ALMA Memo 353
Investigation of suppression of sidelobes by simple displacement of clustered groups of regularly spaced antennas

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Abstract

A means of suppressing beam sidelobes by simple displacements of subgroups of regularly spaced antennas is proposed and investigated. Previous experiments in the optical domain suggest that this should be possible, and the principle is developed first by analogy and then by a simple analysis to encompass the radio astronomical situation. Considering the specific example of a hexagonal array with the additional constraint of compactness, some improvement over a completely regularly placed configuration is demonstrated. An analysis suggests why the improvement is less than that hoped for. Alternative optimisation methods yield compact arrays with lower sidelobes, but the principle outlined here may prove useful in other configurations.

1 Introduction

It has been shown in the optical domain [1,2] that it is possible to reduce replications and aliasing arising from spectral orders in the transfer function of low space-bandwidth (i.e. low number of pixels) pixellated Fourier plane filters in an optical processor through tailoring of the pixels' spatial distribution and aperture function. The case of a hexagonally packed array of radio antennas is closely analogous to the previously analysed case of a square array of pixels in a Fourier plane filter, with the grating response of the antenna array's synthetic beam being the counterpart of the spectral orders mentioned above. The principles used to perturb the pixels' spatial distribution are applied to the antenna array using mainly geometric arguments and the results presented here. Some analysis is included to explain the results obtained. The effect of the antenna aperture function is not investigated.

2 The analogy between the optical and radio astronomical imaging situations

In the earlier work the situation was that of a Fourier transform pair of complex amplitude distributions in the filter plane (or Fourier plane, in the prevailing terminology) and image plane, with respective conjugate coordinates of spatial frequency (ξ, η) and position (x, y). The aim was to remove the spectral orders present in the filter's transfer function (i.e. the Fourier transform of the filter's transmission), which is convolved with the coordinate reversed input object[1] to yield the filtered output that is the image. The spectral orders arise from the underlying periodicity of the regularly spaced square pixels of the filter. With all pixels 'on' (transmittance = 1.0) the transfer function of the filter is governed by the aperture function of the individual pixels, and the overall pattern of placement of the pixels.

In the radio astronomical imaging situation we deal with visibility V and radio brightness I as the Fourier transform pair (subject to certain assumptions– see [3]) with corresponding conjugate coordinates of (u, v) and (l, m), with the analogue of the transfer function being the synthetic (or dirty) beam $B(l, m)$, which is the Fourier transform of the visibility sampling function in the (u, v) plane, $S(u, v)$. In the case of a regular array of antennas we have a 'grating response' of spectral orders other than the main beam (zero order) which we again wish to remove.

Returning to the optical case we discover that by choosing an appropriate probability distribution function for the perturbation of the pixels from their regularly spaced centres we apply an envelope to the filter's transfer function, with a further envelope applied by the Fourier transform of the pixel's aperture function, both envelope functions having moduli less than or equal to one. By setting the fill factor of the filter to 25%, the Fourier transform of the square aperture function has zeroes at the midpoints of the even spectral orders other than the zero order. Further, by applying one of the four possible perturbations shown in Figure 1 to each pixel's position with probability 0.25 of each being chosen, we can contrive for the other envelope function to have zeroes at the middle of the odd spectral orders. Note that this result applies to the ensemble average of all possible arrays which satisfy the displacement probability distribution function– individual members of the ensemble generally display the same features. Thus the higher spectral orders are greatly attenuated.

