NATIONAL RADIO ASTRONOMY OBSERVATORY
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COMPUTER ORIENTED DESCRIPTION OF SOME PROBLEMS OF SPHERICAL ASTRONOMY

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INTRODUCTION

As long as the position accuracy of radio astronomical observations is of the order of minutes-of-arc, all spherical astronomy involved in the data reduction is simple. All the convenient methods, formulae, and appropriate constants can be obtained from most textbooks.

However, as soon as one enters the seconds-of-arc region (as is the case with a large interferometer), many of the "little effects" have to be taken into account. For example, when one compares observed positions with calculated positions, it is not sufficient to include only general precession in the predictions. Nutation is of the order of 17 sec. arc. Aberration is of the order of 20 sec. arc; and the difference between mean and apparent sidereal time is of the order of 1^s of time.

In principle, it is well known how to take these terms into account. Appropriate constants needed to evaluate each term are published for each day in the American Ephemeris, and most of them are available on punched cards or magnetic tape. All these data could be introduced into a computer program for each observation period. This might be sufficient for present day observations.

If a very large array of antennas is observing, an enormous amount of data reduction will have to be done. One certainly wishes to have all the elementary, straight-forward parts of data reduction done completely automatically. As far as spherical astronomy is concerned, it would be very inconvenient to feed-in the Ephemeris data for each day of observation. The process represents a possible source of errors that can be avoided by generating the terms in the computer. It is the purpose of this report to present a convenient, systematic way to generate such data.

All spherical astronomy terms are defined by certain constants and by the time that has elapsed since a certain standard epoch. In a computer program, all constants could be stored in the same way as other mathematical constants are stored. The time interval since the standard epoch can be computed, and therefore, the terms themselves can be computed. However, internal data generated in this manner involves unreasonable amounts of computing time. Instead, one may choose to store the values of the terms for an epoch close to the period of observation (e.g., for the beginning of the current year) so that they assume the role of constants for the whole current year. All formulae then assume a very simple form. If these formulae are available in the form of subroutines, they can use a common set of constants that need to be changed only once a year. Everything else can then be computed internally, and the accuracy of the computations will be clearly defined. Part II of this report deals with these subjects. I have tried to make certain approximations for each term rather than to compute it with full accuracy. Therefore, a limiting accuracy had to be chosen. All effects equal to or greater than 0.2 sec. arc have been included in the present version of the interferometer fringe reduction program. I have chosen an accuracy of about 0.05 sec. arc for such basic variables as nutation, and hence, the Besselian Day Numbers. In the declination range, $-76^{\circ} < \delta < +76^{\circ}$, the errors in the apparent positions (the latter being obtained from mean positions) would not exceed about 0.3 sec. arc with this assumed accuracy of the Besselian Day Numbers. Nutation is presented in a series with monotonically decreasing absolute values of the coefficients. The maximum error that can be made by truncating these series after any term is also given. Each user can select his own accuracy range and therefore reduce his computing expense considerably.

Part I of the report deals with those little famous problems as, for example, "If the calendar date and the local sidereal time on the observer's meridian is given, what is the calendar date and mean time at Greenwich?" In answering questions like this, it is very easy to make errors of the order of almost four minutes of time and multiples of four minutes, and of the order of one day and multiples of one day. In many applications such errors are not very important, but they could be very important for special applications. All these problems can be handled by the simple, accurate and straightforward formulae derived and presented in Part I.

I have not included many of those subjects for which convenient approximate or rigorous formulae are published in textbooks or in the Ephemeris (e.g., the transformation of mean places from one mean equinox to another mean equinox). This report is limited to those problems where the description given in textbooks or other sources is not directly suitable for programming.

PART I

RELATIONS BETWEEN CALENDAR DATE, MEAN TIME, SIDEREAL TIME AND LONGITUDE

1. PERIODIC VARIABLES FOR TIME MEASURING. DEFINITION AND PROPERTIES OF THE "REDUCTION OPERATOR R".

Time on a day can be measured by:

UT = Universal Time
ET = Ephemeris Time*
GMST = Greenwich Mean Sidereal Time
GAST = Greenwich Apparent Sidereal Time
ZT = Zone Time (e.g., Eastern Standard Time)
LMT = Local Mean Solar Time
LMST = Local Mean Sidereal Time
LAST = Local Apparent Sidereal Time

By definition, the numerical values, x_i, of all of these time measures are restricted to the standard interval:

$$0^h \leq x_i < 24^h \tag{1}$$

Instead, throughout this report I will use dimensionless variables, t_i, obtained from x_i by division with the corresponding length of day:

$$0 \leq t_i < 1 \tag{2}$$

The following table shows the time equivalent (in seconds) of a certain error in t:

Error in t	Seconds of Time
10^{-8}	$0^s.001$
10^{-7}	0.009
10^{-6}	0.086
10^{-5}	0.864
10^{-4}	8.640
10^{-3}	$86.400 = 1^m\ 26^s.4$
10^{-2}	$864.000 = 14^m\ 24^s$

*The difference between ET and UT is so small that it has no influence on the various subjects discussed in this report.

Since in most applications one has to deal with UT, GMST, ZT, and LMST, a single letter notation is used:

$$\begin{aligned} \text{UT:} &\quad 0 \le U < 1 \\ \text{GMST:} &\quad 0 \le G < 1 \\ \text{ZT:} &\quad 0 \le Z < 1 \\ \text{LMST:} &\quad 0 \le S < 1 \end{aligned} \tag{3}$$

Occasionally,

$$\text{LMT:} \quad 0 \le M < 1 \tag{4}$$

will be used.

In addition to the variables above, two other variables are important:

L = Geographic Longitude of the observer

L_z = Geographic Longitude of the Zone Time Meridian

$$-\tfrac{1}{2} < L,\ L_z \le +\tfrac{1}{2} \tag{5}$$

One basic task in spherical astronomy is to compute a time variable from other time variables. Normally, the direct result will not be within the standard interval. I will denote this by giving the direct result an index on the upper right side. For example:

$$t_i^r = f(t_1,\ t_2,\ \ldots) \tag{6}$$

means that in order to obtain the variable t_i one has to reduce t_i^r to the standard interval $0 \le t_i < 1$.

Obviously, expressions may also occur where the variables on the right hand side of the equation are not in the standard interval:

$$t_i^r = g(t_1,\ t_2^r,\ t_3^a,\ t_4,\ \ldots) \tag{7}$$

In a computer program as well as in hand computations, the reduction to the standard interval is easily made by decisions like "if $t_i^r \ge 1$, subtract 1 from t_i^r", etc. I will describe these decisions with an Operator $\underline{R}$ (reduction operator) in the following way:

The result of any computation $t_i^r = f(t_1, t_2, ..)$

is a real number:

$$-\infty < t_i^r < +\infty \tag{8}$$

which always can be written in the form:

$$t_i^r = t_i + N(t_i^r), \tag{9}$$

with: $0 \leq t_i < 1$ (10)

and: $N(t_i^r) = 0, \pm 1, \pm 2, ...$ (11)

(t_i, $N(t_i^r)$ being one and only one pair of numbers belonging to t_i^r).

The operator $\underline{R}$ is defined by:

$$\underline{R}.t_i^r = t_i \quad . \tag{12}$$

Application of the operator means that not only t_i but also $N(t_i^r)$ is computed and available.

In a computer program, $\underline{R}$ is realized in form of a subroutine which makes use of such FORTRAN statements as "AMOD" and "INT". The operator $\underline{R}$ is also very helpful in deriving formulae. Therefore, some of its properties have to be discussed although they are all very trivial.

Obviously:

$$\underline{R}.M = 0, \; M = 0, \pm 1, \pm 2, ..., \tag{13}$$

$$\underline{R}.t_i = t_i, \; 0 \leq t_i < 1 \; . \tag{14}$$

The equation (9) can be written and used in either of the following versions:

$$t_i^r = t_i + N(t_i^r) = \underline{R}.t_i^r + N(t_i^r) \tag{15}$$

$$t_i = t_i^r - N(t_i^r) \tag{16}$$

$$N(t_i^r) = t_i^r - t_i = t_i^r - \underline{R}.t_i^r \tag{17}$$

Let us consider the case:

$$t^r = \sum_{i=1}^{n} t_i^r + M, \qquad M=0,\pm 1,\pm 2,\ldots \tag{18}$$

From (15):

$$t + N(t^r) = \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} N(t_i^r) + M \tag{19}$$

$$t = \sum_{i=1}^{n} t_i - N_A \tag{20}$$

or:

$$\boxed{\underline{R}.\left(\sum_{i-1}^{n} t_i^r + M\right) = \sum_{i=1}^{n} (\underline{R}.t_i^r) - N_A} \tag{21}$$

where:

$$N_A = N(t^r) - \sum_{i=1}^{n} N(t_i^r) - M$$

$$N_A = k \text{ if } k \leq \sum_{i=1}^{n} t_i < k+1; \quad k=0,1,2,\ldots n-1 \tag{22}$$

$$\text{i.e., } N_A = \text{largest integer in } \sum_{i=1}^{n} t_i$$

In other words:

Integer terms disappear during reduction and may be omitted before reduction. Reduction of a sum of variables is not necessarily equal to the sum of reductions. (23)

In a similar way one obtains:

For $p > 0$:

$$\underline{R}.(pt^r) = p.(\underline{R}.t^r) + \underline{R}.(pN(t^r)) - N_p \tag{24}$$

