag{8}$$

The sole purpose of sensitivity Equations like (6) and (7) is to evaluate the signal to noise for observations of sources of flux density S_ν and brightness temperature T_b. For this purpose the dependence on reciever temperature and atmosphere is

$$S_\nu/\sigma = T_b/\Delta T_b \propto \exp(-\tau_1 \sec \zeta)/\{T_{rcvr} + T_{atmo}[1 - \exp(-\tau_1 \sec \zeta)]\} \tag{9}$$

so it is obvious that one can define an effective system temperature,

$$T_{sys,eff} = \{T_{rcvr} \exp[\tau_1 \sec \zeta] + T_{atmo} [\exp(\tau_1 \sec \zeta) - 1] \tag{10}$$

that includes the effects of both emission and absorption. If evaluation of signal to noise is desired, Equation (10) should be used in conjunction with the normal sensitivity equations.

In Table 2 we show, for the latitude of the VLA site, tables of $T_{sys,eff}$ (for declinations of 30^o, 0^o and -30^o) as a function of τ_1 and hour (or zenith) angle values. For sensitivity-limited problems the most serious effect of the atmosphere on $T_{sys,eff}$ is the increase in the amount of integration time necessary to achieve the same sensitivity; this is shown in Table 3 where we give observing times, relative to that obtained for a constant system temperature of 166 K, for the same parameters as used in Table 2.

Table 2 - Effective System Temperatures ($T_{sys,eff}$) for Three Declinations

$\delta = 30^{\circ}$

τ_1	HA = 0.0	1.0	2.0	3.0	4.0	5.0	6.0	hrs
	ζ = 4.0	13.3	25.7	38.1	50.3	62.3	73.8	degrees
0.00	100	100	100	100	100	100	100	K
0.05	120	120	122	125	131	143	174	
0.10	140	141	145	151	164	191	263	
0.15	162	163	169	180	201	245	370	
0.20	184	187	194	210	240	304	497	
0.25	208	211	221	242	282	370	649	
0.30	233	237	250	276	328	444	831	
0.35	260	264	280	313	377	526	1049	
0.40	287	293	312	352	431	617	1309	
0.45	317	323	346	393	489	719	1620	
0.50	347	355	382	437	551	833	1992	
0.55	380	389	419	484	619	959	2437	
0.60	413	424	459	534	692	1099	2969	
0.65	449	461	502	588	772	1256	3605	
0.70	487	500	546	645	857	1430	4366	
0.75	526	541	593	705	950	1624	5276	K

$\delta = 0^{\circ}$

τ_1	HA = 0.0	1.0	2.0	3.0	4.0	5.0	hrs
	ζ = 34.0	36.8	44.1	54.1	65.5	77.6	degrees
0.00	100	100	100	100	100	100	K
0.05	124	124	127	134	149	200	
0.10	149	151	157	171	204	326	
0.15	175	178	188	211	266	485	
0.20	204	208	222	255	336	685	
0.25	234	239	258	302	415	938	
0.30	266	273	297	354	504	1258	
0.35	300	308	339	410	604	1662	
0.40	336	346	383	472	717	2171	
0.45	374	387	431	539	845	2815	
0.50	415	429	482	612	990	3627	
0.55	458	475	537	691	1152	4652	
0.60	504	524	596	778	1336	5946	K

$\delta = -30^{\circ}$

τ_1	HA = 0.0	1.0	2.0	3.0	hrs
	ζ = 64.0	65.5	70.0	76.8	degrees
0.00	100	100	100	100	K
0.05	146	149	160	193	
0.10	197	204	229	309	
0.15	255	266	309	453	
0.20	320	336	402	633	
0.25	392	415	509	857	
0.30	473	504	633	1136	
0.35	564	605	777	1483	

Table 3 - Relative Observing Time Needed to Achieve the Same Signal to Noise

$\delta = 30^\circ$

	HA = 0.0	1.0	2.0	3.0	4.0	5.0	6.0^h
	ζ = 4.0	13.3	25.7	38.1	50.3	62.3	73.8°
τ_1							
0.00	0.364	0.364	0.364	0.364	0.364	0.364	0.364
0.05	0.520	0.525	0.539	0.568	0.625	0.746	1.108
0.10	0.715	0.725	0.761	0.836	0.985	1.330	2.527
0.15	0.952	0.972	1.038	1.177	1.466	2.177	4.981
0.20	1.238	1.270	1.376	1.605	2.094	3.366	8.998
0.25	1.579	1.626	1.786	2.133	2.898	4.992	15.350
0.30	1.983	2.050	2.277	2.780	3.915	7.179	25.160
0.35	2.457	2.548	2.861	3.562	5.187	10.080	40.057
0.40	3.009	3.131	3.551	4.502	6.763	13.888	62.399
0.45	3.651	3.810	4.361	5.626	8.703	18.843	95.589
0.50	4.393	4.596	5.308	6.960	11.075	25.249	144.537
0.55	5.246	5.504	6.409	8.538	13.962	33.482	216.309
0.60	6.225	6.547	7.686	10.396	17.460	44.013	321.065
0.65	7.345	7.744	9.162	12.578	21.683	57.431	473.405
0.70	8.622	9.112	10.863	15.131	26.764	74.467	694.287
0.75	10.074	10.672	12.819	18.111	32.860	96.035	1013.775

