

Delay Errors in Single- and Double-Sideband Interferometer Systems

A. R. Thompson (NRAO, Charlottesville), L. R. D'Addario (JPL, Pasadena),
D. T. Emerson (NRAO, Tucson), and P. J. Napier (NRAO, Socorro)

Sept. 20, 2006

Introduction

In many tracking interferometers and synthesis arrays the instrumental delays that are inserted to compensate for the varying geometric delays are implemented digitally, and are variable only in finite increments. The incremental nature of the delays results in phase errors that are periodic functions of time that can degrade the signal-to-noise ratio, i.e. the sensitivity of the instrument. This memorandum is concerned with specification of these minimum increments so that the degradation is no more than $\sim 1\%$, and includes consideration of methods of mitigation that can reduce the effect of the phase errors. Systems like ALMA, with wide receiver bandwidth and long baselines, require very fast digital hardware to provide small enough delay increments, so careful choice of the relevant design parameters is necessary to provide optimum performance. Considerations of single- and double-sideband (SSB and DSB) receiving systems are different in some important details. The characteristics of SSB and DSB systems are discussed in detail in ALMA Memo. 304 (Thompson and D'Addario 2000) for the case where delay errors are negligibly small. An earlier discussion of delay errors in ALMA Memo. 255 (Thompson, 1999) was based on an IF of 0-2 GHz, which was subsequently changed to 2-4 GHz. Note that in this memorandum, DSB refers to systems in which the signals from both sidebands are processed together throughout the system. Systems with DSB front ends in which the sidebands are subsequently separated, or one sideband is suppressed, are regarded as special SSB cases.

Delay Errors

The difference between the propagation times in the wave paths to two antennas that track a source is compensated by variable instrumental delays so that the overall delays of the signals at the correlator input are closely equal. The small inequality in these delays is the delay error¹ $\Delta\tau$. In most current interferometer systems the instrumental compensating delays are implemented digitally, after the signals from the final IF amplifiers have been digitized. Coarse delays that are multiples of the digital sampling interval are implemented in a FIFO (first-in-first-out) memory, possibly using RAM (random access memory), and are usually associated with the correlator. Fine delay increments are necessary to provide the required accuracy, and are typically in the range of $1/4$ to $1/32$ of the Nyquist sample interval (i.e. $1/8$ to $1/64$ of the reciprocal IF bandwidth). In ALMA these are implemented by stepping the phase of the clock pulses that control the digital samplers². The total delay, that is, the geometric (space propagation) delay plus the instrumental delay for each antenna is compared with the geometric delay for an arbitrary reference position³. If the geometric delay is increasing with time as the antennas track a source, then whenever the total delay for an antenna falls below that of the reference by $\tau_0/2$, the compensating delay for the antenna is increased by the minimum increment τ_0 . This causes the delay error to change sign from $-\tau_0/2$ to $+\tau_0/2$. Thus for any antenna the delay error is a sawtooth function of time in which the error jumps from $-\tau_0/2$ to $+\tau_0/2$, then falls approximately linearly to $-\tau_0/2$, and the pattern repeats. The probability distribution of the error is uniform within a range $\pm\tau_0/2$. For a pair of antennas it can usually be assumed that the times of delay adjustment are unrelated (in general the rates of change of the geometric delay will be different for each antenna), so the probability distribution of the difference in their delay errors is a triangular function with

¹Here we are concerned with errors in the compensating delays and do not consider atmospheric effects, etc. in the geometric delays.

²Other digital implementations of fine delay adjustment are possible but are not considered in this memo.

³The magnitude of the instrumental delays required is minimized by choosing for the reference the antenna that is reached last by the approaching wavefront. This could be a significant consideration in systems using analog delays but is generally unimportant with digital ones.

extreme values of $\pm\tau_0$, as in Fig. 1. The rms value of this delay error is:

$$\left[\frac{\int_0^{\tau_0} p(\Delta\tau)\Delta\tau^2 d\Delta\tau}{\int_0^{\tau_0} p(\Delta\tau)d\Delta\tau} \right]^{\frac{1}{2}} = \frac{\tau_0}{\sqrt{6}}, \quad (1)$$

where $p(\Delta\tau)$ is the expression for the probability distribution of $\Delta\tau$ in Fig. 1.