N_p = largest integer in the sum of the first two terms on the right hand side ($N_p = 0,1,2,\ldots$)

Frequently, there will be more than one expression for the same variable. Therefore, one has to use different reduction indices for each equation. For example:

$$\begin{aligned} t_i^r &= f(t_1,t_2) \\ t_i^a &= g(t_3,t_4) \end{aligned} \tag{25}$$

In this case, $N(t_i^r)$ must not necessarily be equal to $N(t_i^a)$

On the other hand, if

$$t_1^r = t_2 + t_3 \tag{26}$$

one obtains the conversions

$$t_2^r = t_1 - t_3 \tag{27}$$

and

$$t_3^r = t_1 - t_2 \tag{28}$$

In this case:

$$\begin{aligned} N(t_1^r) &= -N(t_2^r) \\ N(t_2^r) &= -N(t_3^r) \end{aligned} \tag{30}$$

2. CALENDAR DATE AND JULIAN DATE

A Calendar date, DAT, can be represented by a six digit number:

$$DAT = yymmdd \qquad (1)$$

If DAT is changed by adding or subtracting a number of n days, I will denote this by:

$$DAT_1 = DAT \pm n \; (n=0,\pm 1,\pm 2,\ldots) \qquad (2)$$

or:

$$DAT_1 = DAT + (DAT_3 - DAT_2) \qquad (3)$$

It has to be understood with this sort of operation that the variable number of days in a month and in a year is handled properly. In a computer, a corresponding subroutine could be used.

Usually, two different dates are needed:

$$\boxed{\begin{array}{l} GDAT = \text{Calendar Date on the Greenwich Meridian} \\ ZDAT = \text{Calendar Date on the Zone Time Meridian} \end{array}} \qquad \begin{array}{r} (4) \\ (5) \end{array}$$

A certain instant of time is defined by a pair of numbers such as ZDAT,Z or GDAT,U. Because of the ambiguity of sidereal time (see Section 7), however, it would not be defined in a unique way by pairs such as ZDAT,S or GDAT,G. Throughout this report, any given instant of absolute time is defined by ZDAT,Z or GDAT,U; and it is understood that both pairs describe the same instant of time. In other words, if ZDAT,Z is given, then GDAT,U can be computed and vice versa. Or if <u>x is a variable depending on absolute time</u>, then:

$$\underline{x(ZDAT,Z) = x(GDAT,U)} \qquad (6)$$

If more than one time instant is involved:

$$x(ZDAT_i,Z) = x(GDAT_i,U)$$

but not:

$$x(ZDAT_i,Z) = x(GDAT_k,U)$$, and so on.

Frequently, the value of such a variable for 0^h ZT or 0^hUT has a special interest. The second argument can then be omitted:

$$x_o = x(\mathrm{GDAT},0) = x(\mathrm{GDAT}) \tag{7}$$

$$x_1 = x(\mathrm{ZDAT},0) = x(\mathrm{ZDAT})$$

In this case, $x_o \neq x_1$ because GDAT,0 and ZDAT,0 define different instants of absolute time.

Instead of, or in addition to the above time definition, the so-called Julian Date is in use. T e (JD) defines a certain instant of absolute time. It is, by definition, a date assigned to the Greenwich Meridian, and therefore directly connected with GDAT,U. The Julian Day begins with GDAT, 0.5 = GDAT, 12^h UT. We can write:

$$\boxed{\mathrm{JD} = \mathrm{JD}(\mathrm{GDAT},\mathrm{U})} \tag{8}$$

Of particular interest is:

$$\boxed{\mathrm{JD}_o = \mathrm{JD}(\mathrm{GDAT},0) = \mathrm{JD}(\mathrm{GDAT})} \tag{9}$$

With this notation:

$$\mathrm{JD} = \mathrm{JD}_o + \mathrm{U} \tag{10}$$

JD_o can be found in the <u>American Ephemeris</u>, Table I, where the Julian Day number:

$$\underline{\mathrm{JDN}(\mathrm{GDAT}) = \mathrm{JD}(\mathrm{GDAT},\ 0.5)} \tag{11}$$

is given for the zeroth day of each month. Therefore, knowing JDN,

$$\boxed{\mathrm{JD}_o(\mathrm{GDAT}) = \mathrm{JDN}(\mathrm{GDAT}) - 0.5} \tag{12}$$

Example:

GDAT = 651228 (1965, Dec. 28):

Table I in the <u>American Ephemeris</u>:

$$\begin{array}{rl} \mathrm{JDN}(651200) = & 243\ 9095.0 \\ + & \underline{\qquad\ \ 27.5} \\ \mathrm{JD}_o(651228) = & 243\ 9122.5 \end{array}$$

For computer applications, Dr. Barry Clark has developed a straight forward formula for computing JDN(GDAT).

Using JD for different dates, the difference in days is easily found by $JD_1 - JD_2$, this being an ordinary arithmetic subtraction.

3. NUMERICAL RELATIONS BETWEEN MEAN SOLAR TIME AND MEAN SIDEREAL TIME

In all time conversion problems, the following constants are involved: (k_1 to k_4).

$$\text{Ratio} \quad \frac{\text{Mean Sidereal Day}}{\text{Mean Solar Day}} = \boxed{k_1 = 0.99726\ 95664\ 14 \ -0.00000\ 00000\ 586 \times T} \tag{1}$$

$$\text{Ratio} \quad \frac{\text{Mean Solar Day}}{\text{Mean Sidereal Day}} = \boxed{k_2 = 1.00273\ 79092\ 65 \ +0.00000\ 00000\ 589 \times T} \tag{2}$$

Where: T = number of Julian centuries elapsed since 1900 JAN 0.5 (see equ. 5/2). (3)

1 Julian century = 36525 Mean Solar Days.

Neglecting the small variations of the constants, we can obtain the following constants and relations:

$$\text{Increase of mean sidereal time per mean solar day} = \boxed{k_3 = 0.00273\ 79093 \ (= 3^m\ 56^s\!.55536)} \tag{4}$$

$$\text{Decrease of mean solar time per mean sidereal day} = \boxed{k_4 = 0.00273\ 04336 \ (= 3^m\ 55^s\!.90946)} \tag{5}$$

$$k_1 \cdot k_2 = 1 \tag{6}$$

$$k_3 = k_2 - 1 \tag{7}$$

$$k_4 = 1 - k_1 \tag{8}$$

$$k_4 = k_1 \cdot k_3 \tag{9}$$

$$k_3 = k_2 \cdot k_4 \tag{10}$$

If ΔS and ΔM are the mean sidereal time interval and the mean solar time interval, respectively, both equivalent to the same absolute time interval:

$$\boxed{\Delta S = k_2 . \Delta M, \qquad \Delta M = k_1 . \Delta S} \tag{11}$$

4. RELATIONS BETWEEN DATE, TIME AND POSITION

The basic definitions and relations are compiled in this section. Each special problem involving longitude, date and mean time can be solved with these relations.

Definitions:

L = Geographic longitude of observer, $\boxed{-\tfrac{1}{2} < L \leq +\tfrac{1}{2}}$ (1)

L_z = Geographic longitude of Zone Time Meridian, $\boxed{-\tfrac{1}{2} < L_z \leq +\tfrac{1}{2}}$ (2)

Observer:

M = LMT (Local Mean Solar Time) (3)

Z = ZT (Zone Time) (4)

ZDAT = Calendar date on the Zone Time Meridian (5)

S = LMST (Local Mean Sidereal Time) (6)

Greenwich:

U = UT (Universal Time) (7)

G = GMST (Greenwich Mean Sidereal Time) (8)

GDAT = Calendar Date in Greenwich (9)

$$\boxed{0 \leq M, Z, S, U, G, < 1} \qquad (10)$$

ZDAT,M,Z, and GDAT,U define the same instant of absolute time.

$G_o = G_o$(GDAT) = G(GDAT,0) = GMST for 0^h UT on GDAT (11)

$S_o = S_o$(ZDAT) = S(ZDAT,0) = LMST for 0^h ZT on ZDAT (12)

$$\boxed{0 \leq G_o, S_o < 1} \qquad (13)$$

$\Sigma = \underline{R}.(S-S_o)$ = Interval of LMST elapsed since 0^h ZT (14)

$\Gamma = \underline{R}.(G-G_o)$ = Interval of GMST elapsed since 0^h UT (15)

$$\boxed{0 \leq \Sigma, \Gamma < 1} \qquad (16)$$

In the definition of Σ and Γ, the ambiguity of sidereal time is ignored, so that if S is ambiguous the solution 1 is assumed. (See Section 7)

Using the definition of the $\underline{R}$ operator (Section 1), the following relations hold:

Relations:

$$U^a = Z + L_z \qquad Z^a = U - L_z \qquad N(U^a) = -N(Z^a) \tag{17}$$

$$G^a = S + L \qquad S^a = G - L \qquad N(G^a) = -N(S^a) \tag{18}$$

$$U^b = M + L \qquad M^b = U - L \qquad N(U^b) = -N(M^b) \tag{19}$$

$$U = k_1 . \Gamma = k_1 . \underline{R} . (G - G_o) \tag{20}$$

$$Z = k_1 . \Sigma = k_1 . \underline{R} . (S - S_o) \tag{21}$$

In these equations the solution 1 is assumed in an ambiguous case. (See section 7)

$$G^b = G_o + k_2 . U , \quad G = \underline{R} . G^b \tag{22}$$

$$S^b = S_o + k_2 . Z , \quad S = \underline{R} . S^b \tag{23}$$

These equations are never ambiguous.

