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**Phased Array Feed Model Equations corresponding to
two definitions of embedded beam pattern**

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Abstract

In this report, we present the phased array feed (PAF) model equations for two definitions of embedded beam patterns. In Roshi & Fisher (2016), we presented the PAF model by defining the embedded beam pattern as the beam pattern due to a 1 V excitation to one port and all other ports short circuited. This embedded beam pattern is referred to as voltage-embedded-beam (VEB). The embedded beam pattern can also be defined as the beam pattern due to a 1 A excitation to one port and all other ports open circuited. This definition is usually used in engineering literature and we refer to the pattern as current-embedded-beam (CEB). Here we derive the relationship between the two embedded beam patterns and present the corresponding model equations.

1 Introduction

1.1 Embedded beam pattern

It is convenient to express the radiation pattern of PAF in terms of the *embedded beam pattern* (see Roshi & Fisher 2016). The embedded beam pattern can be defined in different ways. A definition used in Roshi & Fisher (2016) is : the j^{th} embedded beam pattern, $\vec{\mathcal{E}}_j^e$ is the beam pattern of the PAF when j^{th} port is excited with 1 V (i.e. $v_{0_j} = 1$ V) and all other ports are short circuited (i.e. $v_{0_i} = 0$ V for $i \neq j$; see Fig. 1a). The source impedance for excitation is considered to be equal to z_0 , the characteristic impedance of the transmission line connected to the dipole. There will be M embedded beam patterns for a M element PAF, which are represented conveniently as a vector $\vec{\mathcal{E}}^e$,

$$\vec{\mathcal{E}}^e = [\vec{\mathcal{E}}_1^e, \vec{\mathcal{E}}_2^e, \dots]. \quad (1)$$

These beam patterns are functions of the position vector \vec{r} with the origin of the coordinate system located at the center of the PAF (see Roshi & Fisher 2016). The beam patterns are specified at the far-field ie $|\vec{r}| \gg \frac{2D_{array}^2}{\lambda}$, where D_{array} is the maximum physical size of the PAF and λ is the wavelength of operation of the PAF. The radiation pattern when the PAF is excited by an arbitrary set of port voltages is obtained by scaling the embedded beam patterns with the port voltage and summing them up. Hence the dimension of the embedded beam pattern in this definition is m^{-1} . At the far-field, the beam pattern can be described by an outgoing spherical wave,

$$\vec{\mathcal{E}}_i^e(\vec{r}) = \vec{E}_i^e(\theta, \phi) \frac{e^{jk \cdot \vec{r}}}{r}, \quad (2)$$