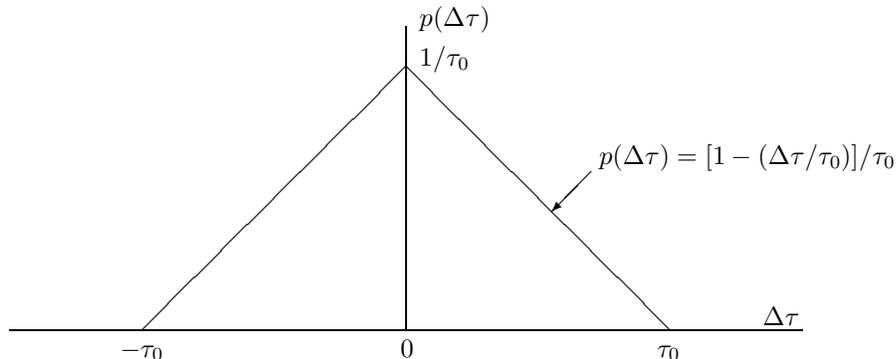


Figure 1: Probability distribution $p(\Delta\tau)$ of the delay error $\Delta\tau$ for a pair of antennas. τ_0 is the minimum increment of the instrumental compensating delay. The expression shown for $p(\Delta\tau)$ applies to the part of the probability function for which $\Delta\tau \geq 0$.

Phase Errors and Degradation of Sensitivity

A delay error results in a phase error in a signal equal to $2\pi\Delta\tau\nu$ where, for analog delays, ν represents frequency in the IF band in which the delays are inserted. For digital delays, ν is a frequency in the final IF band, the signal from which is digitized⁴. (Note that in the remainder of this memorandum IF will refer to the *final* IF band.) For an SSB system the main effect of a phase error is to cause a rotation of the correlation vector as indicated in Fig. 2a, resulting in an error in the correlator output phase. There is also a relatively small decrease in the amplitude, which results from the variation of the phase error with frequency across the IF band, and is proportional to⁵ $\text{sinc}(\Delta\nu\Delta\tau)$ where $\Delta\nu$ is the IF bandwidth. For a DSB system, the delay error causes the components of the correlation vector resulting from the two sidebands to rotate in opposite directions in the complex plane, as shown in Fig. 2b, where the line AB represents the phase angle when the delay error is zero⁶. The amplitude of the vector sum of the two components is proportional to $\cos(2\pi\nu_0\Delta\tau)$ where ν_0 is the IF center frequency, but the phase of the correlation is not changed by a variation in the instrumental delay. The instantaneous sensitivity loss (i.e. the loss for individual samples before time averaging of the correlation) is greater than that for the SSB case. If we are considering delay

⁴In the case where the IF band extends from $N\Delta\nu$ to $(N+1)\Delta\nu$, where N is an integer, it is sometimes useful to regard the digitized signal as representing the baseband response 0 to $\Delta\nu$, i.e. to regard the digitization as involving down conversion to baseband with, for N odd, reversal of the spectrum. It might therefore appear that the phases introduced by digital-delay errors should be calculated for frequency components in the 0 to $\Delta\nu$ range. However, consider the fine delay steps. The delay errors are no greater than the minimum increment and result from the fact that timing of the sampler pulses is not infinitely adjustable. Thus the delay errors can be seen to result from offsets of the sampler pulses from the precisely desired capture times of the waveform. A time shift t in the sample time is equivalent to time shift $-t$ in the input waveform to the sampler, so clearly the resulting phase errors correspond to the frequencies of the IF waveform at the input to the sampler.

⁵ $\text{sinc}(x) = \sin(\pi x)/\pi x$.