If GDAT,U is given: $Z = \underline{R}.Z^a = \underline{R}.(U-L_z)$

$$ZDAT = GDAT + N(Z^a) \tag{24}$$

If ZDAT,Z is given: $U = \underline{R}.U^a = \underline{R}.(Z+L_z)$

$$GDAT = ZDAT + N(U^a) \tag{25}$$

It is important to know G_o or S_o. Formulae will be developed in the next two sections.

5. COMPUTATION OF G_o = GMST FOR 0^h UT

Given: GDAT = Calendar Date in Greenwich

Wanted: $G_o = G(GDAT,0)$ = GMST for 0^h UT.

For hand computations, G_o is found in the American Ephemeris for each day of the current year. (Table: Universal and Sidereal Times). These values are defined by and computed from the following equation:

$$G_o^r = c_1 + c_2.T + c_3.T^2 \tag{1}$$

T is the number of Julian centuries elapsed on GDAT, 0^h UT since the standard epoch 1900 JAN 0.5.

Let $JD_o(GDAT)$ be the Julian Date for GDAT, 0^h UT; then T is found from:

$$T = \frac{JD_o(GDAT) - 2415020.0}{100.JY} \tag{2}$$

$$JY = 365.25 \text{ Mean solar days} \tag{3}$$

$$= \text{Length of the Julian year}$$

The constants in equ. (1) have the following values:

$$\begin{aligned} c_1 &= 0.27691\ 93981\ 45\ (=6^h\ 38^m\ 45^s.836) \\ c_2 &= 100.00213\ 59027\ 77\ (=8640184^s.542) \\ c_3 &= 0.00000\ 10752\ 31\ (= \quad 0^s.0929) \end{aligned} \tag{4}$$

Of course, one could use the equ. (1) in a computer and have a completely accurate representation of the G_o values as published in the Ephemeris. However, this is inconvenient for two reasons: 1. Since $T \approx 0.7$ for the present observation period, the second order term would have to be included if the accuracy has to be $0^s.05$ or better. 2. c_2 involves 100 "revolutions", and, therefore, considerable effort in reduction of G_o to the standard interval.

This inconvenience can be easily avoided by choosing another epoch, T_A, instead of 1900 JAN 0.5 and by using a time variable other than T.

Most reasonable definition of T_A:

$$\boxed{T_A = \text{corresponding to the beginning of the Besselian Year next to observation period.}} \tag{5}$$

With:

$$\boxed{G_A = G_o(T_A)} \tag{6}$$

equ. (1) becomes:

$$G_o^r(GDAT) = G_A + c_2.(T - T_A) \tag{7}$$

During the period 1967-1977, the neglected second order term will always be:

$$c_3.(T^2 - T_A^2) \le 0\overset{s}{.}0007 \tag{8}$$

The astronomical definition of c_2 is:

c_2 = increase of the mean longitude of the sun per Julian century, at 1900 JAN 0.5. (9)

The daily increase of the mean longitude is, therefore:

$$\Delta L = \frac{c_2}{100.JY} \quad \text{(at 1900 JAN 0.5)} \tag{10}$$

On the other hand, the time interval during which the sun's mean longitude increases by exactly 360° (=1, in the units used in this report), is:

$$TY = \text{length of the tropical year}$$
$$= 365.24219\ 879 - 0.00000\ 614 \times T \tag{11}$$

$$\boxed{TY_o = TY(1900.0) = 365.24219879} \tag{12}$$

From equ. (10):

$$c_2 = 100.\frac{JY}{TY_o} \tag{13}$$

The term $c_2(T-T_A)$ in equ.(7) can be written in the form:

$$c_2(T-T_A) = c_2 \frac{JD_o - JDA}{100.JY} \tag{14}$$

with:

$$\boxed{JDA = JD(\text{epoch } T_A)} \tag{15}$$

If one defines the new variable:

$$\boxed{\tau(GDAT) = \frac{JD_o(GDAT) - JDA}{TY_o}} \tag{16}$$

one obtains $G_o(GDAT)$ from:

$$\boxed{G_o^r(GDAT) = G_A + \tau(GDAT)} \tag{17}$$

<u>Remarks on JDA and G_A:</u>

a. If JDA is chosen to be the beginning of the Besselian Year next to the observation period, then τ is the same variable which is used for many other purposes, for example, in the computation of apparent places.
The constant G_A has to be taken from the <u>American Ephemeris</u> for each current year.
The accuracy of equ. (17) will be better than $0^s.001$ for the period 1967-1977.

b. Of course, one has the choice of using a constant epoch JDA throughout all applications.
For instance, if:

$$\boxed{JDA = 243\ 9491.541 = 1967\ \text{JAN}\ 1.041} \tag{18}$$

which is the beginning of the Besselian Year, 1967.0, one obtains:

$$\boxed{G_A = 0.27777\ 87126} \tag{19}$$

In this case, τ increases during 1967-1977 from about 0 to about 10. The error in equ. (17) will increase from 0 to about $0^s.013$ during this period.

c. The constant G_A, which is generally <u>not</u> a GMST for 0^h UT, must be taken from the <u>American Ephemeris</u> in the following way.

Let GDAT,U be the time instant equivalent to JDA. In the Ephemeris:

$$G_o(\mathrm{GDAT}) \text{ in h,m,s.} \tag{20}$$

Convert this to a fraction of a day. Also, express U in the fraction of a day. Then:

$$G_A^r = G_o(\mathrm{GDAT}) + k_3.U \tag{21}$$

<u>Daily Increase of τ and G_o:</u>

$$\Delta G_o(1^d) = \Delta\tau(1^d) = \frac{1}{TY_o} = 0.00273\ 79093 = k_3 \tag{22}$$

This result is already known from Section 3. We could have derived equ. (17) in a much shorter way, starting with the definition of k_3. However, since the constants k_1 to k_4 are derived from the analysis of the mean longitude of the sun rather than being "original" constants, the above derivation of equ. (17) seems to be more appropriate. Another trivial relation follows from equ. (22):

$$\boxed{\begin{aligned} \text{If:}\quad \mathrm{GDAT} &= \mathrm{GDAT}_o + n, \quad n=0,\pm1,\pm2,\ldots, \\ \tau_o &= \tau(\mathrm{GDAT}_o), \\ \text{then:}\ G_o^r(\mathrm{GDAT}) &= G_o(\mathrm{GDAT}_o) + nk_3 \\ \tau(\mathrm{GDAT}) &= \tau_o + nk_3 \end{aligned}} \tag{23}$$

As far as the computation of τ itself is concerned, it is more convenient to use k_3 as the basic constant than to use TY_o:

$$\boxed{\tau = k_3.(JD_o - JDA)} \tag{24}$$

6. COMPUTATION OF LMST FOR GIVEN ZT

Given: ZDAT, Z, L, L_z

Wanted: S = LMST(ZDAT,Z)

The usual way of solving this "problem", as explained in the *American Ephemeris*, consists of the following steps:*

a. Find: GDAT and U corresponding to ZDAT and Z

b. Find: G_o(GDAT) from the *American Ephemeris*

c. Compute: $G = \underline{R}.(G_o + k_2U)$

d. Compute: $S = \underline{R}.(G - L)$

Instead, a straight forward formula shall be derived. The formula includes a minimum amount of numerical and logical operations and does not explicitly use the Greenwich Meridian.

The following equations are available:

Equ.	4/18	$S^a = G - L$	(1)
"	4/22	$G^b = G_o + k_2U$	(2)
"	4/17	$U^a = Z + L_z$	(3)
"	5/17	$G_o^r = G_A + \tau(\text{GDAT})$	(4)
"	4/25	$\text{GDAT} = \text{ZDAT} + N(U^a)$	(5)
"	5/23	$\tau(\text{GDAT}) = \tau(\text{ZDAT}) + k_3.N(U^a)$	(6)

Making use of equ. (1/15), one obtains:

$$S^a = G_A + \tau(\text{ZDAT}) + k_3.N(U^a) + k_2.(Z+L_z) - k_2.N(U^a) - L - N(G^b) - N(G_o^r)$$

Remembering that $k_3 - k_2 = -1$ and that all integer terms can be neglected, it follows that:

$$S^a = G_A + k_2.L_z - L + \tau(\text{ZDAT}) + k_2.Z \qquad (7)$$

* In our notation.