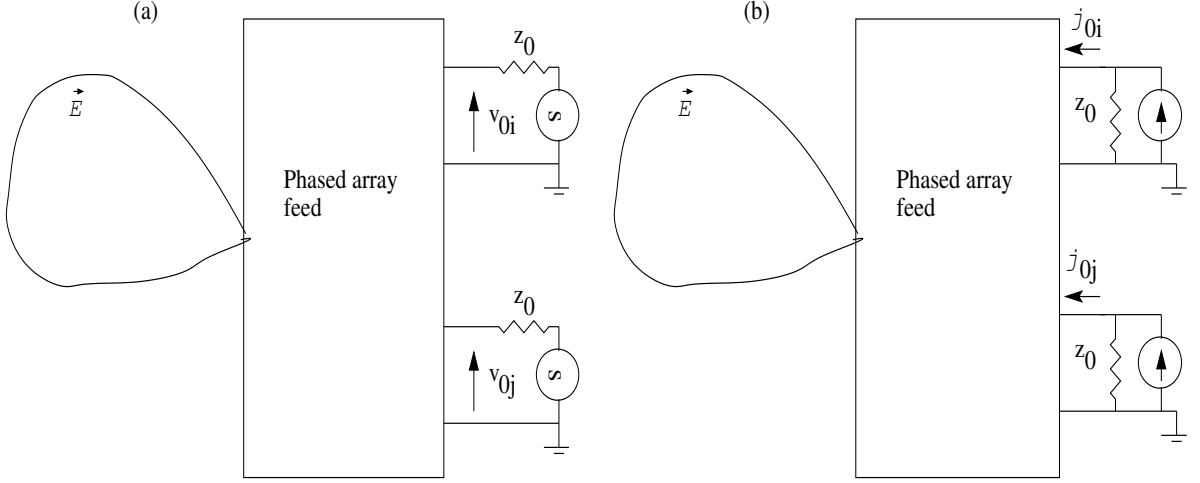


Figure 1: PAF in transmitting mode. The net radiation pattern of the PAF can be expressed as a ‘weighted’ sum of the embedded beam patterns (see Eq. 4 & 9). The ‘weights’ can be either the port voltages along with the VEB, $\vec{\mathcal{E}}^e$ (left) or port currents along with the CEB, $\vec{\psi}^e$ (right).

where \vec{E}_i^e is the i^{th} embedded beam pattern, r and \hat{r} are the magnitude and the unit vector in the direction of \vec{r} respectively, $\vec{k} = \frac{2\pi}{\lambda} \hat{r}$ is the propagation vector. Here \vec{E}_i^e depends only on the coordinates θ, ϕ . The geometric phase due to the location of elements (or in other words the excitation current distribution) away from the co-ordinate center is included in \vec{E}_i^e . From the definition of embedded pattern it follows that \vec{E}_i^e is dimensionless. The fields here are harmonic quantities, and for simplicity we omit the term $e^{j\omega t}$. The radiation pattern of the PAF when excited by a set of arbitrary port voltages is

$$\begin{aligned} \vec{\mathcal{E}}(\vec{r}) &= \sum_{i=1, M} v_{0_i} \vec{E}_i^e(\vec{r}), \\ &= \mathbf{V}_0^T \vec{\mathcal{E}}^e, \end{aligned} \quad (3)$$

where \mathbf{V}_0 is the vector of port voltages v_{0_i} (see Fig. 1a). The radiation pattern $\vec{\mathcal{E}}$ has units V/m. In the far-field, the (θ, ϕ) dependence of the radiation pattern can be written in a similar fashion,

$$\vec{E}(\theta, \phi) = \mathbf{V}_0^T \vec{\mathcal{E}}^e. \quad (4)$$

The unit of \vec{E} is V. In this report, we refer to both $\vec{\mathcal{E}}^e$ and \vec{E}^e as embedded beam pattern VEB.

Another definition for embedded beam pattern is: the j^{th} embedded beam pattern, $\vec{\psi}_j^e$ is the beam pattern of the PAF when j^{th} port is excited with 1 A and all other ports are open circuited, i.e.

$$\begin{aligned} \mathcal{J}_{0_i} &= 1 \text{ A for } i = j, \\ \mathcal{J}_{0_i} &= 0 \text{ A for } i \neq j, \end{aligned} \quad (5)$$

where \mathcal{J}_{0_i} are the port currents. The source impedance for excitation is considered to be equal to z_0 . As before there are M embedded beam patterns, which are represented conveniently as a vector $\vec{\psi}^e$,

$$\vec{\psi}^{eT} = [\vec{\psi}_1^e, \vec{\psi}_2^e, \dots]. \quad (6)$$

The beam pattern at far field can be written as,

$$\vec{\psi}_i^e(\vec{r}) = \vec{\Psi}_i^e(\theta, \phi) \frac{e^{j\vec{k}\cdot\vec{r}}}{r}, \quad (7)$$

The radiation pattern of the PAF when excited by a set of arbitrary port currents is

$$\begin{aligned} \vec{\mathcal{E}}(\vec{r}) &= \sum_{i=1, M} \mathcal{J}_{0_i} \vec{\psi}_i^e(\vec{r}), \\ &= \mathbf{I}_0^T \vec{\psi}^e, \end{aligned} \quad (8)$$

where \mathbf{I}_0 is the vector of port currents \mathcal{J}_{0_i} (see Fig. 1b). The radiation pattern $\vec{\mathcal{E}}$ has the unit V/m and $\vec{\psi}_i^e$ has unit V/A/m. As before, the (θ, ϕ) dependence of the far-field radiation pattern can be written as,

$$\vec{E}(\theta, \phi) = \mathbf{I}_0^T \vec{\Psi}^e. \quad (9)$$

The unit of \vec{E} is V and that of $\vec{\Psi}^e$ is V/A. In this report, we refer to both $\vec{\psi}^e$ and $\vec{\Psi}^e$ as the embedded beam pattern CEB.

