INTERNAL TWIST AND LEAST-SQUARES ADJUSTMENT OF 4-CORNERED SURFACE PLATES FOR REFLECTOR ANTENNAS

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Summary

Surface plates with four adjustment screws, one at each corner, allow four degrees of freedom for their adjustment, including an internal non-planar twist or warp. The surface deformation of trapezoidal plates resulting from an enforced twist is investigated, and a simple approximation is derived which is well confirmed by measurements of four different experimental plates. This approximation is then used to develop a least-squares procedure for obtaining those adjustment amounts which minimize the rms deviation between the plate's surface and the desired telescope paraboloid. The application of this procedure is studied with seven experimental plates in the lab. Measurements of six surface plates on a radio telescope (which needs readjustment) showed that the inclusion of the internal twist and its treatment by the proposed procedure would yield an additional improvement of 18% for the surface deviations.

I. INTRODUCTION

In the error budgets of most existing or newly designed radio telescopes, the adjustment errors of the surface plates are one of the major contributions. It is thus an important task to make them as small as possible, small as compared to the manufacturing tolerances of the plates and the deformations of the telescope.

The surface adjustment consists of four steps. First, the measurement of the surface is still a difficult problem, regarding both the extreme accuracy/distance ratios required, and the unambiguous and exact definition of a reference coordinate system. Second, a "desired paraboloid" must be selected which may be different from the design paraboloid. A paraboloid of revolution has six degrees of freedom, but some of them will be fixed for structural reasons; for example, most of our telescopes are Cassegrain systems allowing axial adjustments at the prime focus, and in this case the direction and location of the optical axis are given by existing equipment locations, fixing four parameters and leaving only two free ones: vertex height and focal length. The vertex height may be chosen in accordance with the telescope's present best-fit paraboloid for minimizing the adjustments. But a new focal length should be the best-fit one for the internal <u>curvature</u> of all plates, independent of their present heights and slopes. Third, after the desired paraboloid has been selected, a computer procedure is needed to obtain the single amounts of adjustment for all plate corners. Fourth, the mechanical adjustments of the corners by the computed amounts should be done without introducing significant errors, which is again a difficult task, especially if it must be done from underneath the surface high above ground.

The third step is the subject of this paper, treating two problems: (a) Telescopes have frequently been adjusted such that the adjustment points at the plate corners will coincide with the desired paraboloid, which we call a "true-corner adjustment". The best method, however, is a "least-squares adjustment" where the rms deviation of the plate surface from the desired paraboloid is minimized, using <u>all</u> available degrees of freedom. (b) Most telescopes are designed with a radial surface pattern, with trapezoidal plates having four adjustment points, one at each corner. This allows four degrees of freedom per plate. But using all four degrees requires the knowledge of a plate's surface deformation resulting from an enforced non-planar twist or warp (three corners level and one raised). The problem is that no detailed description of twisted plates could be found in the engineering literature, neither in the standard textbooks [Ref. 1, 2, 3] nor in the journals [4, 5].

Thus, we developed a description of our own, confirmed by measurements of four different experimental plates. This description is of a simple form, and it is then used for deriving the wanted least-squares procedure for 4-cornered plates, a procedure which is recommended for all telescopes with trapezoidal surface plates.

II. SHAPE OF A TWISTED PLATE

Figure 1,a shows a trapezoidal plate; one of its parallel sides (b_1) is held at $z = 0$, while the other one (b_2) is subject to a torsional twist (T). We want to know the resulting surface shape, $z(x,y)$.