Introducing the new constants:

$$\boxed{\begin{array}{l} \Delta = L - L_z \quad , \\ G_B = \underline{R}.(G_A + k_3.L_z - \Delta) \end{array}} \qquad \begin{array}{r}(8)\\(9)\end{array}$$

the solution can be written in the following form:

Given:	ZDAT, Z, G_B, JDA	(10)
Compute:	JD_o(ZDAT)	
	$\tau = k_3.(JD_o - JDA)$	
	$S = \underline{R}.(G_B + \tau + k_2.Z)$	

Frequently, the following case will appear:

Given:	ZDAT, Z, τ_o(DAT), n = ZDAT - DAT	(11)
	In other words, for a certain date, DAT, the value of τ has already been computed and is still available.	
Compute:	$S = \underline{R}.(G_B + \tau_o + k_3.n + k_2.Z)$	

Special values of G_B:

If we take the longitude of Green Bank, as published in the American Ephemeris, and the longitude of the Eastern Standard Time Meridian, we have:

$$L = 5^h 19^m 20^s\!.7 \quad , \quad L_z = 5^h \tag{12}$$

and in our notation:

$$L_z = +0.20833\ldots, \quad k_3.L_z = +0.000570397762$$
$$\Delta = +0.01343402777 \tag{13}$$

Therefore:

$$\boxed{\begin{array}{l} G_B = \underline{R}.(G_A - 0.01286\ 36300) \\ \text{for } L = +5^h 19^m 20^s\!.7, \quad L_z = +5^h \end{array}} \tag{14}$$

If the origin of τ is the epoch 1967.0, G_A obtains the value of equ. (5/19):

$$\boxed{\begin{array}{l} G_B = 0.26491\ 50826 \\ \text{for } L = +5^h 19^m 20\overset{s}{.}7, \quad L_z = +5^h \\ JDA = 243\ 9491.541 = 1967.0 \end{array}} \tag{15}$$

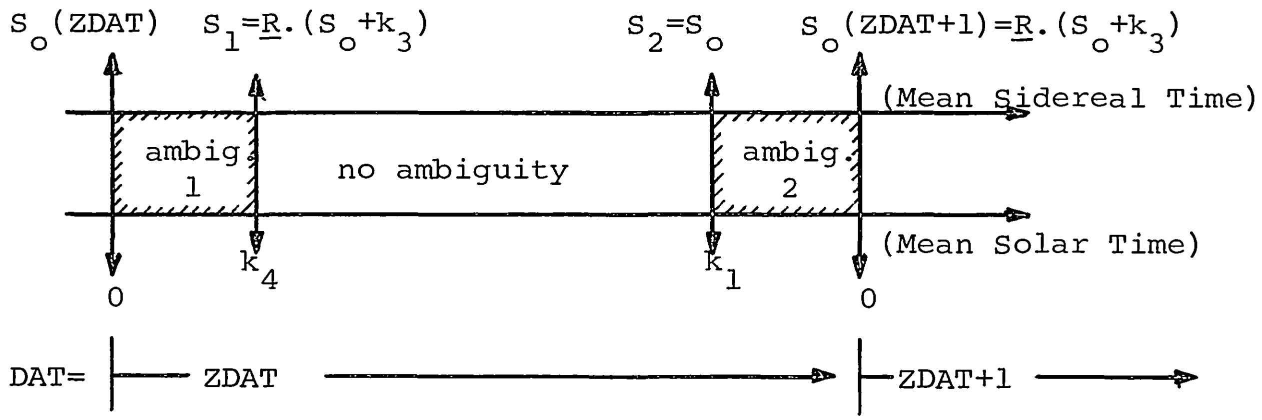
<u>LMST for 0^h ZT:</u>

This is obtained by putting Z=0 in equ. (10):

$$\boxed{S_o(ZDAT) = \underline{R}.(G_B + \tau\ (ZDAT))} \tag{16}$$

7. THE AMBIGUITY OF SIDEREAL TIME WITH RESPECT TO CIVIL TIME

Since the mean sidereal day is shorter than the mean solar day, a certain range of values of mean sidereal time will occur twice a day. The value S_o (LMST for 0^h ZT) is defined by equ. (6/16). Its daily increase is given by k_3, and the corresponding interval of mean solar time is k_4. The conditions are illustrated in the following figure:



In the interval, $k_4 \leq Z < k_1$, each value of S occurs only once. However, the S values in the interval, $0 \leq Z < k_4$ occur a second time on the same day in the interval $k_1 \leq Z < 1$.
The ambiguity can be described in terms of $\sum$, the LMST interval since 0^hZT (equ. (4/14)), as follows:

<u>Given</u>: ZDAT, $0 \leq S < 1$, $0 \leq S_o < 1$. (1)

$$\sum = \underline{R}.(S-S_o)$$

For: $k_3 \leq \sum < 1$ a unique solution Z(S) exists. (2)

For: $0 \leq \sum < k_3$ one obtains $Z_1 = Z(S)$, but there is another solution $Z_2 = Z_1 + k_1$ possible on the same day. Z_1 is the solution near the beginning of the day and Z_2 is the one near the end of the day, and it is: (3)

$$0 \leq Z_1 < k_4 \;, \quad k_1 \leq Z_2 < 1 \tag{4}$$

Solution of an Ambiguous Case

a. Only ZDAT and S are given: No solution

b. If, in addition, a control is given which, for example is 0 in the first part of ZDAT and 1 in the second part, then:

$$Z = Z_1 \text{ if control} = 0$$

$$Z = Z_2 \text{ if control} = 1$$

c. If a table of consecutive time instants is given, each of them consisting of a pair ZDAT, S:

$$\begin{array}{ll} ZDAT_1, & S_1 \\ ZDAT_2, & S_2 \\ \vdots & \\ ZDAT_i, & S_i \\ \vdots & \end{array}$$

(Telescope Tape)

then the ambiguity can be solved, at least in most practical cases. Normally, $\underline{R}_\circ(S_{i+1}-S_i)$ will be larger than k_3. Therefore,

$$\text{If } ZDAT_{i+1} = ZDAT_i: \quad Z(S_i) = Z_1(S_i)$$

$$\text{If } ZDAT_{i+1} = ZDAT_i+1: \quad Z(S_i) = Z_2(S_i)$$

(An example of where the ambiguity cannot be solved would be the case $ZDAT_{i+1} = ZDAT_i+1, \sum_i < k_3$, for all i. But this is only of an academic interest.)

d. No ambiguity would occur if one stores either ZDAT,Z or SDAT,S on the tape, where SDAT = Local Mean Sidereal Date, which is easily related to the Greenwich Sidereal Date.

8. COMPUTATION OF ZT FROM GIVEN LMST

Given: ZDAT,S and information for solving an ambiguity.

Wanted: Z = ZT(ZDAT,S)

With equ. 4/21, 5/23, and 6/16, there are many possibilities to write down the solution. Some such ways are compiled below without further explanations:

Given:	ZDAT, S, S_o	
Compute:	$\sum = \underline{R}.(S-S_o)$	(1)
	$Z = k_1 . \sum$	
	If $0 \le \sum < k_3$: $Z_1 = Z$, $Z_2 = Z_1 + k_1$	

Given:	ZDAT, S, G_B, $\tau = \tau$ (ZDAT)	
Compute:	$\sum = \underline{R}.(S-G_B-\tau)$	(2)
	$Z = k_1 . \sum$	
	If $0 \le \sum < k_3$: $Z_1 = Z$, $Z_2 = Z_1 + k_1$	

Given:	ZDAT, S, G_B, $\tau_o = \tau$(DAT), n = ZDAT - DAT	
Compute:	$\sum = \underline{R}.(S-G_B-\tau_o-n.k_3)$	(3)
	$Z = k_1 . \sum$	
	If $0 \le \sum < k_3$: $Z_1 = Z$, $Z_2 = Z_1 + k_1$	

In all of these methods, the decision about ambiguity could be made with the value of Z itself rather than with the value of $\sum$. For instance, method (1) could be put in the following form:

Given:	ZDAT, S, S_o	
Compute:	$Z = k_1.\underline{R}.(S-S_o)$	(4)
	If $0 \le Z < k_4$: $Z_1 = Z$, $Z_2 = Z_1 + k_1$	

One further remark should be made although it is a rather trivial one. With the abbreviation:

$$t^r = S - S_o \tag{5}$$

one can write the solution in the form of:

$$Z = k_1 . \underline{R} . t^r \quad . \tag{6}$$

The right side of this equation is not necessarily equal to $\underline{R}.(k_1 t^r)$ as shown in Section 1, equ. (24). Under the conditions in equ. (5), we have:

$$-1 < t^r < +1 \quad . \tag{7}$$

It is easy to see that $N_p = 0$ because $k_1 < 1$.
Therefore:

$$Z = k_1 . \underline{R} . t^r = \underline{R}.(k_1 t^r) + \underline{R}.(k_1 N(t^r)) \tag{8}$$

Since $N(t^r)$ can assume the values 0 or -1, we could make an error of the order $1 - k_1 = k_4$ if we reduce Z to the standard interval __after__ the multiplication with k_1 rather than before.

9. COMPUTATION OF UT FROM GIVEN LMST. COMPUTATION OF AN INTERVAL OF UT ELAPSED SINCE A CERTAIN INSTANT $GDAT_o, 0$.

Given: ZDAT, S, and information for solving an ambiguity. Further: $GDAT_o$, $n = ZDAT - GDAT_o$

Wanted: $U = U(ZDAT,S)$, $GDAT = GDAT(ZDAT,S)$

$$\Delta U = (GDAT,U) - (GDAT_o,0) \tag{1}$$

Purpose of computing ΔU: Extrapolating or interpolating an "Ephemeris Variable" (e.g., a Besselian Day Number) whose value and one, two or more derivatives is given for the epoch $GDAT_o, 0$. Let Q_o be such a variable for this epoch, and $\delta(Q_o)$ its first derivative (daily variation). Let us assume that extrapolation over n days could be made linearily. Then:

$$Q(ZDAT,S) = Q_o + \delta(Q_o).\Delta U \tag{2}$$

(It is obvious that ΔU is not restricted to the standard interval.)