The relationship between the two embedded beam patterns can be obtained using the network relationship between the port voltages and currents, $\mathbf{V}_0 = \mathbf{Z}\mathbf{I}_0$. Substituting this relationship in Eq. 3, we get

$$\vec{\mathcal{E}} = \mathbf{V}_0^T \vec{\mathcal{E}}^e = \mathbf{I}_0^T \mathbf{Z}^T \vec{\mathcal{E}}^e. \quad (10)$$

From Eq. 8 & 10 it follows

$$\vec{\psi}^e = \mathbf{Z}^T \vec{\mathcal{E}}^e \quad (11)$$

For a reciprocal PAF, $\mathbf{Z}^T = \mathbf{Z}$, and so the above equation can also be written as

$$\vec{\psi}^e = \mathbf{Z} \vec{\mathcal{E}}^e \quad (12)$$

2 PAF model equations corresponding to the two embedded beam patterns

The PAF model equations are somewhat simplified when written in terms of CEB $\vec{\psi}^e$. Essentially in almost all relevant equations the impedance matrix \mathbf{Z} is absorbed in the embedded beam pattern when $\vec{\psi}^e$ is used. For example, the open circuit voltage vector (see Eq. 37 in Roshi & Fisher 2016) at the output of the PAF for VEB, $\vec{\mathcal{E}}^e$, and CEB, $\vec{\psi}^e$, is given by Eqs. 13 & 14 respectively;

$$\mathbf{V}_{oc} = \mathbf{Z} \int_{A_{free}} \left(\vec{\mathcal{E}}^{eT} \times \mathcal{I} \vec{\mathcal{H}}_r - \mathcal{I} \vec{\mathcal{E}}_r \times \vec{\mathcal{H}}^e \right) \cdot \hat{n} \, dA, \quad (13)$$

$$= \int_{A_{free}} \left(\vec{\psi}^{eT} \times \mathcal{I} \vec{\mathcal{H}}_r - \mathcal{I} \vec{\mathcal{E}}_r \times \vec{\mathcal{J}}^e \right) \cdot \hat{n} \, dA. \quad (14)$$

Here $\vec{\mathcal{H}}^e$ and $\vec{\mathcal{J}}^e$ are the magnetic field patterns corresponding to the VEB, $\vec{\mathcal{E}}^e$ and the CEB, $\vec{\psi}^e$ respectively, $\vec{\mathcal{E}}_r$ and $\vec{\mathcal{H}}_r$ are the incident electric and magnetic fields on the PAF respectively, \mathcal{I} is the identify matrix, and the integration is over a region outside the PAF (see Fig. 2 in Roshi & Fisher 2016). The notation used in Eq. 14 is explained in Appendix J of Roshi & Fisher (2016).

A list of model equations corresponding to the VEB, $\vec{\mathcal{E}}^e$ (left) and the CEB, $\vec{\psi}^e$ (right) is given below.

$$\mathbf{R}_{spill} = \frac{4k_B T_g}{z_f} \mathbf{Z} \mathbf{C}_{Ce1}, \mathbf{Z}^H \quad \mathbf{R}_{spill} = \frac{4k_B T_g}{z_f} \mathbf{C}_{C\psi1}, \quad (15)$$

$$\mathbf{R}_{signal} = \frac{2S_{source}}{z_f} \mathbf{Z} \mathbf{C}_{Ie}, \mathbf{Z}^H \quad \mathbf{R}_{signal} = \frac{2S_{source}}{z_f} \mathbf{C}_{I\psi}, \quad (16)$$