⁶To measure the phase of the cross correlation of both sidebands in combination it is necessary to insert a $\pi/2$ phase shift into the IF signal of one antenna, or not to stop the fringes and fit a sine wave to the fringe function.

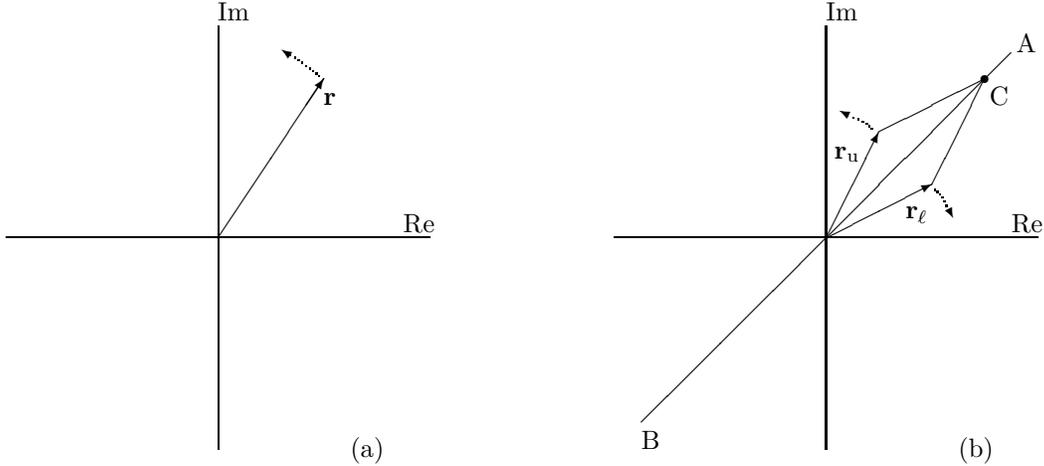


Figure 2: Representation in the complex plane of the output vector \mathbf{r} of a correlator for one antenna pair with (a) single-sideband and (b) double-sideband receiving system. Subscripts u and ℓ indicate the two sideband responses and the point C represents the sum of the upper- and lower-sideband outputs. The dotted arrows indicate variation of the phase angle of the correlation vectors resulting from a phase error, and they reverse when the delay error changes sign. The figure is from Thompson and D’Addario (2000), and the expressions for the response of SSB and DSB systems on which it is based can be found, for example, in Thompson, Moran, and Swenson (1986, 2001) and Woody (2003).

errors that are constant, or only vary slowly with time, the tolerance on the errors is more stringent in the DSB case. Such errors were more important in early interferometers with analog delay systems using coaxial cable or ultrasonic elements (see, e.g. Coe 1973), which could be temperature sensitive and difficult to calibrate accurately. In digital systems the delays are controlled by a highly accurate master clock and the only significant errors result from the incremental nature of the adjustment.

Now consider the effect of time averaging (integration) of the data at the correlator output. If the delay error varies only by a small fraction during a single period of the averaging, then the relative sensitivity is as considered above for the instantaneous effects. This may be the case for the shorter-baseline antenna pairs. For delay errors that vary more rapidly the considerations are different. Consider a long-baseline case in which the geometric delay is varying rapidly enough that the delay error changes sign several times during the minimum averaging time at the correlator output. For an SSB system (Fig. 2a), the phase of the correlation vector \mathbf{r} will swing back and forth, following the difference of the sawtooth error patterns for the two antennas. Components of the correlation that are normal to the vector time-average will cancel. Thus the magnitude of the averaged correlation is proportional to the time average of the cosine of the instantaneous vector phase relative to the average phase. Similarly, for a DSB system (Fig. 2b) the phase angles of the vectors representing the two sideband responses move in opposite senses, following the delay error variations. Components of the correlation vector normal to the direction AB cancel. The magnitude of the correlation is proportional to the time-average of the cosine of the vector angles with respect to AB . Clearly, with the rapidly varying delay error, the loss in sensitivity is effectively the same for the SSB and DSB systems. Note, however, that in the SSB case the loss in sensitivity occurs in the averaging, whereas in the DSB case the loss occurs immediately in the correlation process.