In most actual applications, one will explicitly need the Zone Time.

If

$$Z = Z(ZDAT,S) \tag{3}$$

has been computed by one of the formulae in Section 8, we obtain from equ. (4/25):

$$\boxed{\begin{array}{l} U^a = Z+L_z, \quad U = \underline{R}.(Z+L_z) \\ \qquad\qquad GDAT = ZDAT + N(U^a) \end{array}} \tag{4}$$

Furthermore, from equ. (1):

$$\begin{aligned} \Delta U &= GDAT + U - GDAT_o \\ &= ZDAT + N(U^a) + U - GDAT_o \end{aligned}$$

Since:

$$U + N(U^a) = U^a = Z + L_z \text{ and } ZDAT - GDAT_o = n$$

we obtain:

$$\boxed{\Delta U = Z + L_z + n} \quad (5)$$

These results can be compiled as follows:

<u>Given:</u> ZDAT, S, $n = ZDAT - GDAT_o$, and information for solving an ambiguity.

<u>Compute:</u> $Z = Z(ZDAT,S)$ (Section 8)

1. <u>Wanted:</u> GDAT, U and ΔU

   <u>Compute:</u>

$$U^a = Z + L_z$$
$$U = \underline{R}.U^a$$
$$GDAT = ZDAT + N(U^a)$$
$$\Delta U = U^a + n$$

(6)

2. <u>Wanted:</u> ΔU

   <u>Compute:</u> $\Delta U = Z + L_z + n$

In case No. 1, one has to carry out <u>two</u> reductions (for Z and U).
In case No. 2, one has to carry out only the <u>one</u> for Z.

PART II

COMPUTATION OF NUTATION, EQUATION OF THE EQUINOXES, BESSELIAN DAY NUMBERS, AND VELOCITY COMPONENTS OF THE EARTH

10. THE MEAN MOTION OF THE SUN AND THE MOON

Many effects in spherical astronomy are represented in terms of Fourier Series. The arguments in these series are connected with the mean motion of the sun and the moon and are given by the following "basic arguments" or by combinations of them:

Table 1

Basic Arguments

Notation: Here	Notation: Explanatory Supplement of the American Ephemeris	Explanation
A1	L	Geometric mean longitude of the sun, mean equinox of date.
A2	Γ	Mean longitude of perigee, mean equinox of date.
A3	g	Mean anomaly of the sun.
A4	☾	Mean longitude of the moon, measured along ecliptic from mean equinox of date to the ascending node of the moon's orbit, and then along the orbit.
A5	☊	Longitude of the ascending node of the moon's orbit, measured from mean equinox of date.
A6	☾ $-\Gamma'$	Mean anomaly of the moon.
ECC	e	Eccentricity of the earth's orbit.
EPS	ϵ	Obliquity of the ecliptic.

Obviously, the arguments are function of Ephemeris Time. The original equations are published in the Explanatory Supplement, page 98 and page 107. The "T" in these equations is the same as in our equ. (5/2). Following our procedure in section 5, we introduce instead of "T" the variable τ, the time elapsed since the beginning of the Besselian Year next to observation period, expressed in fractions of a tropical year. Over a time interval of one year, all the terms with T^2 and T^3 in the original equations become less than 1". As shown in the following sections, the basic arguments have to be computed with an accuracy of about $0\overset{\circ}{.}01$*. The terms in T^2 and T^3 can therefore be neglected. For the arguments A1, A2, ...A6, one has the following equations:

$$\boxed{A_i = A_{iA} + a_i \cdot \tau} \qquad , \ \tau = k_3 .(JD - JDA) \qquad (1)$$

where the A_{iA} are the values for the beginning of the Besselian Year next to observation epoch. JD is the Julian Date corresponding to the instant of time for which A_i has to be computed. JDA is the Julian Date of the beginning of the Besselian Year. The A_{iA} can be taken almost directly from pages 50-51 of each current volume of the <u>American Ephemeris</u>.

If α_i is the coefficient of T, expressed in degrees per day, one gets the value of a_i in equation (1) from the following relations depending on the units chosen for A_i:

$$A_i \text{ in radians:} \quad a_i = \alpha_i . TY_o . \pi/180 \quad = \quad \alpha_i . \ 6{,}374678936$$

$$A_i \text{ in degrees:} \quad a_i = \alpha_i . TY_o \quad = \quad \alpha_i . 365{,}2421988 \qquad (2)$$

$$A_i \text{ in } 1/360^\circ\text{:} \quad a_i = \alpha_i . TY_o/360 \quad = \quad \alpha_i . \ 1{,}014561663$$

In a computer program, since sin and cos of the arguments have to be computed, one will prefer radians. In FORTRAN sin and cos subroutines, the reduction of angles to standard interval is done implicitly. However, it could happen that one has to use special high speed sin-cos-subroutines which do not include reduction. In that case, the units $1/360^\circ$ could be preferable and the reduction could then be made with the subroutine for the operator <u>R</u> which must be available for other purposes.

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*In the computation of τ, therefore, UT can be used instead of ET.

The next table contains the coefficients in all three units:

Table 2

Coefficients a_i

Argument	Radians	1/360°	Degrees
A1	+ 6.283185307	+ 1.0	+ 360.0
A2	+ 0.0003000459	+ 0.00004775379	+ 0.01719137
A3	+ 6.282885261	+ 0.9999522459	+ 359.9828087
A4	+83.99529741	+13.36826678	+4812.576041
A5	- 0.3375642524	- 0.05372501937	- 19.34100698
A6	+83.28513217	+13.25524047	+4771.886570

The annual variations of ECC and EPS are so small that they can be kept constant over a year - in most applications even over a longer period.

For the beginning of the Besselian Year 1967, the constants A_{iA}, ECC, and EPS are compiled in the next table:

Table 3

Constants $A1_A$, $A2_A$,...,$A6_A$, ECC, EPS (1967.0)

Argument	Radians	1/360°	Degrees
A1	4.8870266	0.7777945	280.0060
A2	4.9283368	0.7843692	282.3729
A3	6.2418751	0.9934253	357.6331
A4	2.7429856	0.4365597	157.1615
A5	0.7560801	0.1203339	43.3202
A6	5.8749004	0.9350195	336.6070
ECC	0.016723	-----------not used-----------	
EPS	0.4091677	0.06512106	23.44358

As stated above, the accuracy of the arguments should be 0°.01. This corresponds to about 0.0002 radians or 0.00003 of the 1/360° units. In the constants above, five significant digits will be good for most applications. In the coefficients, the accuracy depends on the value which τ can reach. If τ is restricted to about ½ (which would correspond to the conventions), then the number of digits after the decimal point, which must be taken into account, is about 4 for radians, 5 for the 1/360 units, and 2 for the degrees. A general rule would be: Use 6 significant digits for all, then the obtained accuracy will be a little bit better tha the accuracy needed.

11. NUTATION

If the accuracy of position measurements becomes better than about one minute of arc, one has to take into account <u>nutation in longitude</u> ($\Delta\psi$) and <u>nutation in obliquity</u> ($\Delta\epsilon$). These variables enter, for example, in the Besselian Day Numbers and in the equation of equinoxes (= difference between apparent and mean sidereal time). Obviously, one can generate both in the computer by:

$$\boxed{\begin{aligned}\Delta\psi &= \sum_n b_n.\sin B_n \\ \Delta\epsilon &= \sum_n c_n.\cos B_n\end{aligned}} \qquad (1)$$

Here, the B_n are linear combinations of the A_i (see last section), and the b_n, c_n are constants (some of them very slowly changing with time; their variations are so small that we can keep them constant).

In order to obtain $\Delta\psi$ and $\Delta\epsilon$ with the same accuracy as in the <u>American Ephemeris</u>, one would have to compute 69 terms for $\Delta\psi$ and 40 terms for $\Delta\epsilon$. Now we will discuss the accuracy which can be reached by a reasonable reduction of the number of terms. The terms and the arguments are published on pages 44-45 of the <u>Explanatory Supplement</u>. We can arrange them in the sequence:

$$\Delta\psi = \sum_i b_i \sin B_i \;,\; \Delta\epsilon = \sum_i c_i \cos C_i$$

$$|b_i| \leq |b_{i-1}| \;,\; |c_i| \leq |c_{i-1}| \text{ for all } i \qquad (2)$$

If we extend the summation to a term i = r, the error can be estimated by:

$$R(\Delta\psi) \le \sum_{r+1}^{69} |b_i| \,,$$
$$R(\Delta\varepsilon) \le \sum_{r+1}^{40} |c_i| \,. \qquad (3)$$

The actual errors will be smaller since many of the neglected terms will have opposite signs and therefore cancel. Since in most applications (Besselian Day Numbers, Equation of Equinoxes), $\Delta\psi$ appears with factors $\sin\varepsilon$ or $\cos\varepsilon$, we should also consider $R.\sin\varepsilon$ and $R.\cos\varepsilon$. The results are shown in the next tables. The term R in a line i always means that R is the upper limit of the error which is made if the terms i=1,2,...i are included and all the remaining terms are neglected. The number n is the current number of the nutation terms in Explanatory Supplement (not printed there). The lines i=0 show the upper limit of the whole $|\Delta\psi|$ or $|\Delta\varepsilon|$.