$$T_{spill} = T_g \frac{\mathbf{w}_1^H \mathbf{Z} \mathbf{C}_{Ce1} \mathbf{Z}^H \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{Z} \mathbf{C}_{Ce} \mathbf{Z}^H \mathbf{w}_1}, \quad T_{spill} = T_g \frac{\mathbf{w}_1^H \mathbf{C}_{C\psi1} \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{C}_{C\psi} \mathbf{w}_1}, \quad (17)$$

$$T_A = \frac{S_{source}}{2k_B} \frac{\mathbf{w}_1^H \mathbf{Z} \mathbf{C}_{Ie} \mathbf{Z}^H \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{Z} \mathbf{C}_{Ce} \mathbf{Z}^H \mathbf{w}_1}, \quad T_A = \frac{S_{source}}{2k_B} \frac{\mathbf{w}_1^H \mathbf{C}_{I\psi} \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{C}_{C\psi} \mathbf{w}_1}, \quad (18)$$

$$\eta_{app} = \frac{1}{A_{ap}} \frac{\mathbf{w}_1^H \mathbf{Z} \mathbf{C}_{Ie} \mathbf{Z}^H \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{Z} \mathbf{C}_{Ce} \mathbf{Z}^H \mathbf{w}_1}, \quad \eta_{app} = \frac{1}{A_{ap}} \frac{\mathbf{w}_1^H \mathbf{C}_{I\psi} \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{C}_{C\psi} \mathbf{w}_1}. \quad (19)$$

Here \mathbf{R}_{spill} , \mathbf{R}_{signal} are the open circuit voltage correlations due to spillover noise and that due to radiation from source respectively, T_{spill} is the spillover temperature and T_A is the antenna temperature due to the source, η_{app} is the aperture efficiency, \mathbf{w}_1 is the weight vector applied on the open circuit voltage correlations (see Roshi & Fisher 2016),

$$\mathbf{C}_{Ce1} \equiv \int_{\Omega_{spill}} \vec{\mathbf{E}}^e \cdot \vec{\mathbf{E}}^{eH} d\Omega, \quad (20)$$

$$\mathbf{C}_{C\psi1} \equiv \int_{\Omega_{spill}} \vec{\Psi}^e \cdot \vec{\Psi}^{eH} d\Omega, \quad (21)$$

$$\mathbf{C}_{Ce} \equiv \int_{4\pi} \vec{\mathbf{E}}^e \cdot \vec{\mathbf{E}}^{eH} d\Omega, \quad (22)$$

$$\mathbf{C}_{C\psi} \equiv \int_{4\pi} \vec{\Psi}^e \cdot \vec{\Psi}^{eH} d\Omega, \quad (23)$$

$$\mathbf{C}_{Ie} \equiv \left(\int_{A_{pap}} \vec{\mathcal{E}}_{pap}^e dA \right) \cdot \left(\int_{A_{pap}} \vec{\mathcal{E}}_{pap}^e dA \right)^H, \quad (24)$$

$$\mathbf{C}_{I\psi} \equiv \left(\int_{A_{pap}} \vec{\psi}_{pap}^e dA \right) \cdot \left(\int_{A_{pap}} \vec{\psi}_{pap}^e dA \right)^H, \quad (25)$$

k_B is the Boltzmann constant, T_g is the ground temperature, z_f is the free space impedance, S_{source} is the flux density of the observed source and $\vec{\mathcal{E}}_{pap}^e$ and $\vec{\psi}_{pap}^e$ are the aperture fields (see Roshi & Fisher 2016) due to the VEB, $\vec{\mathcal{E}}^e$ and the CEB, $\vec{\psi}^e$ respectively. In Eqs. 20 & 21 the integration is over the parts of the beam solid angle, Ω_{spill} , seeing the ground radiation field and in Eqs. 24 & 25 the integration is over the aperture plane, A_{pap} , of physical area A_{ap} . The model equations that are not affected by the embedded beam pattern definition are

