EVLA Memo 177
Polarimetric calibration and dynamic range issues

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Introduction

This memo considers some dynamic range issues resulting from limitations in the polarimetric calibration solutions used in typical observations. Whereas we focus on “traditional” radio interferometric telescopes such as the VLA, ATCA or ALMA, the analysis approach can be used for other interferometer arrays. The memo is mostly concerned with the principles: rather than addressing real data, this memo focuses on analysis. The discussion generally applies to any feed system, including linearly or circularly polarized feeds and arrays of heterogeneous (i.e. dissimilar) feeds. Generally we consider the calibration information available from an observation of a point source calibrator at a single frequency at the field center at a single instant in time. However the added constraints provided by parallactic rotation are also considered. We will consider the effect when this calibration information is applied to the data of a target source. For brevity, we will assume that the Stokes total intensity of the calibrator source is 1 Jy, and that the target source is also 1 Jy.

This calibration will have systematic imperfections. We are interested in the effect of these imperfections when the calibration information is used to correct an observation (potentially a long observation). Although the simplicity of this calibration seems limiting, there is some discussion on improvements to the calibration when more calibration observations or sources are used. Additionally the principles wideband calibration tend to be straightforward generalizations of the single frequency calibration. Widefield polarimetry introduces new issues, which we will not address here.

This memo builds on the Jones matrix-based analysis in Sault, Hamaker & Bregman (A&AS 1996), hereafter called SHB. In their analysis they note after the observation of a single point source with unknown polarimetric properties, six real-valued parameters (‘degrees of freedom’) remain unsolved for in the antenna-based Jones matrices (SHB mention a seventh parameter, which is the flux scale. As this memo assumes a 1 Jy calibrator, there is no uncertainty of the flux scale). It is these six parameters that are of interest in this memo. These parameters fall into two categories:

- Instrument-based parameters: Three parameters result from only a single observation of the calibrator being used. In some circumstances these parameters manifest themselves as a ‘leakage offset’ and an “RL” or “XY” or ‘cross’ phase.
- Calibrator-based parameters: Three parameters result from not knowing the polarization of the calibrator. Knowing the three Stokes parameters of the calibrator does pin down these calibration parameters.

It is important to understand the effect of these two categories of calibration parameters as they systematically affect all the calibrated data. They will affect the polarimetric purity and dynamic ranges of resultant images.

For the instrument-based parameters, these are in general in some sense hard to measure, and so their values are more uncertain in the calibration. As SHB note, three observations where the (I, Q, U, V) Stokes vectors are linearly independent are needed to pin them down. Amongst other possibilities, these observations could be achieved with three well-chosen cuts of a known linearly-polarized calibrator with antennas which show parallactic rotation (e.g. traditional alt-az antennas). However this requires that the antenna gains do not change between cuts. If the gains do, an additional unpolarized calibrator is needed to track antenna gains. This was one of the approaches used in Memo 170 to determine the EVLA leakage and cross-phase offsets. Given that their measurement takes multiple observations, Jones-matrix self-calibration of these parameters will be problematic if they vary with time.

For the calibrator-based parameters, even if the Stokes parameters of the calibrator are nominally known (or solved for), it is still important to understand what the effect is of imperfections in this knowledge. When aiming for very high dynamic range images, even small imperfections can be important.

Some formalities

The analysis will use coherency and Jones matrices. For calibrator coherency matrix

$$\boldsymbol{C} = \begin{pmatrix} V_{pp} & V_{pq} \\ V_{qp} & V_{qq} \end{pmatrix},$$

for baseline $k - l$ (with antenna Jones matrices of $\boldsymbol{J}_{\mathrm{a},k}$ and $\boldsymbol{J}_{\mathrm{a},l}$), the measured calibrator coherency matrix C'_{kl} will be

$$\boldsymbol{C}'_{kl} = \boldsymbol{J}_{\mathrm{a},k} \boldsymbol{C} \boldsymbol{J}^H_{\mathrm{a},l}.$$

It is convenient to express some of the analysis in terms of the three so-called 2×2 Pauli spinor matrices (called simply spinors here):

$$\begin{aligned} \boldsymbol{\sigma}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

These matrices have a number of properties which make them useful in our analysis, including:

- The matrices are equal to their inverse and their Hermitian (i.e. $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i^{-1} = \boldsymbol{\sigma}_i^H$). They share this property with the identity matrix, $\mathbb{1}$.
- The product of any two spinors is $\pm i$ times the third.
- The matrices of interest to us are readily expressed using spinors.