For a long bar of constant rectangular cross section (of width b and thickness t) the twist angle θ at its end is given [Ref. 1, page 194, case 4] by

$$\theta = \frac{Ta}{GK} \tag{1}$$

where T = twisting moment, a = length of bar, G = modulus of rigidity, and where K is approximately

$$K = \frac{1}{3} b\, t^3 \,[1 - 0.630(t/b) + 0.0525(t/b)^5], \tag{2}$$

with an error not larger than 4%. For our case of relatively thin plates, the last term of (2) is negligible. The twist angle at the end of a rectangular plate thus is

$$\theta = \frac{3\,T}{G\,t^3}\;\frac{a}{b - 0.630\,t} \tag{3}$$

For the trapezoidal plate we have, first, a varying cross section with $b(y) = b_1 + (b_2-b_1)y/a$. Second, we must write (3) in differential form. Using the abbreviations $b_o = b_1 - 0.63t$, $w = (b_2-b_1)/b_o$, and $Q = 3T/(G\,t^3 b_o)$, we have

$$d\theta = Q\,\frac{dy}{1 + wy/a}\,. \tag{4}$$

Third, this is integrated to yield

$$\theta(y) \quad = \quad Q\,\frac{a}{w}\,\ln(1 + wy/a). \tag{5}$$

These equations hold under the conditions that the plate is long, $a >> b$, and that both sides b_1 and b_2 are held on straight lines. Actually, telescope surface plates are somewhat but not very long, mostly with $a/b \approx 2$; and a side is not held as a whole but is held only at its corners, which could result in a small S-shaped deviation from a straight line. A further condition is that the rigidity G is isotropic, the same in all directions, which is true for a solid plate but may be different for honeycombs and skin-and-rib plates.

We now introduce three simplifications. First, we assume that equation (5) is a satisfactory approximation for all $a \geq b$. Second, we assume that z is linear in x for all y, meaning $z(x,y) = x\,\theta(y)$, with $\theta(y)$ from (5). We call z_o = corner difference = $z(A) - z(B)$ in Figure 1. The shape of the twisted solid plate then is

$$z(x,y) \quad = \quad z_o\,\frac{x}{b_2}\,\frac{\ln(1+wy/a)}{\ln(1+w)}\,. \tag{6}$$

As a third simplification, we assume tentatively that the trapezoidal plate may be approximated by a rectangle, meaning that $\ln(1+w)$ may be replaced by w, which is permissible if $w = (b_2-b_1)/(b_1-0.63t) << 2$. Equation (6) then turns into the easiest possible one of all twisted shapes:

$$z(x,y) \quad = \quad z_o\,\frac{xy}{b_2 a}\,. \tag{7}$$

In addition to (6) and (7), another relevant shape should be mentioned. If the plate has a strong rib structure underneath, it will mostly have dominant long radial ribs and short perpendicular ones. In this case all radial lines (meeting at the telescope's center) will have only linear translations to a very high approximation, and the resulting shape then is, with $w_o = (b_1-b_2)/b_1$,

$$z(x,y) = z_o \frac{x}{b_2} \frac{(1+w_o)\ y/a}{1 + w_o y/a} . \tag{8}$$

We have now three different descriptions for the twisted surface shape $z(x,y)$ of trapezoidal plates: equation (6) as our approximation for a solid plate, equation (7) which is of the simplest possible form, and equation (8) which holds for plates with strong radial ribs. We call z_6 the value of $z(x,y)$ from equation (6), and similar for z_7 and z_8. It can be shown that for all possible x, y and w, there is always

$$z_8 \geq z_6 \geq z_7. \tag{9}$$

All three equations become identical for $w \to 0$, for the approach to rectangular plates. Our next question thus is: how different are these three shapes from each other in realistic cases, and mainly: could the easiest one, (7), be considered a satisfactory approximation for both the other two?

As a practical example, we have chosen the surface arrangement of Figure 2 (NRAO design for a 25-meter telescope for 1 mm wavelength). For simplicity, we neglect the difference between b_o and b_1, thus $w = w_o$. If the central hole for the equipment cabin has a radius equal to the plate length, then $w \approx 1/i$, with i = ring number. We ask for the root-mean-square differences, rms$(z_6 - z_7)$ and rms $(z_8 - z_7)$, integrating in x and y over the whole plate. The results are shown in Table 1.