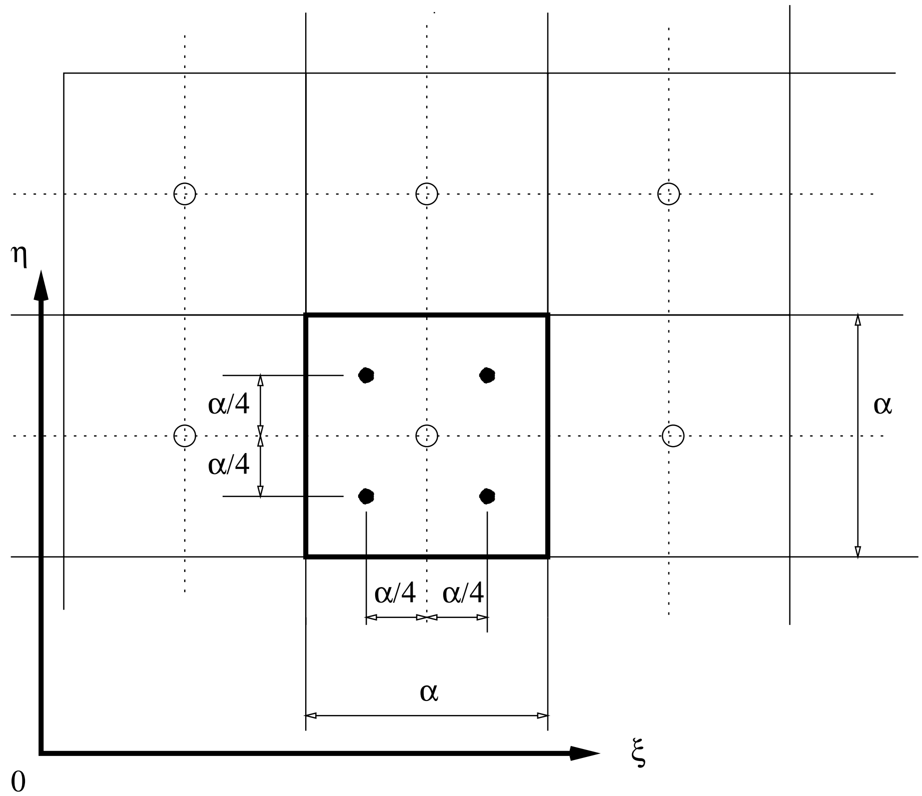
Can we reduce sidelobes in the synthetic beam of a radio telescope array in this manner? When dealing with a regular square or hexagonal grid of antennas it seems a possibilty, but the analogy requires to be pondered further.

An important difference is that the optical filter may be represented by the convolution of the pixel aperture function with an array of delta functions at the pixel centres, which Fourier transforms to a product of the aperture function's FT and the FT of the array of delta functions. This allows us to analyse separately the two envelope functions which apply, one arising from each product term, and to simplify the analysis by now ignoring the aperture function, as this is effectively what will happen when switching to the radio astronomical case. We *hope* that the FT of the aperture function will do something about the even orders. Considering the array of delta functions, this can be represented as an infinite array of delta functions multiplied by an 'array function' which is zero outside the extent of the filter aperture and unity within. This then transforms to a convolution of an infinite array of delta functions with the transform of the array function, which determines the finite width of the spectral orders.

However, in the radio astronomical case we deal not with the transform of the delta function representation of the antenna centres, but with the transform of the differences between the centres, the baselines which make up the uv coverage. Noting that the uv coverage, $S(u, v)$, is given by the autocorrelation of the delta-function representation of the antennas[2], we gratefully invoke the Wiener-Khintschine theorem [4] to claim that when this is Fourier transformed, the attenuation of the spectral orders/sidelobes will be as in the optical case, but squared... though in actual fact the attenuation is effectively the same, as the repeated reference to spectral

[1] The coordinate reversal mentioned is a consequence of the input object having undergone an earlier forward Fourier transform by the time it reaches the filter plane

[2] This would appear to be an approximation, given that mosaicing seems to succeed by virtue of extra baselines resulting from the autocorrelation of dishes of finite width



○ Centre of cell (i.e, the site of the pixel centre in the corresponding regular array)