Using spinors, the coherency matrix of our 1 Jy calibrator, with polarization parameters α_1, α_2 and α_3, is given by

$$\boldsymbol{C} = \mathbb{1} + \alpha_1 \boldsymbol{\sigma}_1 + \alpha_2 \boldsymbol{\sigma}_2 + \alpha_3 \boldsymbol{\sigma}_3.$$

The following analysis holds for arbitrary dual feed systems where the feeds are at least partially orthogonal. It does not assume a homogeneous system of nominally orthogonal feeds. However the relationship between α_1, α_2 and α_3 and Stokes parameters depends on a polarimetric representation adopted.

This requires some explanation: fractional Stokes Q, U and V can be thought of as coordinates in a three-dimensional space. The Poincaré sphere is a useful visualization of this space. The parameters α_1, α_2 and α_3 can represent any other rectilinear coordinate system in the same space, with the same origin[1] but which is potentially rotated/tilted relative to the usual Poincaré sphere axes. When a telescope has nominally orthogonal feeds of a particular type, then there is a "natural" convention that would commonly be adopted to map between α_1, α_2 and α_3 and Stokes parameters. Such a choice will make the antenna Jones matrices diagonally dominant. For example, the natural convention for circularly-polarized feeds would have α_1, α_2 and α_3 corresponding to (fractional) Stokes Q, U and V respectively. The natural convention for linearly-polarized feeds would have α_1, α_2 and α_3 corresponding to U, V and Q respectively.

There is no in-principle reason that the natural convention be used. Indeed for an array of heterogeneous feeds, there is possibly no natural convention. The result if the 'natural' convention is not followed is simply that the off-diagonal terms of the antenna Jones matrices are comparable in magnitude to the on-diagonal terms. In a sense, a coordinate conversion matrix would be folded into the antenna Jones matrix.

When generality is desired, we will refer to polarization states as $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$. In some instances, particularly related to parallactic rotation, the actual Stokes parameters are the important measure.

Instrument-based parameters

For the time being, we assume that the Stokes parameters of the calibrator are known, i.e. that we know the coherency matrix of the calibrator.

We quote a result from SHB: For a given calibrator coherency matrix $\boldsymbol{C}$, there exists a family of Jones matrices, $\boldsymbol{J}_{\mathrm{err}}(\phi, \theta, \zeta)$ (i.e. parameterized by ϕ, θ and ζ) for which

$$\boldsymbol{C} = \boldsymbol{J}_{\mathrm{err}} \boldsymbol{C} \boldsymbol{J}_{\mathrm{err}}^H.$$

Specifically, if we were to replace all the antenna Jones matrices with ones multiplied by a specific instance of $\boldsymbol{J}_{\mathrm{err}}$

$$\boldsymbol{J}'_{\mathrm{a},k} = \boldsymbol{J}_{\mathrm{a},k} \boldsymbol{J}_{\mathrm{err}}$$

then the effect on the observed coherency matrix would be

$$\begin{aligned} \boldsymbol{C}' &= \boldsymbol{J}'_{\mathrm{a},k} \boldsymbol{C} \boldsymbol{J}'^H_{\mathrm{a},l} \\ &= \boldsymbol{J}_{\mathrm{a},k} \boldsymbol{J}_{\mathrm{err}} \boldsymbol{C} \boldsymbol{J}_{\mathrm{err}}^H \boldsymbol{J}_{\mathrm{a},l}^H \\ &= \boldsymbol{J}_{\mathrm{a},k} \boldsymbol{C} \boldsymbol{J}_{\mathrm{a},l}^H. \end{aligned}$$

That is, the result is the same as if $\boldsymbol{J}_{\mathrm{err}}$ had not been introduced at all. $\boldsymbol{J}_{\mathrm{err}}$ has no discernible effect. So from this observation of the calibrator it is not possible to estimate $\boldsymbol{J}_{\mathrm{err}}$, or specifically ϕ, θ and ζ. However even

[1] The origin corresponds to an unpolarized source

though it does not affect this calibrator observation, $\boldsymbol{J}_{\rm err}$ does in general affect the results derived from calibrated data of the target source. In general it does affect the dynamic range and polarimetric purity of the system.