TABLE 1

Checking the simplest approximation (7) against the twisted shape of trapezoidal solid plates (6), and against that of plates with strong radial ribs (8), for the surface plates shown in Figure 2.

Ring	$rms(z_6 - z_7)/z_o$	$rms(z_8 - z_7)/z_o$
1	0.0135	0.0264
2	.0089	.0176
3	.0066	.0132
4	.0053	.0105
5	.0044	.0088
6	.0038	.0075
7	.0033	.0066
8	.0029	.0059

It is very pleasing to learn from Table 1 that the differences between our three descriptions of twisted plates are only rather small, mostly one percent or less of the corner twist z_o, and less than three percent even in the most extreme case. Considering these differences completely negligible, we conclude that the simple form of equation (7) is a satisfactory approximation for both the solid plate and the skin-and-rib plate.

III. LEAST-SQUARES ADJUSTMENT OF PLATES

A four-cornered plate has four degrees of freedom which can be defined in two ways. For an "external" definition, the vertical adjustments at the corners (A,B,C,D) yield one degree each. For an "internal" definition, we say that any offsets A,B,C,D will cause a change of the surface shape which can be described by four independent parameters:

$$\left.\begin{array}{l} p_1 = \ell = \text{parallel vertical lift} \\ p_2 = \alpha = \text{tilt angle about y-axis} \\ p_3 = \beta = \text{tilt angle about x-axis} \end{array}\right\} \quad \text{rigid-body translation} \tag{10}$$

$$p_4 = \tau = \text{internal twist} \qquad \text{non-planar deformation}$$

We ask for a least-squares procedure yielding these four parameters from the measurements, and for a transformation of the result into the wanted amounts of adjustment at the four plate corners. We presuppose that the desired paraboloid has already been defined.

After the telescope surface has been measured, we call $z(x,y)$ the vertical surface deviation

$$z(x,y) = (\text{measured height}) - (\text{desired paraboloid}) \tag{11}$$

where x and y refer to a system within each plate as shown in Figure 1,b, letting the y-axis coincide with the line of symmetry such that, for simplicity,

$$\overline{x} = \overline{xy} = \overline{xy^2} = 0 \ . \tag{12}$$

But the height of the x-axis is left arbitrary, because two different means of y will be needed: the ordinary average $\overline{y}$, and an x^2-weighted one, called y_o:

$$y_o = \overline{x^2y} \ / \ \overline{x^2} \tag{13}$$

where $y_o = \overline{y}$ for rectangles, but $y_o > \overline{y}$ for trapezoids.

Averages shall always be defined by a summation over all measured points of this plate (as opposed to an integration over its surface). If the points are not equal-distributed, all averages should use weights equal to the area represented by the point. If there are only a few long plates along the telescope radius, the weights should also include the illumination taper.

Using the parameters (10), the measured deviations (11) can be written as

$$z(x,y) = \ell + \alpha x + \beta (y-\overline{y}) + \tau x (y-y_o) + R(x,y) \pm \varepsilon \qquad (14)$$

where $R(x,y)$ is the remaining residual describing the plate's internal bumpiness, and ε is the measuring error. If written this way, all four parameters are "decoupled". Without subtraction of $\overline{y}$, a tilt $\beta \neq 0$ would also change the height ℓ, whereas subtraction of $\overline{y}$ decouples β and ℓ; similarly, the subtraction of y_o is needed to decouple τ and α.

The advantage of decoupled parameters is that they are obtained from the measurements in a most direct way, without inverting a matrix (instead of "decoupled" we could have said "orthogonal"). In the present case, the least-squares solution (minimum residual) is obtained, as can be shown, in the following way

$$\begin{aligned} p_1 &= \ell = \overline{z} \\ p_2 &= \alpha = \overline{xz} \,/\, \overline{x^2} \\ p_3 &= \delta = (\overline{yz} - \overline{y}\,\overline{z}) \,/\, (\overline{y^2} - \overline{y}^2) \\ p_4 &= \tau = (\overline{xyz} - \overline{xz}\, y_o) \,/\, (\overline{x^2y^2} - \overline{x^2}\, y_o^2). \end{aligned} \qquad (15)$$