Table 4

$$\Delta\psi = \sum_i b_i.\sin B_i$$

i	n	R	R.sinε	R.cosε	b_i	B_i
0	-	19".36	7".70	17".76	-	-
1	1	2.13	0.85	1.95	-17".2449	B_1 = A5
2	8	0.85	0.34	0.78	- 1.2730	B_2 = 2.A1
3	2	0.64	0.26	0.59	+ 0.2088	B_3 = 2.A5
4	24	0.44	0.18	0.41	- 0.2037	B_4 = 2.A4
5	9	0.31	0.13	0.29	+ 0.1259	B_5 = A3
6	25	0.25	0.10	0.23	+ 0.0675	B_6 = A6
7	10	0.20	0.08	0.18	- 0.0496	B_7 = 2.A1 + A3
8	26	0.16	0.07	0.15	- 0.0342	B_8 = 2.A4 - A5
9	27	0.14	0.06	0.13	- 0.0261	B_9 = 2.A4 + A6
10	11	0.12	0.05	0.11	+ 0.0214	B_{10} = 2.A1 - A3
11	28	0.10	0.04	0.09	- 0.0149	B_{11} = 2.A1 - 2.A4 + A6
12	12	0.09	0.04	0.08	+ 0.0124	B_{12} = 2.A1 - A5
13	29	0.08	0.03	0.07	+ 0.0114	B_{13} = 2.A4 - A6

Table 5

$$\Delta\varepsilon = \sum_i c_i . \cos C_i$$

i	n	R	c_i	C_i
0	-	10".04	-	-
1	1	0.83	+ 9".2106	B_1
2	8	0.28	+ 0.5520	B_2
3	2	0.19	- 0.0904	B_3
4	24	0.10	+ 0.0884	B_4
5	10	0.08	+ 0.0216	B_7
6	26	0.06	+ 0.0183	B_8
7	27	0.046	+ 0.0113	B_9

All terms whose coefficients have absolute values $\geq$ 0".01 are included in the above compilation. Taking these terms into account means that both $\Delta\psi$ and $\Delta\varepsilon$ are computed with an error smaller than 0".05. For many applications just the first term will be sufficient because it reduces the maximum errors 19", 10" to 2", 0".8, respectively. Considering the number of terms which should be included, one has to be aware of the fact that in computation of Besselian Day Numbers, nutation is multiplied by tgδ or secδ. The errors made in the computation of nutation will increase with these factors (see Section 16).
In practical applications the situation will frequently be the following: On a certain day, many observations are stored on a telescope tape. All these would have to be reduced to mean places. Therefore, one would need nutation for each single time point. Instead of using the above formulae, one would like to extrapolate nutation over a period of one day. This could be done if the first derivatives of $\Delta\psi$ and $\Delta\varepsilon$ were known. For example, let

$\Delta\psi_o$, $\Delta\varepsilon_o$ be the values of $\Delta\psi$ and $\Delta\varepsilon$ for one given time t_o and, $\delta(\Delta\psi)$, $\delta(\Delta\varepsilon)$ their daily variations for the same time. Then, with t being the time interval from t_o in units of a day:

$$\boxed{\begin{aligned}\Delta\psi_t &= \Delta\psi_o + \Delta t.\delta(\Delta\psi)\\ \Delta\varepsilon_t &= \Delta\varepsilon_o + \Delta t.\delta(\Delta\varepsilon)\end{aligned}} \tag{4}$$

The daily variations of $\Delta\psi$ and $\Delta\varepsilon$ can easily be obtained from equ. (1):

$$\begin{aligned}\delta(\Delta\psi) &= \sum_n \beta_n \cos B_n \quad , \quad \beta_n = b_n.\delta(B_n)\\ \delta(\Delta\varepsilon) &= \sum_n \gamma_n \sin B_n \quad , \quad \gamma_n = -c_n.\delta(B_n)\end{aligned} \tag{5}$$

and the $\delta(B_n)$ are linear combinations of the $\delta(A_i)$. The following table shows the $\delta(B_i)$:

Table 6

Daily Variations of the B_i

i	n	B_i	$\delta(B_i)$ (degrees)	$\delta(B_i)$ (rad)
1	1	= A5	- 0.0530	- 0.00092
2	8	2.A1	+ 1.9713	+ 0.03441
3	2	2.A5	- 0.1059	- 0.00185
4	24	2.A4	+26.3528	+ 0.45994
5	9	A3	+ 0.9856	+ 0.01720
6	25	A6	+13.0650	+ 0.22803
7	10	B_2 + A3	+ 2.9569	+ 0.05161
8	26	B_4 - A5	+26.4057	+ 0.46087
9	27	B_4 + A6	+39.4178	+ 0.68797
10	11	B_2 - A3	+ 0.9857	+ 0.01720
11	28	B_2-B_4+A6	-11.3165	- 0.19751
12	12	B_2 - A5	+ 2.0242	+ 0.03533
13	29	B_4 - A6	+13.2878	+ 0.23192

The $\delta(B_i)$ for the remaining terms have also been computed but are not given here. If one arranges the terms in equ. (5) in order of decreasing absolute values of the coefficients β_i, γ_i, the following table is obtained:

Table 7

$$\delta(\Delta\psi) = \sum_i \beta_i . \cos B_i'$$

i	n	R	R.sin ε	R.cos ε	β_i	B_i'
0	-	0."24	0."10	0."22	-	-
1	24	0.15	0."06	0."14	- 0."0937	B_4
2	8	0.10	0.04	0.09	- 0.0438	B_2
3	27	0.09	0.04	0.08	- 0.0180	B_9
4	1	0.07	0.03	0.06	+ 0.0159	B_1
5	26	0.05	0.02	0.05	- 0.0158	B_8
6	25	0.04	0.02	0.04	+ 0.0154	B_6

Table 8

$$\delta(\Delta\varepsilon) = \sum_i \gamma_i . \sin C_i'$$

i	n	R	γ_i	C_i'
0	-	0."10	-	-
1	24	0.06	- 0."0407	B_4
2	8	0.04	- 0.0190	B_2

The lines i=0 show the maximum error which can be made by putting the values $\Delta\psi_o$ and $\Delta\varepsilon_o$ constant over a time interval of one day.

The value of R in Tables 7 and 8 probably gives only a rough estimate of the accuracy since we have neglected the second order terms and higher terms in the Taylor development. For instance, the second order term corresponding to β_1 would be:

$$\tfrac{1}{2}\beta_1 . \delta(B_4) = 0.''021$$

The remaining second order terms in Tables 7 and 8 are all less than $0''.01$. For the sum of all second order terms (the factor ½ being included), we obtain from the Explanatory Supplement:

Sum of second order terms in the
Taylor Development:

$\Delta\psi$: $0''.045$

$\Delta\epsilon$: $0''.017$

In neglecting the second order terms, no error will be made larger than the error made by truncating the series for the first order terms after i=6 (for $\Delta\psi$) and i=2 (for $\Delta\epsilon$).

Accuracy needed for computation of the arguments

The largest coefficient in all of these formulae is the $b_1=17''.2$ in Table 4. If we compute the corresponding argument B_1 with an accuracy of $0^\circ.1$, the accuracy of the term $(b_1.\sin B_1)$ becomes $0''.03$. The next largest coefficient is c_1 in Table 5 - $c_1=9''.2$. In order to obtain $0''.03$ accuracy for the term, the argument B_1 has to be computed with only $0^\circ.2$ precision. For all remaining terms in either formula, the argument must be computed with an accuracy less than 1°. Allowing for the summation of these errors, one should follow the rule that all arguments should be computed with about $0^\circ.01$ in order to always be more accurate than that corresponding to the smallest R in the tables ($0''.03$).

Procedure for computing and extrapolating Nutation

Given: ZDAT, Z_i (i=1,2,3,...)

Wanted: $\Delta\psi$ and $\Delta\epsilon$ for $Z=Z_i$

1. Compute GDAT, U for ZDAT, 0 (section 9)

$$U^a = L_z, \quad U = \underline{R}.U^a, \quad GDAT=ZDAT+N(U^a) \qquad (6)$$

2. Compute

$$JD = JD(GDAT, U)$$
$$\tau = k_3.(JD - JDA) \qquad (7)$$

3. Compute

$A1, A3, \ldots\ldots, A6$ from equ. (10/1):

$$A_i = A_{iA} + a_i \cdot \tau \qquad (8)$$

4. Compute the B_i as functions of the A_i (Table 6 in section 11, column 3). (9)

5. Compute $\sin B_i$ and $\cos B_i$ (10)

6. Compute $\Delta\psi_o = \Delta\psi(\text{GDAT, U})$, $\Delta\epsilon_o = \Delta\epsilon(\text{GDAT, U})$ (11) with the $\sin B_i$ and $\cos B_i$ using the coefficients in Tables 4 and 5.

7. Compute $\delta(\Delta\psi)$ and $\delta(\Delta\epsilon)$ with the $\sin B_i$ and $\cos B_i$ (12) using the coefficients in Tables 7 and 8.