SHB quote the form of $\boldsymbol{J}_{\rm err}$ when the calibrator is unpolarized (i.e. $\boldsymbol{C} = \mathbb{1}$):

$$\boldsymbol{J}_{\rm err} = \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\zeta} & 0 \\ 0 & e^{-i\zeta} \end{pmatrix}$$

(for some added consistency with the notation here, the order of the matrices has been changed from SHB - this is of no importance). This can be expressed using spinors as

$$\boldsymbol{J}_{\rm err} = \boldsymbol{J}_1 \boldsymbol{J}_2 \boldsymbol{J}_3$$

where

$$\begin{aligned} \boldsymbol{J}_1 &= \mathbb{1}\cos\phi + i\boldsymbol{\sigma}_1 \sin\phi, \\ \boldsymbol{J}_2 &= \mathbb{1}\cos\theta + i\boldsymbol{\sigma}_2 \sin\theta, \\ \boldsymbol{J}_3 &= \mathbb{1}\cos\zeta + i\boldsymbol{\sigma}_3 \sin\zeta. \end{aligned}$$

A small amount of manipulation will show that, for these matrices

$$\boldsymbol{J}_i^{-1} = \boldsymbol{J}_i^H$$

and so

$$\boldsymbol{J}_i \mathbb{1} \boldsymbol{J}_i^H = \mathbb{1}.$$

Hence these have no discernible effect on the observation of a unpolarized source.

For a homogeneous array, ζ corresponds to cross phase. For small angles θ and ϕ, the combination of $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ approximates a leakage offset, $\theta + i\phi$. As a (perhaps misleading) shorthand, we will refer to the effect of the Jones matrices $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ as 'leakage offsets', even though we are not invoking small angle approximations.

For a general calibrator of known polarization (which can be strongly polarized), there still remain three parameters that an observation of a single calibrator cannot determine. However these parameters are not so easily identified as cross phase or 'leakage offsets', even for a homogeneous array. To consider this case, we assume a calibrator with fractional polarization of α and coherency matrix $\boldsymbol{C} = \mathbb{1} + \alpha\boldsymbol{\sigma}$. Here $\boldsymbol{\sigma}$ is one of the spinors. There is no loss of generality by assuming a single spinor: we can always find a polarization basis where we can express the source properties in this fashion[2]. For this coherency matrix, some algebra will show that

$$\boldsymbol{J}_{\rm err} = \boldsymbol{J}_{\rm C}^{-1} \boldsymbol{J}_1 \boldsymbol{J}_2 \boldsymbol{J}_3 \boldsymbol{J}_{\rm C},$$

where, for $\psi = \frac{1}{2}\sin^{-1}\alpha$,

$$\begin{aligned} \boldsymbol{J}_{\rm C} &= (\cos 2\psi)^{-\frac{1}{2}} \left(\mathbb{1}\cos\psi - \boldsymbol{\sigma}\sin\psi\right), \\ \boldsymbol{J}_{\rm C}^{-1} &= (\cos 2\psi)^{-\frac{1}{2}} \left(\mathbb{1}\cos\psi + \boldsymbol{\sigma}\sin\psi\right). \end{aligned}$$

That is, for a calibrator with arbitrary polarization, there is a $\boldsymbol{J}_{\rm err}$ family that cannot be determined from an observation. Note that the parameterization of this family continues to be ϕ, θ and ζ: the parameter ψ is not a parameterization of this family, but a characteristic of the calibrator. For a weakly polarized calibrator, $\boldsymbol{J}_{\rm C}$ approximates to

$$\boldsymbol{J}_{\rm C} \approx \mathbb{1} - \frac{1}{2}\alpha\boldsymbol{\sigma}.$$

[2] Expressing this another way, we can always introduce an (invertible) coordinate rotation and work in a rotated coordinate space.

More generally, for a weakly polarized source with coherency matrix $\mathbb{1} + \alpha_1\boldsymbol{\sigma}_1 + \alpha_2\boldsymbol{\sigma}_2 + \alpha_3\boldsymbol{\sigma}_3$,

$$\boldsymbol{J}_\mathrm{C} \approx \mathbb{1} - \frac{1}{2}(\alpha_1\boldsymbol{\sigma}_1 + \alpha_2\boldsymbol{\sigma}_2 + \alpha_3\boldsymbol{\sigma}_3).$$

It is noteworthy that

$$\begin{aligned} \boldsymbol{J}_\mathrm{C}(\mathbb{1} + \alpha\boldsymbol{\sigma})\boldsymbol{J}_\mathrm{C}^H &= (\cos 2\psi)\mathbb{1}, \\ \boldsymbol{J}_\mathrm{C}^{-1}\boldsymbol{J}_\mathrm{C}^{-1H} &= (\cos 2\psi)^{-1}(\mathbb{1} + \alpha\boldsymbol{\sigma}), \end{aligned}$$

and so

$$\boldsymbol{J}_\mathrm{err}(\mathbb{1} + \alpha\boldsymbol{\sigma})\boldsymbol{J}_\mathrm{err}^H = (\mathbb{1} + \alpha\boldsymbol{\sigma}).$$

In some sense the Jones matrix $\boldsymbol{J}_\mathrm{C}$ can be thought of as 'converting' the coherency matrix of the calibrator to an identity matrix, after which the results of the unpolarized calibrator can be used. $\boldsymbol{J}_\mathrm{C}$ can be thought of in some sense as the inverse square root of the calibrator coherency matrix.

