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GRAVITATIONAL DEFORMATIONS AND APERTURE EFFICIENCY WITH POLAR MOUNTS

SEBASTIAN VON HOERNER

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Sebastian von Hoerner

Summary

It is suggested to describe the efficiency (as a function of declination D and hour angle H) in terms of those seven parameters which are physically relevant: three structural deformation prameters h_x, h_y, h_z; the pointing angles D_o and H_o of maximum efficiency, and the rms surface error σ_o at that pointing. And the efficiency η_∞ at very long wavelengths (but with the same illumination and taper as used now).

A way of obtaining these seven parameters from astronomical observations is also suggested.

1. Gravitational Surface Deformations

The deformations of an alt-azimuth radio telescope, and the derivation of its best adjustment elevation, have been described in NRAO Engineering Memo 152 of January 1984. For a polar mount, the results still hold for deformations at the meridian and for the best adjustment angle. But an extended treatment is now needed for the deformations at non-zero hour angles.

First, we define a coordinate system, fixed in the telescope dish. Let the telescope point at south horizon, and call:

$$\begin{aligned} x &= \text{elevation axis}, \quad + \text{ is east}, \\ y &= \text{vertical}, \quad + \text{ is down}, \\ z &= \text{optical axis}, \quad + \text{ is back}. \end{aligned} \tag{1}$$

Second, we define three structural deformation parameters. Let the telescope surface be a perfect paraboloid of revolution in the absence of gravity. Switch on gravity in x-direction, find the best-fit paraboloid to the deformed surface, and call h_x the rms deviation between surface and paraboloid. (In this and all following procedures, the proper illumination and taper should be used as weighting functions, in the best-fit and also for the mean in the rms, if the procedures are actually performed in a structural analysis.) Next, switch on gravity in y-direction, find the best-fit paraboloid for <u>this</u> deformation, and call h_y the rms deviation between surface and paraboloid. Finally, do the same with gravity in z-direction for obtaining h_z.

Third, we call declination D, hour angle H, elevation E, and geographical latitude B (38.4° at Green Bank). We give index zero to that pointing (D_o, H_o, E_o) where the efficiency is maximum. We find the best-fit paraboloid for this pointing and call σ_0 the rms deviation between surface and paraboloid. We call η_0 the maximum efficiency for the wavelength used, and we call η_∞ the efficiency for very long wavelenths (with the same illumination and taper as used now).

Fourth, the components of gravity g, in system (1), can be derived as:

$$\begin{aligned} g_x &= -g \cos B \sin H, \\ g_y &= \ \ g (\sin B \cos D - \cos B \sin D \cos H) \\ g_z &= \ \ g (\sin B \sin D + \cos B \cos D \cos H). \end{aligned} \tag{2}$$

Since the angular term in the parentheses of the last line is just sin E, we define in a similar way

$$\sin Y = \sin B \cos D - \cos B \sin D \cos H \tag{3}$$

where Y = angle of y-axis below horizon. The components (2) then read

$$\begin{aligned} g_x &= -g \cos B \sin H, \\ g_y &= \ \ g \sin Y, \\ g_z &= \ \ g \sin E, \end{aligned} \tag{4}$$

where

$$g_x{}^2 + g_y{}^2 + g_z{}^2 = g^2. \tag{5}$$

Fifth, we shall assume that the telescope is almost east-west-symmetric, meaning that H_o is only small. For example, with H_o = 0.5 hours we have $1 - \cos H_o = 0.009$ which may be neglected, but $\sin H_o = 0.13$ which may not. We then have, at optimum pointing, the gravitational components

$$\begin{aligned} g_{xo} &= -g \cos B \sin H_o, \\ g_{yo} &= \ \ g \cos E_o, \\ g_{zo} &= \ \ g \sin E_o. \end{aligned} \tag{6}$$

At this optimum pointing the surface will have an rms deviation σ_o from its best-fit paraboloid (the optimum or intrinsic surface error), and at other pointings a gravitational component σ_g will be added quadratically (the deviation from homology), to yield the total rms deviation σ of the surface from its best-fit paraboloid at that pointing:

$$\sigma^2 = \sigma_o^2 + \sigma_g^2, \tag{7}$$

where

$$\sigma_g^2 = h_x^2 \cos^2 B(\sin H - \sin H_o)^2 + h_y^2(\sin Y - \cos E_o)^2 + h_z^2(\sin E - \sin E_o)^2. \tag{8}$$