● Allowed position for centre of pixel within the cell

Figure 1: Possible pixel positions within a square $\alpha \times \alpha$ cell

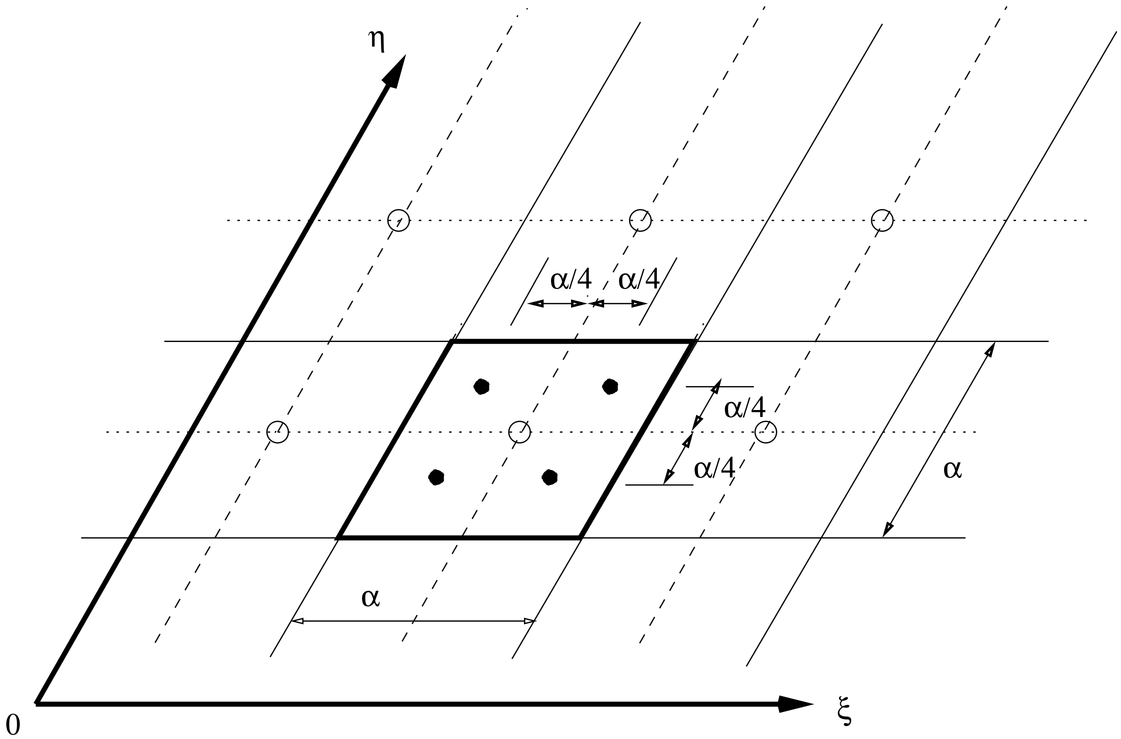
orders conveys the fact that in the optical case we are interested in the squared modulus of the amplitude distribution of the output image, as this intensity distribution is a useful, measurable, quantity. Thus we may use existing ideas to try and reduce the sidelobes.

The other difference of course is that the regular array that we start off with is hexagonal, for close packing purposes. This may be viewed as a skewed square or rectangular array, and to avoid becoming bogged down in coordinate transformations, we simply skew the allowed displacements for the square grid by a corresponding amount, as shown in Figure 2. This is still in accordance with the interpretation of the shifted pixels or antennas being in antiphase with the pixels or antennas shifted in the opposite direction, due to the spatial separation in ξ and/or η (u and/or v) being $(n \pm \frac{1}{2}) \times \alpha \quad : n \in Z$, where α is the pitch of the regularly spaced pixels or antennas whose phase contributions add constructively when Fourier transformed.

Having avoided any in depth analysis, let us plunge into some simulations to test the theory.

3 Candidate arrays using the above analogy, and resulting synthetic beams

In the optical case studied earlier, the perturbations of the pixels from their regular centres was chosen at random from the set of four allowed perturbations, with each perturbation being



○ Centre of cell (i.e, the site of the pixel centre in the corresponding regular array)

● Allowed position for centre of pixel within the cell

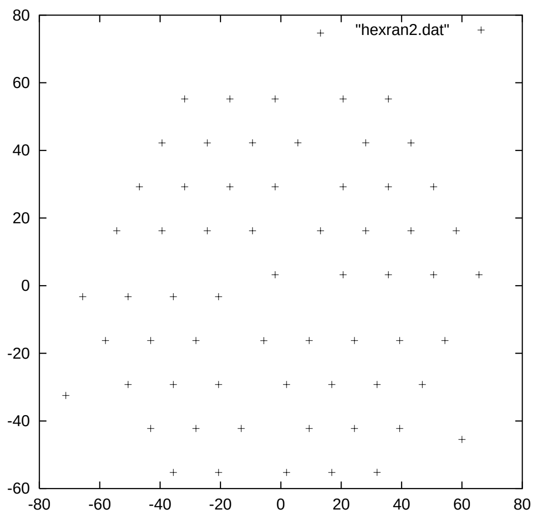
Figure 2: Possible pixel positions within an $\alpha \times \alpha$ cell, for hexagonal array