The wanted corner adjustments Δz we define by $z(\text{after}) = z(\text{before}) + \Delta z$, and we call x_k and y_k the corner coordinates, with $k = A,B,C,D$. The best-fit adjustments then are, using all four degrees of freedom,

$$\Delta z_k = -\{\ell + \alpha x_k + \beta (y_k - \overline{y}) + \tau x_k (y_k - y_o)\} . \qquad (16)$$

We want to check the single steps of improvement by calculating their remaining residuals. First, we define a set of "uncorrected" residuals U_ν which would remain after stepwise removal of one of the parameters (14) at a time, with still the <u>same</u> measuring error ε on each point:

$$
\begin{aligned}
U_o^2 &= \overline{z^2} \\
U_1^2 &= U_o^2 - \ell^2 \\
U_2^2 &= U_1^2 - \alpha^2\,\overline{x^2} \\
U_3^2 &= U_2^2 - \beta^2(\,\overline{y^2} - \overline{y}^2) \\
U_4^2 &= U_3^2 - \tau^2(\,\overline{x^2y^2} - \overline{x^2}\,y_o^2)
\end{aligned}
\tag{17}
$$

Second, we define the "corrected" residuals R_ν as those which would remain after stepwise removal, if the plate would after removal be remeasured at the same points <u>without</u> any new measuring errors, but still keeping in mind that the removed parameters are erroneous because of our original measuring errors of (14), whose rms value we call ε_o. (We presuppose that ε_o is known; the best way to determine ε_o is to measure the telescope twice.) Omitting a lengthy derivation, the result is, for $\nu = 0 \ldots 4$,

$$R_\nu = \left\{ U_\nu^2 - \frac{N-2\nu}{N}\,\varepsilon_o^2 \right\}^{1/2} . \tag{18}$$

Third, we call n the number of measured points on a plate. If no weights are used in the averages of (13), (15) and (17), then $N = n$. If weights W_j are used, then the equivalent number of unweighted points is, as can be shown,

$$N = \left\{ \sum_{j=1}^{n} W_j \right\}^2 \Big/ \sum_{j=1}^{n} W_j^2 . \tag{19}$$

For a verification of (18), consider the two extreme cases. For $N \to \infty$, we have $R_\nu^2 = U_\nu^2 - \varepsilon_o^2$; and for the minimum, $N = n = \nu$, we have $U_\nu = 0$ and $R_\nu = \varepsilon_o$; both as to be expected.

Fourth, we define stepwise improvements as

$$I_\nu = [1 - (R_\nu/R_{\nu-1})]\ 100\%, \tag{20}$$

and the total improvement as

$$I_t = [1 - (R_4/R_o)]\ 100\%. \tag{21}$$

When is the removal of a parameter statistically justified? Consider for example the case where the plate has actually no tilt about the y-axis, $p_2 = \alpha = 0$, but where the measuring errors have feigned an erroneous tilt $\alpha \neq 0$. Its removal according to (17) will still decrease the uncorrected residual by definition. This is different for the corrected residuals, where going from ν-1 to ν will give $R_\nu > R_{\nu-1}$ if the measuring error ε_o is sufficiently large. We thus regard a positive improvement as a criterion for a justified removal, using the convention:

$$\text{If} \quad I_\nu \leq 0, \quad \text{then set} \quad p_\nu = 0, \tag{22}$$

and these cases are not counted for the value of ν in the term N-2ν of equation (18).

Finally, a note of caution. The definition of the residuals (18) assumed the plate, after adjustment, to be remeasured without errors at the <u>same points</u> as before; whereas regarding the performance of the adjusted telescope, the receiver will, so to say, remeasure without error <u>all</u> of the surface <u>area</u>. The latter may yield a somewhat larger residual than the former, if the distribution of measuring points was not dense enough to catch the shortest surface bumps. For treating this effect one would have to know the Fourier power spectrum of the plate's surface shape, which we will not follow up. Thus, equation (18) is used for the residuals, assuming a sufficiently dense measuring grid.