$$\mathbf{R}_{rec} = 4k_B T_0 \left(R_n \mathbf{I} + \sqrt{R_n g_n} (\rho \mathbf{Z} + \rho^* \mathbf{Z}^H) + g_n \mathbf{Z} \mathbf{Z}^H \right), \quad (26)$$

$$T_n = T_{min} + N T_0 \frac{\mathbf{w}_1^H (\mathbf{Z} - Z_{opt} \mathbf{I}) (\mathbf{Z} - Z_{opt} \mathbf{I})^H \mathbf{w}_1}{\text{Re}\{Z_{opt}\} \frac{1}{2} \mathbf{w}_1^H (\mathbf{Z} + \mathbf{Z}^H) \mathbf{w}_1}, \quad (27)$$

$$\mathbf{R}_{cmb} = 2k_B T_{cmb} (\mathbf{Z} + \mathbf{Z}^H), \quad (28)$$

$$\mathbf{R}_{sky} \approx 2k_B T_{sky} (\mathbf{Z} + \mathbf{Z}^H). \quad (29)$$

Here \mathbf{R}_{rec} , \mathbf{R}_{cmb} , \mathbf{R}_{sky} are the open circuit voltage correlations due to the amplifier noise, the cosmic microwave background and the sky background radiation respectively, T_n is the receiver temperature of the PAF, $T_0 = 290K$, R_n , g_n and ρ are the noise parameters of the amplifier, which can equivalently be expressed in terms of the minimum noise temperature T_{min} , Lange invariance N and optimum impedance Z_{opt} (Pospieszalski 2010); T_{cmb} is the cosmic microwave background temperature and $T_{sky} = T_{cmb} + T_{bg,\nu_0} \left(\frac{\nu}{\nu_0}\right)^{-2.7}$ is the temperature of the sky background at the observed off-source position, T_{bg,ν_0} is the galactic background radiation temperature at ν_0 , and ν is the frequency at which \mathbf{R}_{sky} is computed.

3 Embedded beam patterns from the CST far-field patterns

The CST (<https://www.cst.com/>) microwave studio provides the far-field pattern \vec{E}'_j when the j^{th} port is excited and all other ports are terminated with the CST port impedance (in our case it is 50Ω). From Eqs. 4 & 9 we get

$$\vec{E}'_j = \sum_{i=1,M} q_{ij} \vec{E}_i^e, \quad (30)$$

$$\vec{E}'_j = \sum_{i=1,M} \mathcal{J}_{ij} \vec{\Psi}_i^e. \quad (31)$$

Here $q_{ij} = v_{0_i}$ is the port voltage and \mathcal{J}_{ij} is the port current. These voltages and currents are computed below. The elements of the wave amplitude vector for the excitation are

$$\begin{aligned} a_i &= \sqrt{2 P_{stim}} \quad \text{for } i = j \\ &= 0 \quad \text{for } i \neq j \end{aligned} \quad (32)$$

where $P_{stim} = 0.5$ W, is the RMS excitation power in the CST simulation. The wave amplitude vector \mathbf{b} is then

$$\mathbf{b} = a_j \begin{bmatrix} S_{1j} \\ S_{2j} \\ \vdots \\ S_{jj} \\ \vdots \\ S_{Mj} \end{bmatrix}, \quad (33)$$

where a_j is the j^{th} element of the vector \mathbf{a} , $S_{ij}, i = 1$ to M is the j^{th} column of \mathbf{S} . The port voltages and currents are then

$$\begin{aligned} q_{ij} &= \sqrt{z_0}(a_i + b_i) \\ &= \sqrt{z_0}(1 + S_{jj})a_j \quad \text{for } i = j \\ &= \sqrt{z_0}S_{ij}a_j \quad \text{for } i \neq j \end{aligned} \quad (34)$$

$$\begin{aligned} \mathcal{J}_{ij} &= \frac{1}{\sqrt{z_0}}(a_i - b_i) \\ &= \frac{1}{\sqrt{z_0}}(1 - S_{jj})a_j \quad \text{for } i = j \\ &= \frac{-1}{\sqrt{z_0}}S_{ij}a_j \quad \text{for } i \neq j \end{aligned} \quad (35)$$