For large interferometer arrays the tolerance on the delay errors is set by the long baseline consideration,

and is the same for both DSB and SSB cases. In ALMA the IF band extends from 2 to 4 GHz, i.e, from $\Delta\nu$ to $2\Delta\nu$. For the longest baselines the rate at which increments in the compensating delays are inserted is as high as ~ 200 times per second. Thus the relative sensitivity is given by the mean of the cosine of the phase error. Since interferometers are usually designed so that the loss in sensitivity resulting from delay errors is small, we can use the cosine approximation $\cos(\phi) = 1 - \frac{1}{2}\phi^2$, so the relative loss is half the mean squared phase error (in radians). For continuum observations the sensitivity is determined by the mean squared value of the phase error $\langle\phi^2\rangle$ across the full bandwidth, which, noting that the values of $\Delta\tau$ and ν under consideration are independent of one another, is

$$\begin{aligned}\langle\phi^2\rangle &= \frac{(2\pi)^2}{\Delta\nu} \int_{\nu_{min}}^{\nu_{max}} \int_{-\tau_0}^{\tau_0} p(\Delta\tau)(\nu\Delta\tau)^2 d\nu d\Delta\tau = \frac{(2\pi)^2}{\Delta\nu} \int_{\nu_{min}}^{\nu_{max}} \left[\int_{-\tau_0}^{\tau_0} p(\Delta\tau)\Delta\tau^2 d\Delta\tau \right] \nu^2 d\nu \\ &= (2\pi)^2 \langle\Delta\tau^2\rangle \langle\nu^2\rangle,\end{aligned}\tag{2}$$

where $p(\Delta\tau)$ is the probability distribution of the delay error for an antenna pair as in Fig. 1, ν_{max} and ν_{min} are the upper and lower limits of the IF passband, and $\langle\rangle$ indicates the mean value. For a baseband IF in which the frequency response extends from 0 to $\Delta\nu$, $\langle\nu^2\rangle = \Delta\nu^2/3$, and for a frequency response from $\Delta\nu$ to $2\Delta\nu$, as in ALMA, $\langle\nu^2\rangle = 7\Delta\nu^2/3$. From Eq. (1), $\langle\Delta\tau^2\rangle = \tau_0^2/6$. The corresponding values of relative sensitivity loss are given in Table 1, which includes the IF band 0 to $\Delta\nu$ for comparison.

	0 - $\Delta\nu$		$\Delta\nu - 2\Delta\nu$	
τ_0	ϕ_{rms}	SNR loss	ϕ_{rms}	SNR loss
1/(8 $\Delta\nu$)	10.6°	1.7%	28.1°	11.1%
1/(16 $\Delta\nu$)	5.30°	0.43%	14.03°	3.0%
1/(32 $\Delta\nu$)	2.65°	0.11%	7.02°	0.75%
1/(64 $\Delta\nu$)	1.33°	0.027%	3.51°	0.19%

Table 1. Values of the loss in signal-to-noise ratio (sensitivity) for IF responses extending from 0 to $\Delta\nu$ and $\Delta\nu$ to $2\Delta\nu$. Results are shown for four values of the delay step τ_0 , which it is convenient to express as a fraction of the reciprocal bandwidth. The rms value of the IF band is used, which is appropriate for continuum observations.

	0 - $\Delta\nu$		$\Delta\nu - 2\Delta\nu$	
τ_0	ϕ_{rms}	SNR loss	ϕ_{rms}	SNR loss
1/(8 $\Delta\nu$)	18.4°	5.1%	36.7°	20.6%
1/(16 $\Delta\nu$)	9.19°	1.3%	18.4°	5.1%
1/(32 $\Delta\nu$)	4.59°	0.32%	9.19°	1.3%
1/(64 $\Delta\nu$)	2.30°	0.08%	4.59°	0.32%

Table 2. Values of the loss in signal-to-noise ratio as in Table 1, but using the upper edge of the IF band instead of the rms of the intermediate frequency. This is appropriate for observations using a spectral line (multichannel) correlator, and the values shown correspond to the highest frequency channel, i.e. the worst case.