2. Aperture Efficiency

Neglecting the atmospheric extinction, and any spatial correlation of the gravitational deformations (Appendix 1), the aperture efficiency is, according to J. Ruze (IEEE Proc. 54, 633, 1966):

$$\eta = \eta_\infty e^{-A(4\pi\sigma/\lambda)^2} \tag{9}$$

with $A = 1$ for a very flat dish, and $A = 0.76$ for our NRAO telescopes of $F/D = 0.43$. At the optimum pointing, we have

$$\eta_o = \eta_\infty e^{-A(4\pi\sigma_o/\lambda)^2} \tag{10}$$

thus (9) can be written as

$$\eta = \eta_o e^{-A(4\pi/\lambda)^2 \sigma_g^2} \tag{11}$$

with σ_g from (8).

This then is the suggested description of the efficiency in terms of physical parameters. For a given wavelength λ, there are six parameters: three structural prameters h_x, h_y, h_z, and the maximum efficiency η_o at optimum pointing E_o, H_o.

But in general, for any wavelength λ, we have <u>seven</u> parameters, replacing η_o by η_∞ and σ_o, using (10).

3. Observational Parameter Determination

First, we ask only for η_∞ and σ_o. We assume that E_o and H_o do not depend on wavelength. We presuppose that the maximum efficiency η_o has been measured at many different wavelengths λ_i ($i=1...n \geq 3$), and we call $\eta_{oi} = \eta_o(\lambda_i)$. The logarithm of (10) is

$$\ln \eta_{oi} = \ln \eta_\infty - \sigma_o^2 \, A(4\pi/\lambda_i)^2 \tag{12}$$

which shall be written in the normal form for <u>linear regression</u>:

$$y_i = a + bx_i . \tag{13}$$

For this purpose we call the two unknowns

$$a = \ln \eta_\infty \quad \text{and} \quad b = \sigma_o^2 , \tag{14}$$

and we call the experimental data

$$x_i = A(4\pi/\lambda_i)^2 \quad \text{and} \quad y_i = \ln \eta_{oi} . \tag{15}$$

As well known, the solutions are

$$b = \frac{\overline{xy} - \bar{x}\,\bar{y}}{s_x^2} \quad \text{and} \quad a = \bar{y} - b\,\bar{x} \tag{16}$$

and the correlation coefficient r is

$$r = b\, s_x/s_y \tag{17}$$

with $s_x^2 = \overline{x^2} - \bar{x}^2$ and $s_y^2 = \overline{y^2} - \bar{y}^2$.

We also want the mean errors, which are not so well known. We call

$$P = s_y \sqrt{\frac{1 - r^2}{n - 2}} \quad \text{and} \quad Q = \sqrt{\overline{x^2}} , \tag{18}$$

and

$$W = \sqrt{1 + \left\{\frac{x-\bar{x}}{s_x}\right\}^2} . \tag{19}$$

The scatter of the experimental data then is

$$\mathrm{rms}(y_i - a - bx_i) = P\sqrt{n}, \tag{20}$$

and the mean errors of a and b are found as

$$\varepsilon_b = \varepsilon(b) = P/s_x \qquad \text{and} \qquad \varepsilon_a = \varepsilon(a) = \varepsilon_b Q. \tag{21}$$

Finally, if for some given x we predict y(x) from (13), using a and b from (16), then this predicted y will have the mean error

$$\varepsilon_p = \varepsilon(y\{x\}) = P\,W. \tag{22}$$

Equation (22) holds even for far-out extrapolations, provided the true physical relation between x and y indeed is linear.

We now return to our original quantities. The rms scatter ε of the observed efficiencies is obtained from (20) and (15) in the form

$$\varepsilon / \eta = P\sqrt{n}. \tag{23}$$

The intrinsic surface error σ_o and the long-wave efficiency η_∞ are

$$\sigma_o = \sqrt{b} \qquad \text{and} \qquad \eta_\infty = e^a, \tag{25}$$

and their mean errors are

$$\varepsilon_o = \varepsilon(\sigma_o) = P/(2s_x\,\sigma_o) \tag{26}$$

and

$$\varepsilon_\infty = \varepsilon(\eta_\infty) = \eta_\infty PQ/s_x. \tag{27}$$

If we predict, for some given wavelength λ, the maximum efficiency η_o from (10), using η_∞ and σ_o from (25), then the mean error is