8. Compute for each Z_i:

$$\Delta\psi_i = \Delta\psi_o + Z_i \cdot \delta(\Delta\psi), \Delta\epsilon_i = \Delta\epsilon_o + Z_i \cdot \delta(\epsilon) \qquad (13)$$

One might wish to avoid the computation of the $\Delta\psi_o$, $\Delta\epsilon_o$. In this case, one has to take their values from the <u>American Ephemeris</u>. They are given there for each day, 0^h ET; $\Delta\psi_o$ is the "Nutation in Long." in the ephemeris of the sun; and ϵ_o is "-B", the Besselian Day Number B, with the opposite sign. The small difference between ET and UT can be ignored here since we have limited our accuracy to about 0".03. In this case, only B_1, B_2, B_4, B_6, B_8, and B_9 and their trigonometric functions have to be computed. The time factor in equ. (13) is <u>not</u> Z_i, but the time interval elapsed at ZDAT, Z_i since the time for which the $\Delta\psi_o$, $\Delta\epsilon_o$ are taken from the Ephemeris. The formulae to be used for the computation of this interval are given in Section 9.

12. EQUATION OF THE EQUINOXES

The so-called equation of the equinoxes is given by:

$$\boxed{EEQ = \Delta\psi . \cos EPS} \tag{1}$$

$\Delta\psi$ being the nutation in longitude (section 11). It is, on the other hand, the <u>difference between mean and apparent sidereal time</u>, in the sense:

$$\boxed{LAST = LMST + EEQ} \tag{2}$$

According to Table 4, the upper limit of EEQ is:

$$|EEQ| < 19''.4 \cos EPS = 17''.8 = 1^{s}.18 \tag{3}$$

If apparent right ascension is observed and mean sidereal time (=sidereal clock minus clock error) is given, the apparent hour angle follows from:

$$\begin{aligned} \text{App. Hour Angle} &= \text{LAST} - \text{App. Right Ascension} \qquad (4)\\ &= \text{LMST} + \text{EEQ} - \text{App. Right Ascension} \end{aligned}$$

The EQU has to be taken into account in all applications where the position accuracy is of the order of 17" or better.

13. THE BESSELIAN DAY NUMBERS A, B, AND E (PRECESSION AND NUTATION)

These variables are defined by:

$$A = \tau.n + \sin EPS.\Delta\psi$$
$$B = -\Delta\epsilon \qquad (1)$$
$$E = c.\Delta\psi$$

$\Delta\psi$ and $\Delta\epsilon$ are discussed in section 11 and EPS (the obliquity of the ecliptic) in section 10. τ is the usual fraction of the tropical year. The remaining coefficients are rigorously given by:

$$n = \text{annual general precession in declination} \qquad (2)$$
$$= 20''0468 - 0''0085.T$$

$$c = \text{Ratio } \frac{\text{annual planetary precession on the equator}}{\text{annual luni-solar precession}} \qquad (3)$$
$$= 0.002476 - 0.000373.T$$

Obviously, both n and c can be kept constant over a long time range. For 1967.0:

$$\boxed{\begin{aligned} n &= 20''0411 \\ c &= \;\; 0.00223 \end{aligned}} \qquad (4)$$

The day number E is always less than about 0''04; it enters the computation of apparent right ascension as an additive constant. Therefore, if $\Delta\psi$ is computed with an accuracy of not more than 0''04, E can be neglected for most applications. The daily variations are needed for a linear extrapolation of the day numbers. Observing that $\delta(n)$ and $\delta(EPS)$ are neglegible, one obtains:

$$\delta(A) = n.\delta(\tau) + \sin EPS.\delta(\Delta\psi)$$
$$= n.k_3 + \sin EPS.\delta(\Delta\psi)$$
$$\delta(B) = -\delta(\Delta\epsilon)$$

and, obviously, $\delta(E) = 0$. In all of these formulae, sin EPS can be kept constant (see section 10). Using its value for 1967.0 (Table 3), we obtain:

$$\boxed{\begin{aligned} A &= 20''0411.\tau + 0.3978.\Delta\psi \\ \delta(A) &= 0''055 \quad + 0.3978.\delta(\Delta\psi) \\ B &= -\Delta\varepsilon \\ \delta(B) &= -\delta(\Delta\varepsilon) \\ E &= 0.00223.\Delta\psi \\ \delta(E) &= 0 \end{aligned}} \tag{5}$$

14. THE TRUE LONGITUDE OF THE SUN

The mean longitude, A1, of the sun was one of the basic arguments in the computation of nutation and of the Besselian Day Numbers, A and B. The true longitude of the sun has to be used in the computation of the Besselain Day Numbers, C and D (aberration), and of the Radial Velocity of the Earth. It is defined by:

A7 = True longitude of the sun, referred to the true equinox of date,

$$\boxed{A7 = A1 + ECE + \Delta\psi} \tag{1}$$

where $\Delta\psi$ is the nutation in longitude (Section 10), and

$$ECE = \text{"Equation of the Centre"} \quad . \tag{2}$$

ECE contains, so to speak, the elliptic part of the earth's motion; it is the difference between mean and true anomaly. The following Fourier Series can be used for its approximate computation:

$$\begin{aligned} ECE = {} & (2ECC - 1/4\ ECC^3 + \ldots\ldots).\sin(A3) \qquad (3) \\ & + (5/4\ ECC^2 - 11/24\ ECC^4 + \ldots\ldots).\sin(2.A3) \\ & + \ldots\ldots\ldots \end{aligned}$$

The first two terms are sufficient for all applications in this report. Using the value for ECC for 1967.0, we obtain:

$$\boxed{\begin{aligned} ECE &= 1^{\circ}\!.9162 \sin(A3) + 0^{\circ}\!.0200 \sin(2.A3) \\ &= 0.033444^{\text{rad}} \sin(A3) + 0.000350^{\text{rad}} \sin(2.A3) \end{aligned}} \tag{4}$$

The neglected remainder is of the order of about $1''$. In the applications, A7 has to be computed with about $0^{\circ}\!.01$ precision. $\Delta\psi$, which is of the order of $0^{\circ}\!.005$, can be neglected if not available (it is available if Besselian Day Numbers are computed).

For linear extrapolation we need the daily variations of ECE and $\Delta\psi$ (the latter being added only to be consistent at this point):

$$\delta(ECE) = \left\{1\overset{\circ}{.}9162 \cos(A3) + 0\overset{\circ}{.}0200 \cdot 2.\cos(2.A3)\right\} \cdot \delta(A3)$$

$$\boxed{\begin{aligned}\delta(ECE) &= +0\overset{\circ}{.}03296 \cos(A3) + 0\overset{\circ}{.}000688 \cos(2.A3) \\ &\overset{\text{rad}}{=} +0.00057526 \cos(A3) + 0.0000120 \cos(2.A3)\end{aligned}} \qquad (5)$$

$$\boxed{\begin{aligned}\delta(A7) &= +0\overset{\circ}{.}985647 + \delta(ECE) + \delta(\Delta\psi) \\ &\overset{\text{rad}}{=} +0.0172028 + \delta(ECE) + \delta(\Delta\psi)\end{aligned}} \qquad (6)$$

If extrapolation is extended only over one day, one can neglect the second term in equ. (5) and, of course, the $\delta(\Delta\psi)$.

15. BESSELIAN DAY NUMBERS C AND D (ABERRATION)

Before 1960, these variables were computed from the true longitude of the sun. Since 1960, they are computed from the true velocity of the earth, i.e., they contain the periodic planetary and lunar perturbations and the difference between heliocentrum and barycentrum. Since these differences concern the region beyond about 0".01, we can ignore the 1960's change and still use the old method:

$$
\begin{aligned}
C &= -20''.47 \cos(A7).\cos(EPS) \\
D &= -20''.47 \sin(A7)
\end{aligned} \qquad (1)
$$

A7, the true longitude of the sun, is described in the last section. Putting EPS constant = EPS 1967.0, we obtain:

$$
\boxed{\begin{aligned}
C &= -18''.78 \cos(A7) \\
D &= -20''.47 \sin(A7)
\end{aligned}} \qquad (2)
$$

With this approximation, the agreement with the presently published values is of the order of about 0".01.

For linear extrapolation:

$$
\begin{aligned}
\delta(C) &= +18''.78 \sin(A7).\delta(A7) \\
\delta(D) &= -20''.47 \cos(A7).\delta(A7)
\end{aligned} \qquad (3)
$$

An accuracy of 0".01 can be obtained if we include in $\delta(A7)$ only those effects which are about $0.0001^{\text{rad}} = 0^{\circ}.006 = 21''$. Therefore, $\delta(\Delta\psi)$ can be neglected, and only the first term of $\delta(ECE)$ has to be taken into account. Therefore, we obtain:

$$
\boxed{\begin{aligned}
\delta(C) &= +0''.3231.\sin(A7).\left\{1 + 0.0334.\cos(A3)\right\} \\
\delta(D) &= -0.3521.\cos(A7).\left\{1 + 0.0334.\cos(A3)\right\}
\end{aligned}} \qquad (4)
$$

16. REDUCTION OF MEAN PLACES TO APPARENT PLACES

Given: α_o, δ_o = Mean geocentric position of a source, referred to the mean equinox of the beginning of the current Besselian Year, next to observation epoch. The source has no proper motion.

Wanted: $\alpha(\tau)$, $\delta(\tau)$ = Apparent geocentric position of the source, referred to the true equinox of the date. The date is defined by τ, the time elapsed since the date to which α_o, δ_o belong, expressed in a fraction of a tropical year.

