MILLIMETER ARRAY
MEMO NO. 42

Analysis of the Ekers and Rots method of short-spacing estimation

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INTRODUCTION

Some years ago, Ron Ekers and Arnold Rots suggested a scheme for estimating short-spacing information (Ekers and Rots 1979). The essence of the method is to scan an interferometer in position on the sky and record the complex visibility measured as a function of scan position. A transform with respect to the scan position then yields estimates of all spatial frequencies spanned by the surfaces of the two interferometer elements, weighted by a sensitivity function. An elaborated version of this approach will be used in the millimeter array to allow imaging of large fields of view (Cornwell, 1984). This latter method is based upon the Maximum Entropy method of image reconstruction, and is known as "mosaicing".

For the design of the mm array, we must know the sensitivity of the mosaicing method to various types of error in the measured visibility, such as receiver noise, uncertainties in the knowledge of the illumination patterns, pointing errors of the elements, incomplete sampling in both the sky and Fourier planes, cross-talk between receivers, correlated atmospheric emission, etc. Therefore, as a first step in gaining this necessary understanding, I have performed an error analysis of the Ekers and Rots scheme, which is given in this memo.

THE EKERS AND ROTS SCHEME

A simple mathematical description of the Ekers and Rots scheme can be given in terms of the pointing-position-dependent visibility function measured. For simplicity, our analysis will apply only to the one-dimensional case. Let x_p be the pointing position of the elements of the telescope on the sky, and u be the separation of the telescopes measured in wavelengths, as seen from the source. We describe the "primary beam" or sensitivity to emission of an element by the function $A(x - x_p)$, which tapers off to zero for large offsets from the pointing center. We will normalize so that $A(0) = 1$. The visibility function sampled at pointing position x_p is then basically the Fourier transform of the true sky brightness $I(x)$, in Jy/(unit area), weighted by the primary beam A (see e.g. Thompson et al., 1986).

$$V(u, x_p) = \int A(x - x_p) I(x)\ e^{2\pi j u x}\ dx \tag{1}$$

For a single dish, u is zero and then:

$$V(0, x_p) = \int A(x - x_p) I(x)\ dx \tag{1'}$$

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The goal of wide-field imaging is to find $I(x)$ over a region much larger than the width of the primary beam $P(x_p)$. To understand how this is possible, suppose that we perform a Fourier transform of equation (1) with respect to the pointing center position x_p. Let $a(u)$ be the Fourier transform of the primary beam, $A(x)$, and $i(u)$ be the Fourier transform of the true sky brightness. It is easy to show that the Fourier transform of $V(u, x_p)$ with respect to x_p yields what we will call the intermediate visibility function:

$$\begin{aligned} V(u,\xi) &= \frac{\int V(u,x_p)\, e^{j2\pi\xi x_p}\, dx_p}{\int A(x_p)\, dx_p} \\ &= \frac{a(\xi)}{a(0)}\, i(u+\xi) \end{aligned} \tag{2}$$

where ξ is the variable conjugate to x_p. Therefore, by scanning continuously over some region, and then Fourier inverting with respect to the scan position, one can obtain an estimate of the true visibility function weighted by the Fourier transform of the primary beam of the elements.

Sampling requirements

Scanning is not actually required for complete sampling of all the spatial frequencies: we only need samples of $V(u, x_p)$ spaced in x_p by an increment Δx_p equal to $\frac{\lambda}{2D}$ where λ is the observing wavelength, and D is the diameter of the elements. To formalize this, we can introduce a sampling function in x_p-space, similar to the sampling functions in u-space: let $P(x_p)$ be this sampling function, which would typically be a collection of Dirac δ-functions. The Fourier transform of the sampled visibility function yields:

$$V(u,\xi) = p(\xi) * \left(\frac{a(\xi)}{a(0)}\, i(u+\xi)\right) \tag{3}$$

For complete, uniform sampling, this is equivalent to equation (4). The sampling requirements can be deduced from this equation: if $P(x_p)$ is represented by a collection of δ-functions, then so is $p(\xi)$. Aliasing of power along the ξ axis will occur if the extent of the sensitivity function $a(\xi)$, at most twice the element diameter, is greater than the separation of the δ-functions in $p(\xi)$. Hence, to avoid aliasing, the spacing of pointings must obey:

$$\Delta x_p \leq \frac{\lambda}{2D} \tag{4}$$

If the signal-to-noise is sufficiently weak then poorer sampling may be allowed with little consequent degradation in image quality. Note that this limit applies directly to simple single-dish imaging where it is often violated, but usually the illumination of the aperture is arranged to fall off severely near the edge so that the effective diameter is somewhat smaller.

From here on, we will assume that aliasing can be neglected, and that we can use equation(2) rather than equation(3).

Derivation of the visibility function

Equation (2) can, of course, be inverted to obtain the unknown visibility function i:

$$i(u+\xi) = V(u,\xi)\,\frac{a(0)}{a(\xi)} \tag{5}$$

One obvious conclusion can be drawn from this equation: uncertainties in $a(\xi)$ lead to proportionate errors in the derived visibility function, i. The effects of the other types of error are rather more subtle. We will discuss these in turn.

To simplify matters, we will assume that there are N_p pointing centers required to span the object, so that the sampling function in x_p-space is:

$$P(x_p) = N_p^{-1} \sum_{i=1}^{N_p} \delta(x_p - x_p^{(i)}) \tag{6}$$

We can now discuss various sources of error.

Receiver noise: Let the noise for the visibility, $V(u, x_p^{(i)})$ corresponding to the i'th pointing be $\sigma_V(u, x_p^{(i)})$. The covariance of the error in the derived visibility is then:

$$\langle \delta i(u,\xi)\delta i^*(u,\xi')\rangle_R = \frac{|a(0)|^2}{N_p^2\; a(\xi)a^*(\xi')} \sum_{i=1}^{N_p} \sigma_V^2(u, x_p^{(i)})\; e^{\,2\pi j(\xi-\xi').x_p^{(i)}} \tag{7}$$

If the pointings are at regular increments, Δx_p, in x_p, and if all the noises are equal, then this can be approximated :

$$\langle \delta i(u,\xi)\delta i^*(u,\xi')\rangle_R \sim \frac{\sigma_V^2(u,x_p)}{N_p}\left|\frac{a(0)}{a(\xi)}\right|^2 \qquad for \qquad |\xi-\xi'| \ll \frac{1}{N_p \Delta x_p} \tag{8}$$

Note that, as expected, the SNR improves as the square root of the number of pointing centers. Also, the correlation scale in the u-plane is related directly to the size of the sampled region in the x_p-plane.

Pointing errors: Suppose that the i'th pointing is made with a pointing center which is actually $x_p^{(i)} + \delta x_p^{(i)}$. We then have that:

$$V(u,\xi) = N_p^{-1} \sum_{i=1}^{N_p} V(u, x_p^{(i)} + \delta x_p^{(i)})\; e^{2\pi j \xi x_p^{(i)}} \tag{9}$$

since the reconstruction can only be made with the nominal pointing center (unless self-calibration is used to obtain an estimate of the $\delta x_p^{(i)}$). Using the derivative theorem, we have that for small pointing errors, the error in the derived visibility function is:

$$\delta i(u+\xi) = \frac{-j2\pi}{N_p}\frac{a(0)}{a(\xi)} \int \frac{a(\xi')}{a(0)} \sum_{i=1}^{N_p} e^{2\pi j(\xi+\xi')x_p^{(i)}}\; \delta x_p^{(i)}\; \xi\; \hat{i}(u+\xi')\; d\xi' \tag{10}$$

where $\hat{i}$ denotes the true visibility function. Hence if the expected pointing error is zero, then so is the expected error in the derived visibility function. If the errors are independent across pointing centers (probably not a good approximation) then the covariance is given by:

$$\langle \delta i(u,\xi)\delta i^*(u,\xi')\rangle_P = \frac{(2\pi)^2}{N_p^2}\left|\frac{a(0)}{a(\xi)}\right|^2 \int\int \frac{a(\xi'')\ a^*(\xi''')}{|a(0)|^2} \sum_{i=1}^{N_p} \sigma_{x_p}^2(u, x_p^{(i)}) \\ \xi\xi'\ e^{\ 2\pi j(\xi-\xi'+\xi''-\xi''')x_p^{(i)}}\ \hat{i}(u+\xi'')\ \hat{i}^*(u+\xi''')\ d\xi'' d\xi''' \tag{11}$$