chosen the same number of times when considered over the whole array. In the interpretation offered at the end of the previous section, this should ensure net cancellation of the contributions from all pixels at the centres of the spectral orders. As the perturbations were chosen at random, to prevent overlapping of pixels, it was necessary for the extent of the array to be 2 $\times$ (number of pixels across array) $\times$ (width of pixel).

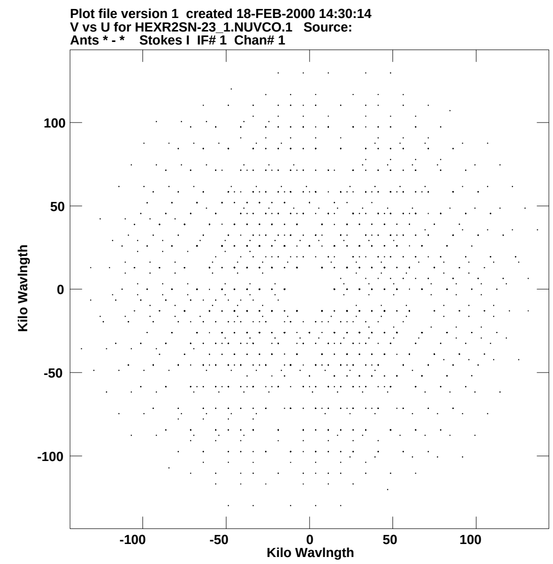
This is at odds with our wish in the radio astronomical case to have as compact an array as possible, so we choose to split the most compact hexagonal configuration into quarters, each quarter containing 16 antennas. We then perturb each of the quarters as an entire block by one of the four allowed displacements, increasing the extent of the array by only half an antenna spacing. While this is not obviously random, it is a member of the ensemble of possible arrays which could arise if the displacements were chosen at random, so is worth persevering with. This gives rise to an array as shown in Figure 3a with the corresponding uv coverage shown in Figure 3b.

The synthetic beam is shown in Figure 3c alongside the synthetic beam from the regular hexagonal array (Figure 3d) for the purposes of comparison.

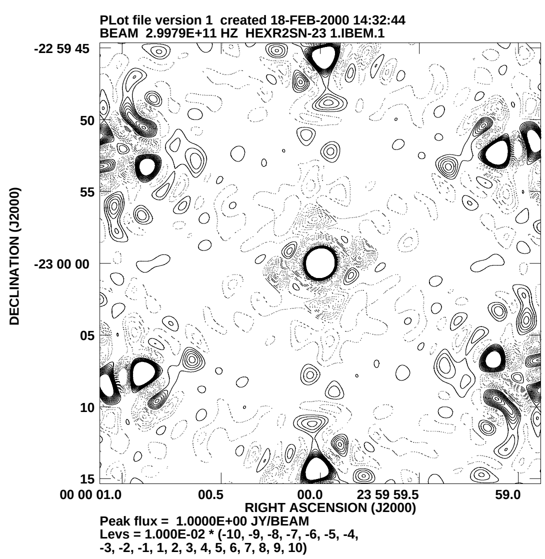
Unhappily the perturbed array is not devoid of large sidelobes. In case this was a consequence of an incorrect choice of allowed perturbations, the simulation was repeated for a square array with the proven perturbations as depicted in Figure 1, with Figure 4a and b showing the array and its uv coverage, 4c showing its synthetic beam and 4d showing the synthetic beam of the unperturbed square array of 64 antennas. The same effect is shown as for the hexagonal arrays of Figure 3.