The set of far-field patterns provided by the CST along with the port voltages and currents can be used to obtain the VEB, \vec{E}^e and the CEB, $\vec{\Psi}^e$. Eq. 30 & 31 for the set of far-field patterns can be concisely written as

$$\vec{E}' = \mathbf{Q} \vec{E}^e, \quad (36)$$

$$\vec{E}' = \mathbf{J} \vec{\Psi}^e. \quad (37)$$

where the elements of the matrix \mathbf{Q} are q_{ij} and that of the matrix \mathbf{J} are \mathcal{J}_{ij} . This equation is valid for each θ, ϕ . Using Eqs. 34 & 35 \mathbf{Q} and \mathbf{J} can be written as

$$\mathbf{Q} = \sqrt{2 z_0 P_{stim}} (\mathcal{I} + \mathcal{S}), \quad (38)$$

$$\mathbf{J} = \sqrt{\frac{2 P_{stim}}{z_0}} (\mathcal{I} - \mathcal{S}). \quad (39)$$

The matrices \mathbf{Q} and \mathbf{J} are also related through the equation

$$\mathbf{J} = \mathbf{Q} \mathbf{Z}^{-1}. \quad (40)$$

The embedded beam patterns are then obtained as

$$\vec{E}^e = \mathbf{Q}^{-1} \vec{E}', \quad (41)$$

$$\vec{\Psi}^e = \mathbf{J}^{-1} \vec{E}'. \quad (42)$$

4 Some sanity checks

4.1 Energy conservation

We verify here whether the computed embedded beam patterns satisfy energy conservation. Details of such a verification for the VEB \vec{E}^e are given in Roshi & Fisher (2016). We consider below the case for CEB, $\vec{\psi}^e$. From the definition of embedded beam pattern $\vec{\psi}_j^e$ the port currents are

$$\begin{aligned} \mathcal{J}_{0_i} &= 1 \text{ A} \quad \text{for } i = j, \\ &= 0 \text{ A} \quad \text{for } i \neq j, \end{aligned} \quad (43)$$

and hence the wave amplitudes are

$$\begin{aligned} \frac{1}{\sqrt{z_0}}(a_i - b_i) &= 1 \quad \text{for } i = j, \\ &= 0 \quad \text{for } i \neq j. \end{aligned} \quad (44)$$

The vector \mathbf{a} can be written as

$$\mathbf{a} = \mathbf{b} + \sqrt{z_0} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad (45)$$

where the non-zero element (which is 1) is located at j^{th} row. Substituting this in the equation $\mathbf{b} = \mathbf{S}\mathbf{a}$ and re-arranging we get

$$\mathbf{b} = \sqrt{z_0} (\mathbf{I} - \mathbf{S})^{-1} \begin{bmatrix} S_{1j} \\ S_{2j} \\ \vdots \\ S_{jj} \\ \vdots \\ S_{Mj} \end{bmatrix}. \quad (46)$$

Power dissipated at the j^{th} port is

$$P_{dis} = \frac{1}{2}(a_j a_j^* - b_j b_j^*), \quad (47)$$

$$= \frac{\sqrt{z_0}}{2}(\sqrt{z_0} + (b_j + b_j^*)). \quad (48)$$

The far-field beam pattern of the PAF for the above excitation is the embedded beam pattern $\vec{\psi}_j^e$ and hence the radiated power is,

$$\begin{aligned} P_{rad} &= \frac{1}{2z_f} \int_{sphere} \vec{\psi}_j^e \cdot \vec{\psi}_j^{e*} dA, \\ &= \frac{1}{2z_f} \int_{4\pi} \vec{\Psi}_j^e \cdot \vec{\Psi}_j^{e*} d\Omega. \end{aligned} \quad (49)$$