For spectral line observations the delay-induced phase errors are greatest for the highest frequency spectral channel, for which we can take the frequency as closely equal to the upper edge of the IF response ν_{max} . The mean squared values of $\Delta\tau$ are the same as in the continuum case, but for frequency ν we use the upper

edge of the IF band, i.e. $\Delta\nu$ and $2\Delta\nu$ for the two cases considered. The corresponding values of relative sensitivity are given in Table 2.

Mitigation of delay Errors

Conceptually the most straightforward way of keeping the loss in sensitivity resulting from delay errors within a tolerable limit⁷ (say, $\sim 1\%$) is to use a small enough value for the minimum delay increment. This may not always be easy in systems with wide bandwidths which require correspondingly high sample rates in the digitization. It is also possible to use a larger value for τ_0 and apply corrections to reduce the phase errors. Such corrections can be inserted either before digitization by adjusting the phase of a local oscillator, or, under certain conditions, at the correlator output by adjusting the phase of the measured correlation. A possible scheme (D’Addario 2003) is one in which whenever a delay increment is inserted or removed, a phase jump of magnitude $2\pi\nu_0\tau_0$, and opposite sign to the delay-induced phase jump, is inserted in the corresponding signal through a local oscillator⁸. Here ν_0 is the IF center frequency, for which the phase error is exactly canceled. The overall effect for the full bandwidth can be found by determining the value of $\langle(\nu - \nu_0)^2\rangle$, that is, the mean squared value of frequency measured with respect to the band center:

$$\langle(\nu - \nu_0)^2\rangle = \frac{1}{\Delta\nu} \int_{\nu_{min}}^{\nu_{max}} (\nu - \nu_0)^2 d\nu = \frac{\Delta\nu^2}{12}. \quad (3)$$

This result applies for both the $0 - \Delta\nu$ and $\Delta\nu - 2\Delta\nu$ IF bands. Comparison with $\Delta\nu^2/3$ and $7\Delta\nu^2/3$ for the cases without mitigation shows that the loss in sensitivity is reduced by factors of 4 and 28, respectively. In the case of a DSB system, the phase adjustments must be inserted on the second local oscillator (or a later one) so that, like the delay-induced phase errors, they are applied with the same sign to each sideband component within the IF. Since the phase changes resulting from the changes in the instrumental delay provide a component of the frequency offset used to stop the interferometer fringes, it is also necessary to replace this by inserting a smooth component in the form of a frequency offset, $2\pi\nu_0 d\tau_g/dt$, where τ_g is the geometric delay. The combination of the inserted phase jumps and the frequency offset provide a sawtooth phase component that, at the band center, exactly cancels the phase sawtooth induced by the delay error. For an SSB system the frequency offset can be combined with the usual fringe-stopping offset on the first local oscillator, but for a DSB system it must be on the second (or a later) local oscillator.

Another possibility for reducing the effect of delay errors is to calculate the induced phase error in the cross correlation of each pair of signals and correct the phase at the correlator output. The delay error for each baseline, at any time, can be calculated accurately, since the timing with which delay increments are inserted is highly precise and the interferometer baselines are accurately calibrated. Thus the computed phase error, averaged over each correlator averaging period, can be used to correct the correlation phase. For a spectral correlator, the correction can be applied to each channel in proportion to its center frequency. Note, however, that the correlator output data are only available for correction at intervals determined by the minimum averaging period that is built into the hardware or software. For short baselines the rate of change of delay may be small enough that the resulting phase error varies by only a few degrees during an averaging period. In such cases there will be very little loss in the magnitude of the measured correlation. For longer baselines where several or many delay increments occur during an averaging period the phase error will average down, but the magnitude of the correlation will suffer the loss indicated by the numbers in columns 3 and 5 of Tables 1 and 2. D’Addario (2003) suggests that this correction be applied routinely to all ALMA baselines. If it were possible to apply the phase correction to averaging periods short enough that from one to the next the phase error never changed by more than a few degrees, then this would provide

⁷Various effects in an interferometer system limit the sensitivity. There are some large effects like aperture efficiency and quantization efficiency, and more numerous smaller ones such as phase irregularities in frequency responses, local oscillator noise, timing errors, delay errors, etc. The combined effect of the smaller losses can become serious, so for each one it is reasonable to aim at a fairly stringent limit like the 1% figure suggested here.