IV. RESULTS OF MEASUREMENTS

Table 2 shows the construction types and sizes of seven experimental surface plates, manufactured (Plate No. 1 and 2) or purchased (No. 3 ... 7) by NRAO in connection with the design of two millimeter-wave telescopes. All plates are trapezoidal and are supported at their four corners. Plates No. 1 ... 4 are described in detail by Findlay and von Hoerner [6]. All plates have been measured with a radial grid similar to Fig. 1b. (Some additional measurements were done with a rectangular grid, without significant differences.)

TABLE 2

Types and sizes of experimental surface plates.
With length a, width b, thickness t, number of measured points n, and $w_o = (b_2 - b_1)/b_1$. See Figure 1a.

Plate No.	Telescope Design	Plate type	Name	size (cm) a	b_1	b_2	t	w_o	n
1	65-m	skin + ribs riveted	II A	191.1	68.9	74.8	10.16	0.085	56
2	65-m	(same)	III	191.1	68.9	74.8	10.16	.085	56
3	65-m	skin + ribs cast alumin.	PF #1	189.1	65.4	74.3	7.62	.136	60
4	65-m	(same)	PF #2	189.1	65.4	74.3	7.62	.136	60
5	25-m	skin + ribs cast alumin.	ANF #1	152.4	72.7	91.4	7.62	.257	60
6	25-m	(same)	ANF #4	152.4	72.7	91.4	7.62	.257	60
7	25-m	honeycomb	H #2	151.5	72.7	91.6	7.62	.260	60

First, we want to check how well the simple equation (7) will approximate the actual shape of a twisted plate. The plate is corner-supported with all four corners adjusted level, and heights h_1 are measured for the n grid points (using scale and level). Then one corner is lifted by z_o = 3.0 mm, the other three corners held level, and heights h_2 are measured. Finally the heights h_3 are measured with all four corners leveled again. The twist deformation then is

$$z = h_2 - (h_1 + h_3)/2. \tag{23}$$

The single measuring error was determined as ε_1 = 12.7 μm = 0.0127 mm. The error of the combination (23) then is

$$\varepsilon = \varepsilon_1\sqrt{3/2} = 15.6\ \mu m. \tag{24}$$

The n values of z and their coordinates x and y are submitted to a program which calculates the ten averages and four parameters of equation (15) and the residuals of (18). The results are shown in Table 3; most plates were measured twice, lifting a corner of the long side (A or B), or one of the short side (C or D). R_o is the rms deformation of the plate, given in percent of z_o; and R_4 is the rms deviation of the measured shape from the shape of equation (7). If equation (7) would hold exactly, then R_o = 1/3 = 33.3% as can be shown, and R_4 = 0 by definition. The agreement may be called quite satisfactory: R_o deviates from 1/3 by only 2% of z_o in the average, and the average of R_4 is (1.15 ± 0.25)% of z_o, which agrees well enough with the theoretical expectations of Table 1, and which is completely negligible for all practical purposes. Thus, equation (7) and the least-squares procedure (15) based on it are satisfactory approximations.

TABLE 3

The shape of twisted plates.

R_o = rms deformation with one corner lifted, R_4 = rms deviation of measured shape from equation (7).

Plate No. (Table 2)	Corner lifted	Residuals (in % of z_o) R_o	R_4
4	B	36.1	0.24
4	D	36.3	.33
5	A	30.52	.83
6	B	37.91	1.88
6	D	34.48	1.39
7	A	37.48	1.53
7	C	34.28	1.71

Second, for the experimental plates of Table 2, we want to find the gradual improvements of equation (20) resulting from the stepwise removal of the four parameters as described in (17). For Plates 1, 2 and 3 we have only one set of measurements, thus $\varepsilon = \varepsilon_1 = 12.7$ µm; for Plate 5 we have two sets and use the average, where $\varepsilon = \varepsilon_1/\sqrt{2} = 9.0$ µm; and for Plates 4, 6 and 7 we have four sets and use the average, with $\varepsilon = \varepsilon_1/2 = 6.4$ µm.