What does this mean in practice? Not much more than the case of the unpolarized calibrator – it is simply a statement to reinforce the point that the issue of underdetermined calibration is not avoided by using a strongly polarized calibrator.

A note on cross phase and leakage offset

As mentioned above, for a homogeneous array observation of an unpolarized calibrator, $\boldsymbol{J}_\mathrm{err}$ amounts to an unknown cross phase and unknown feed 'leakage offset'. It may be a good assumption that the feeds are well engineered and that, averaged over all the antennas, they follow their nominal performance [3]. That is, the sensible choice for 'leakage offset' is one that minimizes the off-diagonal response of the antenna Jones matrices.

The cross phase is usually not constrained by the engineering design: it will be set by small variations in the electrical lengths of the signal paths of the system. There may be no good assumption to set a value for this. Interestingly the difficulty of estimating cross phase is only a characteristic of the polarimetric response of homogeneous arrays when observing unpolarized calibrators. If the calibrator is (strongly) polarized and/or the array feeds (significantly) heterogeneous, then assuming the array feeds on average follow their nominal performance is usually sufficient to constrain the three parameters of $\boldsymbol{J}_\mathrm{err}$. This point was made by SHB and re-enforced by Hamaker (A&AS, 2000). The case of using a strongly polarized calibrator is tie down cross phase is commonly used. An example of using a heterogeneous array is given by Weiler (A&A, 1973), who showed that the so-called 'crossed-linear' configuration (linear feeds where some feeds are rotated by 45° relative to the others) and assumption of good feeds on average is adequate to determine the instrumental cross phase.

Effect on calibrated data of the instrument-based parameters

Having determined a calibration, which is uncertain to a Jones factor of $\boldsymbol{J}_\mathrm{err}$, the important question is what effect this has when this calibration is used with an observation of the target source. The target can have

[3] That the average leakage offset is zero is an assumption that should be questioned. All the feeds of an array probably share the same engineering design, and so while they may show a manufacturing variation, they probably share a design bias.

arbitrary polarization. What are the artifacts resulting in the data of the target from the unknown $\boldsymbol{J}_{\rm err}$? It is important to understand dynamic range issues and polarimetric purity. We will assume that the calibration is perfect within the limits of the unsolvable $\boldsymbol{J}_{\rm err}$ factor. It is useful to treat the unpolarized and polarized calibrator cases separately.

Unpolarized calibrator

The unpolarized calibrator case is partially discussed in SHB and more fully in Hamaker (2000). For this case, $\boldsymbol{J}_{\rm err}$ is the product of $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$. None of these have any affect on the measured value of Stokes I in the target source. Mathematically

$$(\boldsymbol{J}_1\boldsymbol{J}_2\boldsymbol{J}_3)\mathbb{1}(\boldsymbol{J}_1\boldsymbol{J}_2\boldsymbol{J}_3)^H = \mathbb{1}.$$

Reiterating, the calibrated measurement of Stokes I of the target source is unaffected by $\boldsymbol{J}_{\rm err}$. The underdetermined calibration does not limit the Stokes I dynamic range, or show as a closure error or the like (the V_{pp} and V_{qq} correlations are potentially affected by a leakage of the $\boldsymbol{\sigma}_3$ polarization state, but this cancels out when they are summed to form Stokes I).

Polarimetric purity is a different issue: the terms of $\boldsymbol{J}_{\rm err}$ can be thought of as introducing a tilt into the Poincaré sphere: $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ correspond to rotations about the three Poincaré sphere axes. In a real sense the lack of polarimetric purity can be though of as a an error in the polarization reference frame. A little algebra shows that J_1 acts to rotate ('leak') between the $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ polarization states, $\boldsymbol{J}_2$ between $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$ states and finally $\boldsymbol{J}_3$ between $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ states. Importantly there is no leakage from Stokes I into the polarized measures, or the polarized measures into Stokes I.

That the calibration results in a "tilted" Poincaré sphere is an intuitively appealing outcome. Indeed perhaps it is intuitively obvious! No where in the process has there been an observation that ties the frame of the measurements to the frame of the Poincaré sphere in any way. The calibration process can ensure that a self-consistent α_1, α_2 and α_3 result, but it has no reference frame to determine the rotations needed to remove any tilt. The best that can be done is to rely on the nominal instrumental properties to minimize the tilt.