$\delta = 0^\circ$

	HA = 0.0	1.0	2.0	3.0	4.0	5.0 hrs
	ζ = 34.0	36.8	44.1	54.1	65.5	77.6 degrees
τ_1						
0.00	0.364	0.364	0.364	0.364	0.364	0.364
0.05	0.557	0.564	0.591	0.652	0.806	1.453
0.10	0.806	0.825	0.895	1.061	1.511	3.861
0.15	1.120	1.158	1.291	1.619	2.571	8.550
0.20	1.511	1.573	1.796	2.359	4.103	17.096
0.25	1.990	2.085	2.430	3.324	6.259	32.072
0.30	2.571	2.708	3.215	4.562	9.237	57.649
0.35	3.269	3.462	4.179	6.133	13.290	100.572
0.40	4.103	4.365	5.352	8.108	18.744	171.704
0.45	5.092	5.442	6.772	10.572	26.022	288.508
0.50	6.259	6.719	8.479	13.627	35.661	479.003
0.55	7.631	8.225	10.521	17.393	48.357	788.078
0.60	9.237	9.995	12.956	22.018	64.996	1287.571

$\delta = -30^\circ$

	HA = 0.0	1.0	2.0	3.0 hrs
	ζ = 64.0	65.5	70.0	76.8 degrees
τ_1				
0.00	0.364	0.364	0.364	0.364
0.05	0.775	0.806	0.930	1.359
0.10	1.419	1.513	1.910	3.480
0.15	2.369	2.577	3.479	7.490
0.20	3.722	4.114	5.878	14.607
0.25	5.601	6.278	9.436	26.758
0.30	8.160	9.268	14.600	46.988
0.35	11.600	13.338	21.978	80.087

In Tables 2 and 3 we have drawn lines corresponding to the point where

$$T_{sys,eff} \approx 200\text{-}210 \text{ K}$$

because it represents a reasonable break point between where observations are worthwhile vs difficult, given the rapid increase in effective noise for larger values of τ_1 and ζ. In Table 3 we scaled all observing times to those for a system temperature of 166 K, because it is an ideal low value that should be achievable at a very good observing site when (τ_1 sec ζ) = 0.15.

IV. Conclusions

Table 2 and 3 indicate how strongly atmospheric absorption and emission can affect the effective system temperature, and hence the observing time needed to accomplish a particular observation. A number of obvious conclusions can be drawn from these results.

The absorption of signal from the source is an effect which must be corrected for as part of image restoration from aperture synthesis data. For weak sources, for which self-calibration is not possible, one will need to "correct" amplitudes, as a function of time, based upon concurrent or interspersed measurements of τ_1. For strong enough sources self-calibration can be used to correct for inadequacies in this empirical correction for atmospheric absorption.

Observations which are signal to noise limited, or which take too long when there are high effective system temperatures, are critically limited by the transparency characteristics of the observing site. Based upon the criterion that observations are most effective when one has $T_{sys,eff} \leq$ 200-210 K, one can see from Tables 2 and 3 that one can observe higher declination sources for a reasonable amount of time only when $\tau_1 \approx$ 0.1-0.15; and observations of the galactic center are reasonable easy only when $\tau_1 \leq 0.1$. For 230 GHz this means roughly $\leq$ 2 mm^{-1} PWV for higher declination observations and roughly $\leq$ 1.6 mm^{-1} PWV for the galactic center - if the array is located at the VLA site. The bias against the galactic center can be removed with a location with some combination of a lower τ_1 and/or a more southerly latitude.

The system temperatures that should be assumed for array evaluation purposes at 230 GHz should be at least 200 K, assuming a site with $\tau_1 \leq$ 0.1-0.15 for a reasonable amount of the time. If $\tau_1 \approx 0.25$, for example, system temperatures from 300 to 500 K should be assumed. This means that for 1.3 mm observations all the sensitivity numbers in Table 1 (and previous memos assuming 100 K system temperature) should be scaled upward by at least a factor of two, and possibly a factor of 3 to 5.

We probably should set the specifications for the Millimeter Array sites in terms of the fraction of time that τ_1 at 230 GHz is less than or equal to some number. Requiring $\tau_1 \leq 0.1$ for a site, for a considerable fraction of the time, may preclude the VLA site, but a specificaion of $\tau_1 \leq 0.15$ may make it barely acceptable.

Since the proposed Millimeter Array is actually two arrays, the array of movable ~ 10 meter antennas and the Multi-Telescope array of ~ 4 meter antennas, which are be coupled/combined only when their data are merged in the image construction process, one can consider locating them at different sites. The M-T array is, in fact, more sensitivity-limited than the other array because of much less collecting area, so it is a prime candidate for a better mountain-top site. A mountain-top site may be practical only for the M-T array because it does not need an unreasonably large amount of mountain-top space. The M-T array is the component of the Millimeter Array project which can be used for sub-mm operation, so a very good mountain-top site would allow NRAO to provide a national facility with this capability; however, location at anything but a very good 1.3 mm site would preclude this possibility. While Mauna Kea is the obvious candidate for the "best" site for the M-T component, a location at 10,700 ft on South Baldy (near Socorro) may be good enough and still allow the operational support of both arrays from the same people, labs, etc., if the larger array is located at the VLA site. The characteristics of the Aquarius plateau (Utah) and Grand Mesa (Colorado) are presently unknown.

We conclude that we should develop different site criteria for the two millimeter arrays, and that a site with poor atmospheric characteristics will reduce 1.3 mm sensitivity by factors of 2-3. Therefore, depending upon the extent to which the basic sensitivity of the arrays meets the scientific needs, we may need to adopt more stringent site criteria.