TABLE 4

Stepwise removal of the four adjustment parameters, and resulting improvement of the residuals, for the plates of Table 2.

Plate No.	Residuals (µm) R_o	R_4	Improvements (%) I_1	I_2	I_3	I_4	I_t
1	53.4	51.1	1.73	0.43	0.55	1.60	4.24
2	58.8	49.8	12.67	.45	0	2.62	15.34
3	78.5	48.7	21.51	18.65	2.86	0	37.98
4	51.0	30.4	13.17	30.65	0	1.04	40.41
5	40.9	36.7	4.35	0	2.17	3.95	10.12
6	29.5	27.2	0	5.64	.65	1.63	7.77
7	291.0	174.1	39.53	.47	.54	.02	40.15
		rms =	18.4	13.7	1.41	2.03	27.0

Table 4 shows the results. R_o is the original rms deviation of the true-corner adjusted plate from the design paraboloid; R_4 is the rms deviation after the final 4-parameter least-squares adjustment, and I_t is the total improvement thus achieved. For the seven plates, we have rms(I_t) = 27%, a rather substantial improvement. From the single improvements I_ν we learn that I_1 is the largest, which means than an over-all bulging up or down of the plate is its largest mode of distortion; $I_2 >> I_3$ may be explained by the dominance of strong radial ribs which are present in Plates 3, 4, 5, 6; and I_4 = 2% is small but still significant.

Third, we ask for the gradual improvements again, but this time for plates being on an actual telescope which needs readjustment. The surface of the 140-ft telescope has been measured by J. Findlay with a new method (unpublished); the data are still preliminary but sufficient for our present purpose, with about ε = 0.30 mm. The 140-ft has only three rings of relatively long plates (panels), and we select arbitrarily the North and South sectors, treating a sample of six plates. Ring 1 is at the center, Ring 3 at the rim.

TABLE 5

Same as Table 4, but for six plates on the 140-ft telescope which needs readjustment.

Sector	Ring	Residuals (mm)		Improvements (%)				
		R_o	R_4	I_1	I_2	I_3	I_4	I_t
N	1	0.87	0.47	40.5	8.8	0	0	54.7
N	2	1.53	.49	43.4	13.2	15.5	23.4	68.2
N	3	2.06	.95	1.6	23.5	14.0	28.6	53.8
S	1	.83	.52	30.2	0	10.4	0	37.4
S	2	.99	.40	.3	18.1	32.0	26.3	59.1
S	3	1.28	1.05	1.5	15.2	.4	1.4	17.9
rms =		1.33	0.69	27.2	15.1	16.2	18.5	49.7

Results are shown in Table 5. Since the starting condition of R_o is not any more the true-corner adjustment as before, but is now a plate which is maladjusted at all four corners (including a non-planar internal twist), all improvements of Table 5 are larger than those of Table 4 (especially I_4). Provided that the data used are representative of the actual surface of the whole telescope, and that the mechanical adjustments can be done without significant errors, the comparison of R_o = 1.33 mm with R_4 = 0.69 mm would mean that a proper readjustment could improve the surface accuracy by almost a factor of two. And rms(I_4) = 18.5% shows drastically the importance of using the fourth degree of freedom, the internal twist.

It is a pleasure to thank J. Findlay for the 140-ft data, and L. King for the literature search.