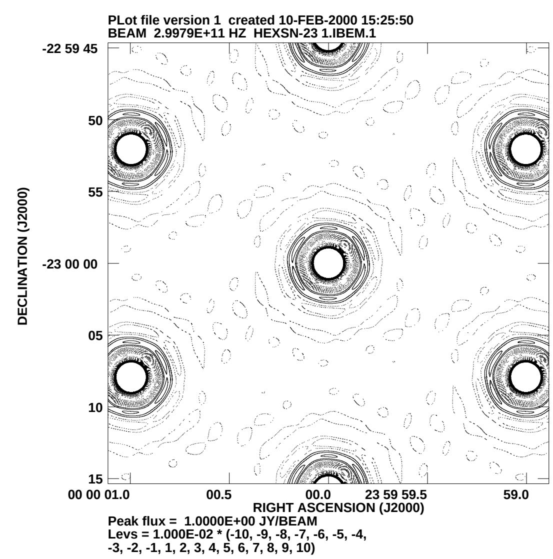
a) positions of array centres



b) uv coverage

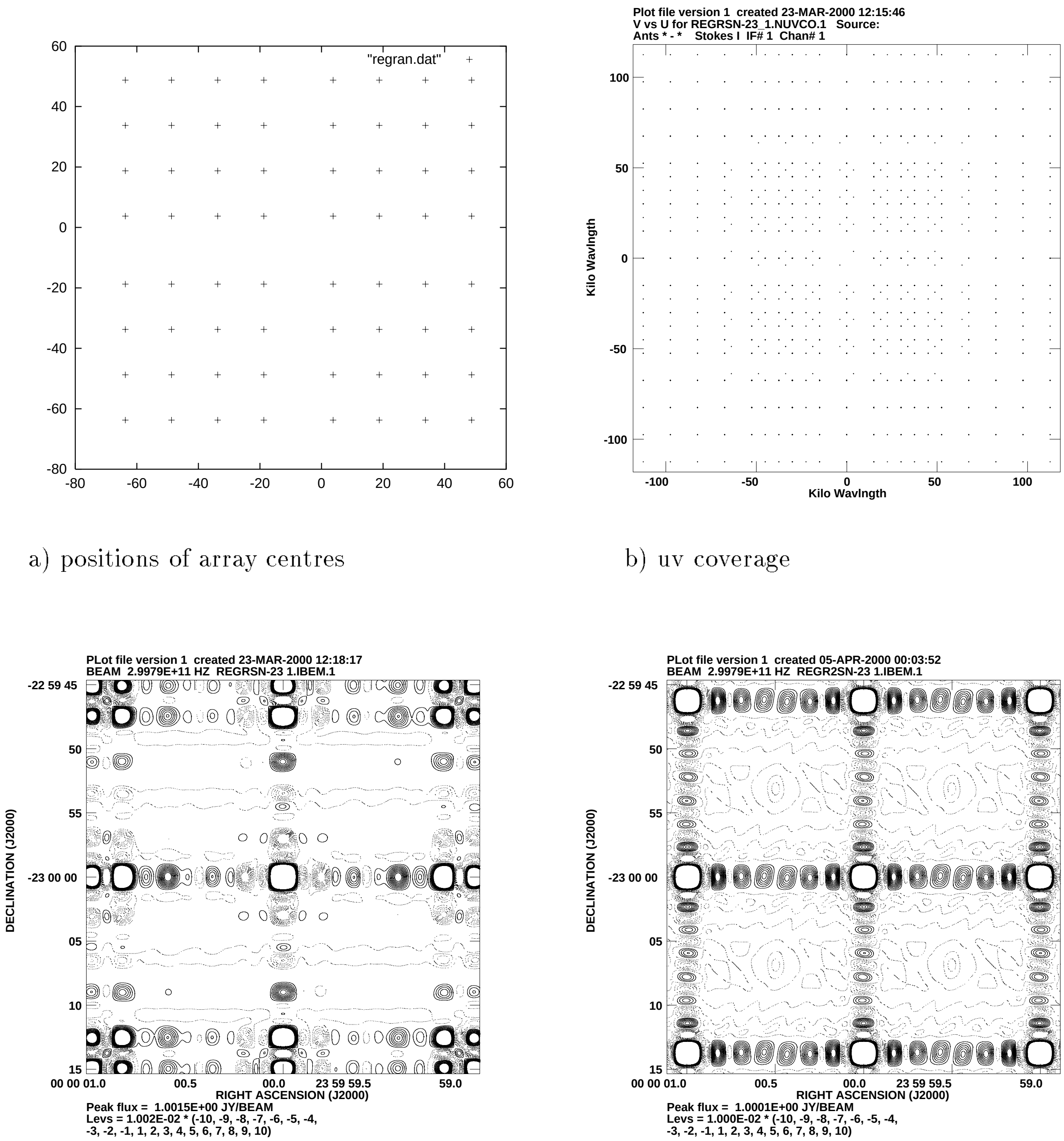


c) synthetic beam corresponding to a)