⁸This method of mitigation of the delay errors was considered but not implemented during the early development of the VLA. The original idea is attributed to B. G. Clark.

another scheme for mitigation of delay errors. Note, however, that this would not work for a DSB system unless sideband separation is used, since for DSB the sensitivity loss occurs in the correlation, not in the averaging.

ALMA

In the early design of ALMA the minimum delay increment was $1/8$ of the Nyquist sample interval ($1/16\Delta\nu$), resulting in a 3% sensitivity loss (Table 1), or 5.1% for the highest frequency channel (Table 2). These losses are considered to be too high. Mitigation of the delay errors by removing the phase steps at the IF center frequency could be used, but the control of the direct digital synthesizer (DDS) would need to be modified to allow the introduction of the phase offsets at small enough time intervals. An alternative, which is preferred (Napier 2006), is to enable a minimum delay increment of $1/16$ of the Nyquist sample interval ($1/32\Delta\nu$). This solution requires less additional software than inserting phase steps in a local oscillator, but requires modification of a restriction on the number of increments that can be inserted in a 48 ms interval. The sensitivity loss for continuum observations then becomes 0.75% (Table 1), and for spectral line observations is 1.3% for the highest frequency channel. For the lowest frequency channel it is $1.3\%/4 = 0.33\%$ ⁹. Thus the general guideline of 1% loss can reasonably be judged to have been met. Another detail of the ALMA design is that coarse delay increments that are multiples of the sample interval can only be inserted or removed on 1 ms timing boundaries. This also applies to the finer delay steps, but it is proposed (Napier, 2006) that the insertion of the coarse and fine steps be synchronized so that in the case where the stepping of the fine delay resets and the coarse delay is adjusted by one sample interval, the timing of these two events will be simultaneous (i.e. be applied to the same sample). If this synchronization is not achieved a delay error of approximately one sample interval will result.

References

- D’Addario, L. R., Notes on Delay-Tracking for ALMA: Resolution and Tolerance, Memorandum, Feb. 2, 2003, <http://edm.alma.cl/forums/alma/dispatch.cgi/GenRepsDocs/showFile/100340/d20030808221610/No/delayTolerance.pdf>.
- Coe, J. R., NRAO Interferometer Electronics, Proc. IEEE, 61,1335-1339, 1973.
- Napier, P. J., Possible Changes to ALMA Fine-Delay and Phase Update Rates, Memorandum, Aug. 7, 2006, <http://edm.alma.cl/forums/alma/dispatch.cgi/f.testdatates/showFile/100086/d20060823140033/No/Delay-PhaseUpdateRateRevC.doc>
- Thompson, A. R., Fine Adjustment of the MMA Delay System, ALMA Memo. 255, March 12, 1999.
- Thompson, A. R. and D’Addario, L. R., Relative Sensitivity of Double- and Single-Sideband Systems for Both Total-Power and Interferometry, ALMA Memo. 304, April 22, 2000.
- Thompson, A. R., Moran, J. M., and Swenson, G. W., Interferometry and Synthesis in Radio Astronomy, Wiley, NY, 1986 and 2001; Wiley-VCH, 2004.
- Woody, D., The interferometer Equation II, Memorandum, July 7, 2003.

⁹The mean squared phase error as a function of frequency is $\langle\phi^2\rangle = (2\pi)^2(\Delta\tau^2)\nu^2$. Thus the loss in sensitivity is proportional to ν^2 .