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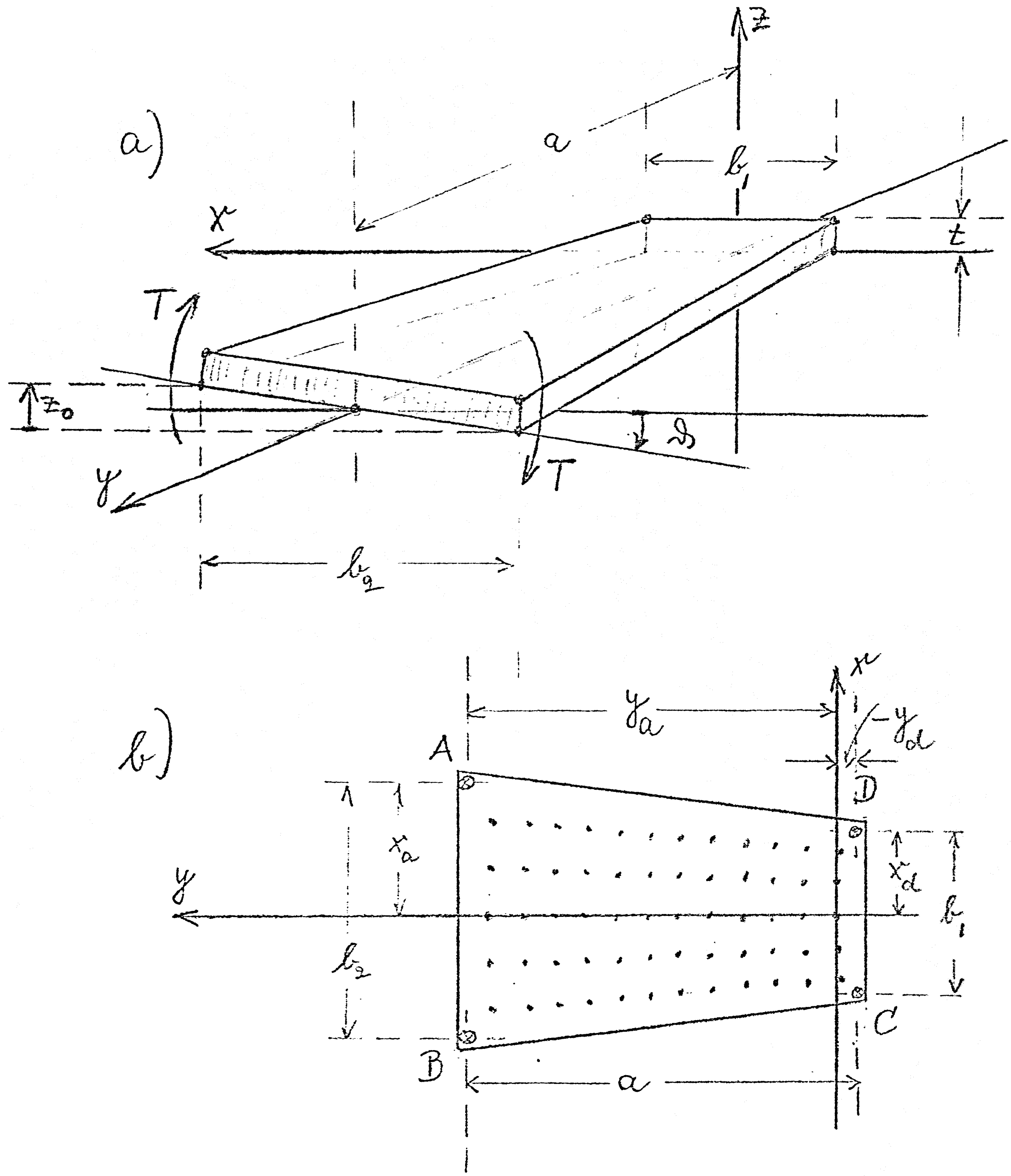
FIGURE CAPTIONS

Fig. 1. Trapezoidal surface plate.

a) Torsional twist of a beam with rectangular cross section ($t \times b$); T = twisting moment, θ = twist angle, b = variable width.

b) Top view of plate; showing coordinate system, radial grid of measuring points, and corner adjustment points (A, B, C, D).