To get to grips with this equation we need to make two simplifying approximations: first, that the pointing error is the same for all pointings, and second, that:

$$\sum_{i=1}^{N_p} e^{\ 2\pi j\xi x_p^{(i)}} \sim N_p\ \delta(\xi) \tag{12}$$

Now consider a one dimensional uniformly illuminated dish of diameter D, observing a point source of flux S. To a reasonable approximation, the variance introduced by pointing errors is:

$$\sigma_i^2(u,\xi) \sim \frac{\pi^2}{N_p}\left|\frac{a(0)}{a(\xi)}\right|^2 \left(\frac{\sigma_{x_p}}{\Delta x_{p_{crit}}}\right)^2 S^2 \left(\frac{\lambda\xi}{D}\right)^2 \tag{13}$$

where $\Delta x_{p_{crit}}$ is the critical sampling rate in the x_p-plane. This is gratifyingly intuitive, given the complexity of equation (11). In particular, as seems reasonable, the error declines as the inverse square root of the number of pointings.

For an extended source, the flux S should be replaced by a weighted average of the squared visibility over the aperture, and thus the error will decrease.

Poor knowledge of the sensitivity pattern: From equation (5), we can show that the covariance of the errors due to uncertainties in the sensitivity pattern is given by:

$$\langle \delta i(u,\xi)\delta i^*(u,\xi')\rangle_S = i(u+\xi)i^*(u+\xi')\ \frac{R_a(\xi,\xi')}{a(\xi)a^*(\xi')} \tag{14}$$

where R_a is the covariance of the uncertainties in the sensitivity function, $a(\xi)$.

For simplicity, we will use a simple Gaussian model for the errors in the illumination pattern, $w(\xi)$:

$$R_w(\Delta\xi) = \sigma_w^2\ e^{-\frac{1}{2}\left(\frac{\Delta\xi}{L_w}\right)^2} \tag{15}$$

so that the correlation scale-size is L_w wavelengths. We then have

$$R_a(\Delta\xi) = \sigma_w^2\ e^{-\left(\frac{\Delta\xi}{L_w}\right)^2} \tag{16}$$

Therefore, for the simple case of a point source, the variance is:

$$\sigma_i^2(u,\xi) \sim \frac{\sigma_w^2}{|a(\xi)|^2}\ S^2 \tag{17}$$

DISCUSSION

At this first look, it seems that both pointing errors and illumination uncertainties will be important. Consider a one-dimensional image of an object requiring 40 pointings, and suppose that we use uniform illumination, and that the pointing is good to one-tenth of the FWHM width of the primary beam. The SNR due to pointing errors on a measurement of the visibility function of a point source is then only about 10 at baselines differing from the nominal by half a dish diameter. For the case of llumination errors, use of interferometers dishs for which the surface is known to $\frac{\lambda}{20}$ will have allow a SNR of about 10, again at baselines offset by half a dish diameter. Unlike these two effects, receiver noise can be reduced to an arbitrarily small level simply by integration over time.

In assessing the importance of these results, we must note that the relationship of the Ekers and Rots scheme to the full mosaicing method is not clear. The general qualitative nature will probably be conserved: for example, the effect of pointing errors will decline with number of pointings. However, given the complexity of the mosaicing method, only simulation will answer some of these questions properly. The next step in this analysis should therefore be simulation of the mosaicing method. The order of importance of factors in the simulation is:

- Pointing errors: both correlated and uncorrelated in time.
- Uncertainty and variability in the sensitivity function.
- Receiver noise.
- Limited sampling in the x_p-plane.

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