d) synthetic beam of unperturbed array, included for comparison

Figure 3: Results for perturbed hexagonal array



a) positions of array centres

b) uv coverage

c) synthetic beam corresponding to a)

d) synthetic beam of unperturbed array, included for comparison

Figure 4: Results for perturbed square array

4 Analysis of results

Disappointingly the synthetic beam of the perturbed array is not the single narrow peak which was desired, but instead exhibits distinct spectral order like structures, albeit curiously bifurcated.

Some analysis is required to understand this effect. First we introduce some notation and illustrate the analogy between the optical and radio astronomical cases. A Fourier transform relationship is denoted by $\Longleftrightarrow$, or by the operator $\mathcal{F}$.

The optical case

We denote the complex amplitude disturbance just after the uniformly illuminated filter array with all pixels 'on' by $t_f(\xi, \eta)$. Thus t_f is the delta function representation of the pixels' centres convolved with the aperture function of the individual pixels. This transforms to an amplitude disturbance in the image plane equal to the transfer function of the filter, $t_i(x, y)$, i.e.

$$t_f(\xi, \eta) \Longleftrightarrow t_i(x, y) \tag{1}$$

To exclude the aperture function of the individual pixels from the analysis we simply take it to be a delta function itself. This results in the envelope function in the transfer function of the filter due to the aperture being identically one, and t_f being simply the delta function representation of the pixels' centres.

We wish to remove spectral orders from t_i by means of perturbing the placement of the pixels to modify t_f, with the quantity measured in the image plane being

$$|t_i|^2 = \mathcal{F}[t_f \star t_f] \tag{2}$$

where $\star$ denotes autocorrelation.

The radio astronomical case

Here t_f denotes the delta function representation of the antenna positions. The sampling function in the uv plane is $S(u, v)$ which transforms to the synthetic beam $B(l, m)$, i.e.

$$t_f \star t_f = S(u, v) \Longleftrightarrow B(l, m) \tag{3}$$

and similarly to before, the measured quantity is

$$B(l, m) = \mathcal{F}[t_f \star t_f] \tag{4}$$

Thus an arrangement of array positions which reduces spectral orders in t_f should also reduce sidelobes in $B(l, m)$.

Explanation of results

Looking at the perturbed arrays used in 1-d, we may split t_f into an additive combination of two functions, identical apart from a displacement added to one of them. We consider each of the functions to be four delta functions evenly spaced by a distance α, and the two functions to be separated by a displacement of 5.5α, i.e. the same situation as is seen in one of the rows of the perturbed square array.

Thus

$$t_f(\xi) = f(\xi) + f(\xi + 5.5\alpha) \tag{5}$$

where

$$f(\xi) = p(\xi) * \sum_{n=0}^{3} \delta(\xi - n\alpha) \tag{6}$$

with $p(\xi)$ being the aperture function of the pixel.

Fourier transforming gives

$$\begin{aligned} \mathcal{F}[t_f(\xi)] = t_i(x) &= (1 + e^{-2\pi i 5.5\alpha x})\mathcal{F}[f(\xi)] && (7) \\ &= e^{-\pi i 5.5\alpha x}(e^{\pi i 5.5\alpha x} + e^{-\pi i 5.5\alpha x})\mathcal{F}[f(\xi)] && (8) \\ \Rightarrow |t_i(x)|^2 &= 4\cos^2(5.5\pi\alpha x)|\mathcal{F}[f(\xi)]|^2 && (9) \end{aligned}$$

which is the measurable quantity. The squared modulus of the Fourier transform of $f(\xi)$ gives rise to the spectral orders, which in this case will be modulated by $4\cos^2(5.5\pi\alpha x)$.