Fig. 2. Example of trapezoidal surface plates, in radial pattern with eight rings; showing 1/4 of telescope.



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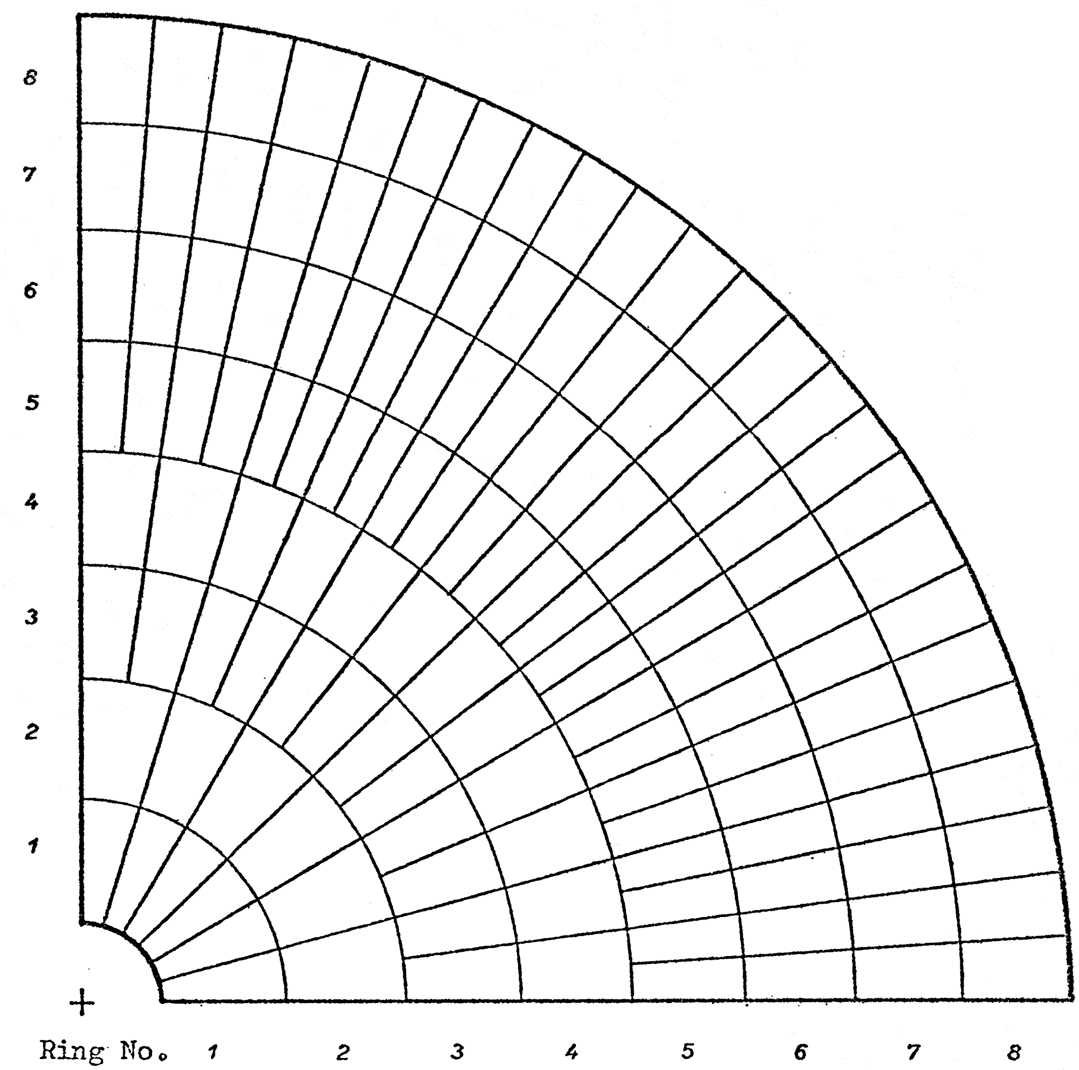

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