$$\varepsilon_p = \varepsilon(\eta_o\{\lambda\}) = \eta_o PW. \tag{28}$$

If the n observations used have different accuracies, we may apply weights. If $\mathcal{E}_i$ is the estimated mean error of the observed efficiency η_{oi}, then the mean error of y_i is $\mathcal{E}_i/\eta_{oi}$, and all the averages in (16), (17) and (18) should use the weights

$$w_i = (\eta_{oi}/\mathcal{E}_i)^2. \tag{29}$$

The number n of observations, in equations (18) through (23), should now be replaced by an equivalent number n_o of equal-weight observations with

$$n_o = (\Sigma w)^2/\Sigma w^2 \tag{30}$$

which, for unequal weights, is smaller than n.

Second, we ask for the remaining five parameters h_x, h_y, h_z, H_o, and E_o. After η_∞ and σ_o have been calculated, we use n observations of the efficy η_i, all over the sky and at different wavelengths λ_i (i = 1...n), and we find for each one the gravitational contribution σ_{gi} from (9) and (7) as

$$\sigma_{gi}^2 = \frac{\lambda_i^2 \ln(\eta_\infty/\eta_i)}{A\ (4\pi)^2} - \sigma_o^2 \tag{31}$$

which is to be used in (8).

Unfortunately, (8) is highly <u>nonlinear</u> regarding H_o and E_o. As a good approach, we suggest to estimate H_o and E_o simultaneously with the maximum efficiencies η_{oi} of the last section; either by eye-inspection, or by fitting a parabola to the central parts of the scans.

We insert these estimates into equation (8), which then becomes a system of n linear equations for the three unknowns

$$a_1 = h_x^2, \qquad a_2 = h_y^2, \qquad a_3 = h_z^2. \tag{32}$$

The matrix of the system is

$$M_{i1} = \cos^2 B\,(\sin H_i - \sin H_o)^2$$
$$M_{i2} = (\sin Y_i - \cos E_o)^2 \tag{33}$$
$$M_{i3} = (\sin E_i - \sin E_o)^2$$

with $i=1\ldots n$. Calling $v_i = \sigma_{gi}^2$ from (31), the system finally reads

$$\sum_{j=1}^{3} M_{ij}\, a_j = v_i, \qquad \text{or} \qquad Ma = v. \tag{34}$$

We call T the transpose of M ($T_{ji} = M_{ij}$), and multiply (34) from left with T, which gives $TM\,a = Tv$. We call $(TM)^{-1}$ the inverse of matrix TM and obtain the solution

$$a = (TM)^{-1}\,Tv. \tag{35}$$

For finding the mean errors, we use the solution a_j from (35) and calculate the residual

$$R = \sum_{i=1}^{n} \left\{ v_i - \sum_{j=1}^{3} M_{ij}\, a_j \right\}^2 . \tag{36}$$

The mean error of a_j then is, using the diagonal elements of $(TM)^{-1}$,

$$\mathcal{E}(a_j) = \sqrt{\frac{R}{n-3}\,(TM)^{-1}_{jj}} \tag{37}$$

and the mean error $\mathcal{E}_j$ of $h_j = \sqrt{a_j}$ then is

$$\mathcal{E}_j = \mathcal{E}(h_j) = \frac{\mathcal{E}(a_j)}{2h_j}. \tag{38}$$

In case of different accuracies, we may again use <u>weights</u>: we multiply M_{ij} of (33) by w_i from (29), call $v_i = w_i\sigma_{gi}^2$, and proceed as described above. In (37), n must again be replaced by n_o from (30).

For short wavelengths, the atmospheric <u>extinction</u> must have been corrected for, and η_i in (31) then is the corrected efficiency without extinction. **With the deformable <u>Cassegrain</u>, h_z must be smaller than at prime focus, but h_x and h_y must be equal (as long as we have no lateral tilt).**

APPENDIX:

Correlation of Gravitational Deformations

Ruze's paper which we quoted with (9) treats also the case that the surface errors are spatially correlated over a correlation length L, under the simplification that the errors are about constant within circles of diameter L, but uncorrelated from one circle to the other. Ruze's result can be written in the form:

$$\eta = \eta_{\infty} e^{-\beta} \left[1 + \frac{1}{\eta_{\infty}} (L/D)^2 S(\beta)\right] \tag{39}$$

with the phase error

$$\beta = A\ (4\pi\sigma/\lambda)^2 \tag{40}$$

and with the following function

$$S(\beta) = \sum_{m=1}^{\infty} \frac{\beta^m}{m\ m!}\ . \tag{41}$$

Table 1. Phase error β and its functions in (39).