Fourier transforming $f(\xi)$ gives

$$\begin{aligned} \mathcal{F}[f(\xi)] &= \mathcal{F}[p(\xi)] \times \sum_{n=0}^{3} \exp^{-2\pi i n\alpha x} && (10) \\ &= \mathcal{F}[p(\xi)] \times \frac{\sin(4\pi\alpha x)}{\sin(\pi\alpha x)} \exp^{-3\pi i\alpha x} && (11) \end{aligned}$$

With reference to the similar equations for the transmittance of a Fabry-Perot etalon, the latter quotient gives rise to peaks spaced by $1/\alpha$, of nominal width $1/(4\alpha)$. This may be compared with the narrower troughs of the modulating $\cos^2$ function of nominal width $1/(5.5\alpha)$, which thus may be responsible for the observed splitting of the peaks in the synthetic beams studied here.

If we consider $p(\xi)$ as a $\delta-$function (effectively ignoring the form of the aperture function), then its Fourier transform is identically one, and our measurable quantity analagous to $B(l)$ becomes

$$|t_i(x)|^2 = 4\cos^2(5.5\pi\alpha x)\left(\frac{\sin(4\pi\alpha x)}{\sin(\pi\alpha x)}\right)^2 \tag{12}$$

which we can plot as Figure 5, setting α equal to 1, and normalising. The comparable slice through the perturbed regular array at 0 degrees is shown as Figure 6.

There are still disparities, such as the curve of Figure 6 having negative values. I believe however that this may be explicable by the default uniform weighting scheme for uv sample points which is applied in AIPS, rather than the natural weighting which has been implicitly assumed in the above analogy between optical and radio astronomical cases. Further differences may arise due to gridding and aliasing effects, as illustrated in [5], suggesting that the explanation offered here for the observed effect is satisfactory.

5 Comments and conclusions

The perturbed arrays considered are not effective at significantly reducing the grating response noted for the corresponding regular hexagonal or square arrays. This arises as a result of the net displacements for the perturbations being significantly larger than the array element spacing, which when Fourier transformed leads to an envelope function with a periodicity significantly