For loss-less PAF $P_{dis} = P_{rad}$. This equality is satisfied in our PAF model computation. Further, for a loss-less antenna,

$$P_{rad} = \frac{1}{2} \mathcal{J}_{0_j}^2 \operatorname{Re}\{Z_{pin_j}\}, \quad (50)$$

where \mathcal{J}_{0_j} is the current flowing to port j , which for the embedded pattern $\vec{\psi}_j^e$ is 1 A and Z_{pin_j} is the input impedance of port j when all other ports are open circuited. The input impedance for this case is given by

$$Z_{pin_j} = z_{jj}, \quad (51)$$

where z_{jj} is the j^{th} diagonal element of the impedance matrix \mathbf{Z} . Thus

$$P_{rad} = \frac{1}{2} \operatorname{Re}\{z_{jj}\}. \quad (52)$$

4.2 PAF in a thermal radiation field

In this Section, we show that the open circuit voltage correlations \mathbf{R}_t , obtained from the two embedded beam patterns, when the the PAF is embedded in a black body radiation field are equal to the result given by Twiss's theorem (Twiss 1955). The correlation \mathbf{R}_t is given by (Roshi & Fisher 2016)

$$\begin{aligned} \mathbf{R}_t &= \frac{4k_B T_0}{z_f} \mathbf{Z} \left(\int_{4\pi} \vec{\mathbf{E}}^e \cdot \vec{\mathbf{E}}^{eH} d\Omega \right) \mathbf{Z}^H, \\ &= \frac{4k_B T_0}{z_f} \mathbf{Z} \mathbf{C}_{Ce} \mathbf{Z}^H, \end{aligned} \quad (53)$$

for the VEB, $\vec{\mathcal{E}}^e$ and

$$\mathbf{R}_t = \frac{4k_B T_0}{z_f} \mathbf{C}_{C\psi}, \quad (54)$$

for the CEB, $\vec{\psi}^e$. For a loss-less antenna the power dissipated at the ports should be equal to the radiated power, which can be used to calculate \mathbf{C}_{Ce} and $\mathbf{C}_{C\psi}$. The energy balance condition gives,

$$\begin{aligned} \frac{1}{2} \left(\frac{\mathbf{V}_0^H \mathbf{I}_0}{2} + \frac{\mathbf{I}_0^H \mathbf{V}_0}{2} \right) &= \frac{1}{2z_f} \mathbf{V}_0^H \mathbf{C}_{Ce} \mathbf{V}_0, \\ \frac{1}{4} \mathbf{V}_0^H \left(\mathbf{Z}^{-1} + (\mathbf{Z}^{-1})^H \right) \mathbf{V}_0 &= \frac{1}{2z_f} \mathbf{V}_0^H \mathbf{C}_{Ce} \mathbf{V}_0, \end{aligned} \quad (55)$$

and

$$\begin{aligned} \frac{1}{2} \left(\frac{\mathbf{V}_0^H \mathbf{I}_0}{2} + \frac{\mathbf{I}_0^H \mathbf{V}_0}{2} \right) &= \frac{1}{2z_f} \mathbf{I}_0^H \mathbf{C}_{C\psi} \mathbf{I}_0, \\ \frac{1}{4} \mathbf{I}_0^H \left(\mathbf{Z} + \mathbf{Z}^H \right) \mathbf{I}_0 &= \frac{1}{2z_f} \mathbf{I}_0^H \mathbf{C}_{C\psi} \mathbf{I}_0. \end{aligned} \quad (56)$$

Since Eqs. 55 & 56 are valid for arbitrary excitations it follows that

$$\frac{1}{2} \left(\mathbf{Z}^{-1} + (\mathbf{Z}^{-1})^H \right) = \frac{1}{z_f} \mathbf{C}_{Ce}, \quad (57)$$

$$\frac{1}{2} \left(\mathbf{Z} + \mathbf{Z}^H \right) = \frac{1}{z_f} \mathbf{C}_{C\psi}. \quad (58)$$

Substituting Eq. 57 in Eq. 53 and Eq. 58 in Eq. 54, we get

$$\mathbf{R}_t = 2k_B T_0 \left(\mathbf{Z} + \mathbf{Z}^H \right), \quad (59)$$

from both Eqs. 53 & 54, which is the voltage correlation given by Twiss's theorem (Twiss 1955).

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