Procedure: Compute the Besselian Day Numbers A, B, C, D, and E, corresponding to τ. Then:

$$\alpha(\tau) = \alpha_o + aA + bB + cC + dD + E + F \qquad (1)$$

$$\delta(\tau) = \delta_o + a'A + b'B + c'C + d'D + G \qquad (2)$$

Here, F and G are the so-called second order corrections, which, in most cases are to be neglected. I will discuss their influence later. The other factors are the star constants:

$$\boxed{\begin{array}{ll} a = \frac{m}{n} + \sin\alpha_o \,\mathrm{tg}\,\delta_o\,, & a' = \cos\alpha_o \\ b = \cos\alpha_o \,\mathrm{tg}\,\delta_o\,, & b' = -\sin\alpha_o \\ c = \cos\alpha_o \sec\delta_o\,, & c' = \mathrm{tg}(EPS_o)\cos\delta_o - \sin\alpha_o\sin\delta_o \\ d = \sin\alpha_o \sec\delta_o\,, & d' = \cos\alpha_o\sin\delta_o \end{array}} \qquad (3)$$

The term $\frac{m}{n}$ is defined by:

$$\frac{m}{n} = 2.29887 + 0.00237 \cdot T \qquad (4)$$

Since the upper limit of A is about 13", the precision of m/n must be of the order of 10^{-4} to obtain an accuracy of 0"01 for α. Therefore, m/n can be kept constant over one year or more:

$$\boxed{1967.0 \quad m/n = 2.30046} \tag{5}$$

<u>Error Consideration</u>

First we will discuss the second order terms, F and G. They are given by:

$$\begin{aligned} F &= J.\mathrm{tg}^2\delta_o \\ G &= J'.\mathrm{tg}\delta_o \end{aligned} \tag{6}$$

and have the following meaning. The effects which enter the transformation of a mean place into an apparent place depend on α, δ rather than on α_o, δ_o. The star constants, however, contain the already known mean places. This was made for computational convenience. The terms F and G correct for this and for the second order differences which are between rigorous computation of precession, nutation, aberration, and the approximate formulae (1) and (2) which is a series development of the rigorous methods. The new numbers J and J' depend on the numbers A, B, C, D, and on the position α_o, δ_o. They can easily be computed by the formulae given in the <u>Explanatory Supplement</u> on page 161. Here we will consider only their influence. J and J' have absolute values less than 0"01. For 1967, the maximum values are about 0"003. Let us assume these maximum values. Let us further assume that we want an accuracy of about 0"05 in the apparent place. Then from equ. (6):

$$\begin{aligned} &\text{For } |\delta_o| \le 76^\circ: \quad |F| \le 0''05 \\ &\text{For } |\delta_o| \le 86^\circ: \quad |G| \le 0''05 \end{aligned} \tag{7}$$

Therefore, the second order corrections can be neglected for declinations between -76° and $+76^\circ$.

Next, we consider the influence of errors in A,B,C,D,E to the apparent places. Since the declination star constants do not contain $tg\delta_o$ or $sec\delta_o$, the accuracy of δ will be of the same order as the accuracy of the A,B,C,D. In right ascension, however, all four star constants contain either $tg\delta_o$ or $sec\delta_o$. The errors in α are given by the errors in A,B,C,D multiplied by $tg\delta_o$. The error in A is the term R.sin(EPS) in Table 4; the error in B is the term R in Table 5. Let us assume that we compute A and B with maximum possible errors of 0".05. We must then include the first 10 terms of Table 4 and all 7 terms of Table 5. The errors of C and D are of the order of 0".01. The error in α can then be estimated by:

$$\text{Error in } \alpha \leq 0''.07 \cdot tg\delta_o \qquad (8)$$

The error in α, therefore, will be less than 0".05, in the range $-35^\circ \leq \delta_o \leq +35^\circ$. In the range, $-76^\circ \leq \delta_o \leq +76^\circ$, in which the second order terms can be neglected, the error in α has the upper limit 0".28.

The above considerations lead to the following, very rough results:

Besselian Day Numbers A,B,C,D, and E are given with errors less than $0''.05$:

Apparent places are computed from:

$$\alpha = \alpha_o + aA + bB + cC + dD$$

$$\delta = \delta_o + a'A + b'B + c'C + d'D$$

Errors in α: $\leq 0''.05$ for $-35^\circ \leq \delta_o \leq +35^\circ$

$\leq 0''.28$ for $-76^\circ \leq \delta_o \leq +76^\circ$

For $|\delta_o| > 76^\circ$, second order terms have to be included, and accuracies of A and B have to be increased if an accuracy of $0''.3$ is not sufficient.

Errors in δ: $\leq 0''.07$ for $-86^\circ \leq \delta_o\ \delta\ +86^\circ$

For larger declinations, second order terms have to be included if this accuracy is not sufficient.

Final Remarks

a. It should be mentioned again, that the mean places enter in the star-constants. They remain constant during the whole year. This is also true if one first computes one apparent place and then makes extrapolations using the daily variations of the Day Numbers.

b. Using a set of A,B,C,D from the American Ephemeris would increase the accuracy of the apparent positions for just the time to which this set belongs. Unless one computes the daily variation with full accuracy, extrapolating the Day Numbers over one or more days will lead to possible errors (see Tables 7 and 8). Therefore, if the accuracy of the approximation presented in Sections 11 and 15 is not sufficient, the only way seems to be the following: A table containing all Besselian Day Numbers is stored for several days in the observation period, and the particular values needed are obtained by rigorous interpolation. This, at least, is the method used in automatized evaluation of high accuracy astrometric work.

c. As far as the error considerations in this section and in Section 11 are concerned, one should keep in mind that they only give <u>upper limits</u>. In the series for nutation, many terms will have different signs. It seems to be difficult to study this in detail other than by numerical methods. Furthermore, in many applications, the position error, $\Delta\alpha$, has less significance than the angular error, $\Delta\alpha \cos\delta$. If we ask for 0".05 of accuracy in $\Delta\alpha \cos\delta$, the δ limits in the above estimates can be increased correspondingly.

d. If the reductions from mean to apparent places or vice versa have to be made with a very high accuracy, the use of the Besselian Day Numbers should be avoided and be replaced by rigorous methods. (See Explanatory Supplement, page 150, last section.)

17. THE RADIAL VELOCITY COMPONENTS CAUSED BY THE EARTH'S MOTION

In a paper written by S. Herrick (Lick Observatory Bulletin, page 470), it is shown how that part of an observed radial velocity caused by the earth's motion is computed. This effect depends on the direction cosines of the object and on the ecliptic velocity components X', Y' of the earth. Here it is shown how the X', Y' can be computed for any given time, with the same accuracy as in Herrick's tables. Of course, the direction cosines (e.g., the position of the source) and the velocity components must refer to the same equinox. Let t_o be the <u>equinox of the positions</u> and t be the date of observation (<u>epoch</u>). The velocity components for epoch t and equinox t are given by:

$$\begin{aligned} X' &= -c'(\sin u + e \sin \pi) \\ Y' &= +c'(\cos u + e \cos \pi) \end{aligned} \qquad (1)$$

where:

- $c' = ck/\sqrt{a(1-e^2)}$,
- c = factor converting $\frac{\text{A.U.}}{\text{day}}$ into $\frac{\text{km}}{\text{sec}}$
- k = Gaussian Gravitation Constant
- a,e = Semi-major axis and eccentricity of the earth's orbit
- u = true longitude of the earth, referred to the true equinox of date
- π = true longitude of the perihelion, referred to the true equinox of date.

(See Herrick's paper, page 88, equ. (3))

First, let us consider the accuracy with which u and π have to be computed. The factor c' is of the order of 30 km/sec. Here we limit our approximation to 0.01 km/sec. To go further down would mean that the perturbations by the moon had to be included. With this limitation, the accuracy of u must be about 0.00033 rad = $0\overset{\circ}{.}02$; the accuracy of π has to be only 1° because of the factor e. Together, aberration and nutation can reach only $0\overset{\circ}{.}01$. Therefore, they can be neglected. This means that with sufficient accuracy, we can represent u and π by our basic arguments A7 (Section 14) and A2 (Section 10).

The relation is:

$$u = A7 - 180^\circ, \quad \pi = A2 - 180^\circ \tag{2}$$

Both A7 and A2 refer to the equinox of date (the difference between true and mean equinox being negligible, as was stated above). In our application, however, they should be referred to the equinox t_o. To obtain this, we have to subtract from both the precession in longitude for the interval $t-t_o$. Observing equ. (2), we obtain:

$$X' = +c'(\sin\lambda + ECC.\sin\gamma)$$
$$Y' = -c'(\cos\lambda + ECC.\cos\gamma)$$
$$\lambda = A7 - p(t-t_o)$$
$$= A2 - p(t-t_o)$$

(A2 and ECC from Section 10, A7 from Section 14)

p = annual precession in longitude

$= 50''\!.2564 + 0''\!.0222.\ T$

$= 50''\!.2415$ for 1967.0 (=0.01396°)

$t-t_o$ = time between epoch of observation and equinox of source positions, expressed in years

In the computation of A7, the term $\Delta\psi$ can be ignored. In ECE, both terms have to be included.

The coefficient c', in Herrick's paper, corresponds to a solar parallax of 8".80. It's value is <u>29.77 km/sec.</u> Using the recent value of $1.496.10^8$ km per A.U.(IAU, 1964, radar measurements), and the corresponding value of the mass of the earth-moon system, we would obtain <u>c' = 29.79 km/sec.</u>