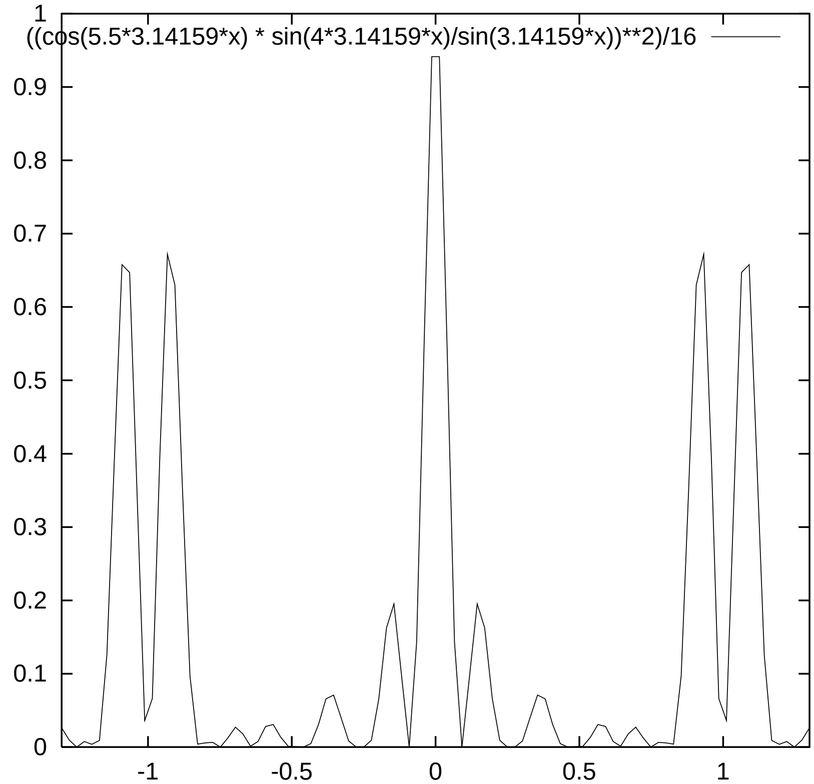


Figure 5: Cross sectional plot of model 1-d perturbed array's power spectrum

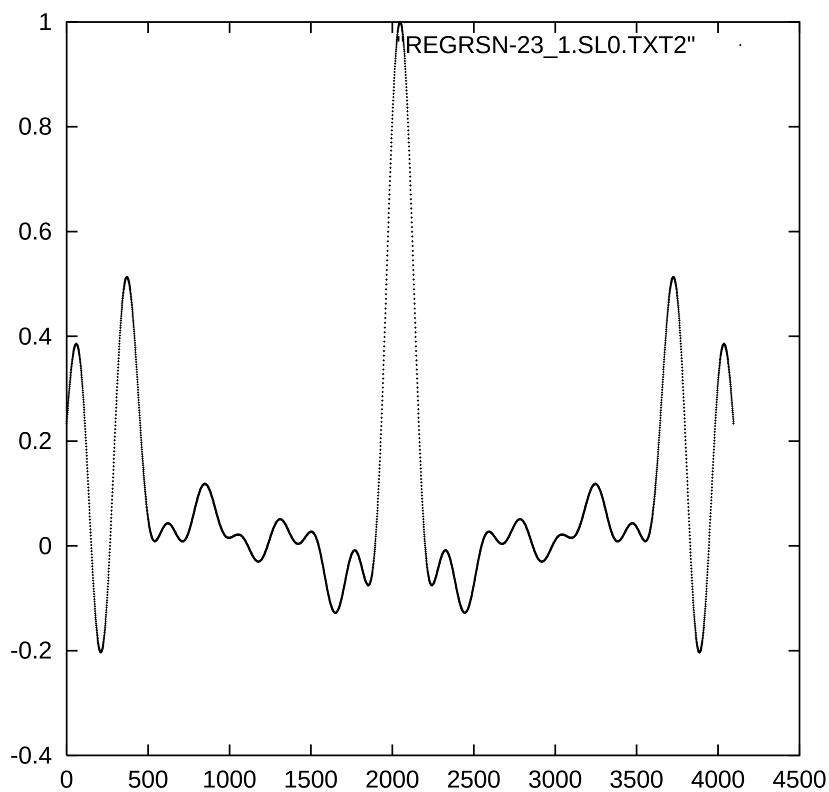


Figure 6: Cross sectional plot of slice at 0 degrees through synthetic beam of perturbed square array

shorter than that of the spectral order like peaks in the grating response. The envelope function therefore serves only to split the peaks, rather than being close to zero across their entire width as would be the case if the perturbations were randomly chosen from the allowed set for each array centre, rather than for the 4 large blocks as considered above. However, this would require the extent of the array to double to prevent antennas striking each other, and so is a not a favoured option when better results may be obtained from more compact arrays designed using other means. The member of the ensemble of sets of possible perturbations chosen here was *not* representative of the ensemble as a whole.

6 Acknowledgement

The uv and synthetic beam results were calculated using Classic AIPS (http://www.cv.nrao.edu/aips/), with gnuplot used for some graphing.

References

1. S.Heddle and R.M.Sillitto, **Reduction and removal of replication in an optical processor by randomisation of pixel positions in the Fourier plane filter**, *Journal of Modern Optics*, **40**(2), *pp299-313, 1993.*

2. S.Heddle, D.G.Vass and R.M.Sillitto, **Reduction of aliasing in correlation using a pixelated Spatial Light Modulator**, *Proceedings of the SPIE, Vol. 1772, pp116-127, 1992.*

3. A.Richard Thompson, **Fundamentals of Radio Interferometry**, *Synthesis Imaging in Radio Astronomy II, ASP Conference Series, Vol. 180, pp11-36, 1999, G.B.Taylor, C.L.Carilli and R.A.Perley (eds).*

4. Ronald N.Bracewell, **The Fourier Transform and its Applications-Second Edition, Revised**, *pp115-116, McGraw Hill, Singapore, 1986.*

5. Daniel S.Briggs, **Imaging**, *Synthesis Imaging in Radio Astronomy II, ASP Conference Series, Vol. 180, pp127-149, 1999, G.B.Taylor, C.L.Carilli and R.A.Perley (eds).*