β	$e^{-\beta}$	$S(\beta)$	$e^{-\beta}S(\beta)$
0.1	0.905	0.103	0.093
.2	.819	.210	.172
.4	.670	.444	.294
.7	.497	.844	.419
1.0	.368	1.318	.485
1.5	.223	2.319	.517
2.0	.135	3.684	.499
3.0	.050	8.257	.411

Some values are shown in Table 1. We see that the correcting term in (39) can be appreciable for large surface errors σ and long correlation lengths L. An exact treatment then would be difficult.

How large will L/D actually be? At the pointing of maximum efficiency, we have only the intrinsic surface error σ_o, caused by the maladjustment of the surface panels and their internal bumpiness. Then L will be about half a panel side, and with a number N of surface panels, we expect about $(L/D)^2 = 1/(4N)$. This will mostly be rather small, for example 1/240 for the 60 panels of the 140-ft. Thus the correlation term in (39) may be completely neglected for the observational determination of σ_o and η_∞. If λ_s is the shortest wavelength used, this neglection then will cause a relative error ε_∞ of η_∞ :

$$\varepsilon_\infty / \eta_\infty \leq \begin{cases} 0.012 & \text{for } \lambda_s = 13.0 \text{ mm,} \\ 0.050 & \text{for } \lambda_s = 8.4 \text{ mm.} \end{cases} \tag{42}$$

This is different for gravitational deformations which will have a longer correlation, depending on the different modes of the deformation (some of which have already been automatically corrected for at the 140-ft). Visualizing the form of the deformations, a rough estimate leads to the values of Table 2:

Table 2. Deformation mode and estimated correlation.

mode	corrected?	correlation	$(L/D)^2$
axial defocussing	yes	L = D/2	1/4
astigmatism	yes	L = D/2	1/4
lateral defocussing	no	L = D/3	1/9
higher orders	no	L = D/4	1/16

From the last two (uncorrected) lines we would expect for the 140-ft at present about

$$(L/D)^2 = 1/12. \tag{43}$$

For small phase errors we have about $S(\beta) = \beta$, and we can bring the whole correction term of (39) into the exponent, where ß then is multiplied by a factor

$$K = 1 - (L/D)^2/\eta_\infty . \tag{44}$$

With our estimate (43) and about $\eta_\infty = 0.61$ we have about

$$K = 0.86. \tag{45}$$

Correlation then is taken care of, in a first-order approximation, if $A = 0.76$ is left unchanged regarding the intrinsic surface error σ_o, but if it is replaced by $A_g = KA = 0.66$ regarding the gravitational contribution σ_g. Thus (9) should be written as

$$\eta = \eta_\infty e^{-A(4\pi/\lambda)^2(\sigma_o^2 + K\sigma_g^2)} \tag{46}$$

and (31) as

$$\sigma_{gi}^2 = \frac{1}{K}\left[\frac{\lambda_i^2 \ln(\eta_\infty/\eta_i)}{A(4\pi)^2} - \sigma_o^2\right] \tag{47}$$

to be used in (8).

Instead of applying these corrections, we suggest instead to <u>omit</u> the correlation completely in our treatment of the data and in the determination of all parameters; this will have <u>no</u> effect on the quality of the parametric descriprion of the efficiency (in first-order approximation, that is). But we must keep only in mind that the three structural parameters h_j will turn out somewhat larger in a structural analysis of the telescope model on a computer, as compared to their observational determination:

$$h_j(\text{analysis}) = h_j(\text{observation})/\sqrt{K} . \tag{48}$$

The structural analysis must take care of corrected and uncorrected modes. For the present state of the 140-ft, this means that the best-fit paraboloid, which in general would have six degrees of freedom (three translations each of vertex and focus), can have four degrees only: three translations of the vertex plus a change of focal length; with two constraints: no lateral movements perpendicular to the axis whose direction is given by the deformed locations of vertex and feed leg apex. And finally, the astigmatic part of the surface error must be subtracted.