To give a readily appreciated example, consider a homogeneous array with circularly polarized feeds where we follow the 'natural' system of $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ polarization states corresponding to Stokes Q, U and V respectively. In this case ζ in $\boldsymbol{J}_3$ corresponds to the RL phase offset. That J_3 rotates between $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ states is saying an error in the RL phase causes rotation between Stokes Q and U. It does not affect Stokes I or V. For circular feeds, the 'leakage offsets' ($\boldsymbol{J}_1$ and $\boldsymbol{J}_2$) cause rotation ('leakage') between Q and V as well as U and V. Again Stokes I is unaffected.

For an equatorial telescope (or a snapshot observation with any sort of mount), leakage between the polarization states will not cause dynamic range or imaging problems: the results will be artefact-free images. However the polarimetric purity will suffer. A polarization image will be the weighted superposition of the different polarization states. The situations where this is important is somewhat limited: if the unknown leakage offset is of order 10^{-2}, then the leakage offset will be important where a source has appreciable linear and circular polarized emission, and polarimetric purity of greater than 1 part in 100 is needed.

The situation is more involved when doing an observation with significant parallactic rotation (e.g. a long synthesis with an alt-az telescope). Parallactic rotation can be visualized in the Poincaré sphere as a rotation about the tilted Stokes V axis combined with a corresponding rotation in the opposite sense of the untilted V axis. Parallactic rotation combined with leakage between linear polarization and Stokes V will introduce dynamic range errors in Q, U and V images at a fractional level below the size of the unknown leakage offset. However Stokes I continues to be unaffected.

Polarized calibrator

When using a calibrator that is polarized, the situation is more involved. $\boldsymbol{J}_{\rm err}$ is no longer easily described in terms of cross phase and 'leakage offset' even for a homogeneous array, although it approaches this as the fractional polarization of the calibrator decreases. In general the unknowns parameters manifested in $\boldsymbol{J}_{\rm err}$ cause leakage between all of I, Q, U and V. The exception is for targets for which the polarization is the same as the calibrator – which is generally not a useful exception.

Again it is possible to visualize the effect of $\boldsymbol{J}_{\rm err}$ using the Poincaré sphere. For a calibrator with coherency matrix $\mathbb{1} + \alpha\boldsymbol{\sigma}$, $\boldsymbol{J}_{\rm err}$ amounts to a shift of $-\alpha$ in the $\boldsymbol{\sigma}$ direction (along with some scale compression in that direction), followed by rotation by $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$, and then finally by a shift of $+\alpha$ in the $\boldsymbol{\sigma}$ direction (and scale correction).

For an equatorial telescope or for a snapshot observation, the polarimetric calibration will not limit the dynamic range of any resultant images. There will be no noticeable closure errors of the data. However the images will not be polarimetrically pure: images formed will be a sum of the different Stokes parameters. This is the case for Stokes I as well as Q, U and V.

The leakage between Stokes I and the polarized measures is a second order effect. For a weakly polarized source with fractional polarization of α, and assuming small instrumental error (cross phase or 'leakage offsets') θ, the leakage between I and the polarized measures is $\sim 2\alpha\theta$. For calibrator polarization of $\alpha \sim 5 \times 10^{-2}$ and instrumental error of $\theta \sim 10^{-2}$, this is likely to be a significant issue when accuracies of the target source polarized emission of better that 10^{-3} of Stokes I are needed. Conversely if the target source fractional polarized emission is $\beta = 10^{-1}$, Stokes I dynamic range may be affected above about $(2\alpha\theta\beta)^{-1} = 10^4$.

For an observation with parallactic rotation, it becomes more involved still. There will be dynamic range issues in the images caused by time-varying leakage between Stokes I and V and the linear polarized measures. But this issue should not manifest itself before dynamic ranges of Stokes I exceed $\sim (2\alpha\theta\beta)^{-1}$. There will be no measurable closure errors on instantaneous measurements of Stokes I in the same way that there are not closure errors on a time-varying source.

Effect of incorrect assumed calibrator polarization

The above analysis considered the effect of no deducing $\boldsymbol{J}_{\rm err}$ in the calibration process. We now turn to the effect of not knowing the calibrator polarization properties, or at least not knowing them perfectly. Clearly if the assumed polarization properties of the calibrator are not correct, then we will introduce systematic errors that will flow through to the calibrated target source. Even if the errors in the assumed calibrator properties are small, they still potentially affect the achievable dynamic range and polarimetric purity of any results.

Thankfully given the analysis above, this problem is not a hard one to analyze. For a calibrator with coherency matrix $\mathbb{1} + \Delta\alpha\boldsymbol{\sigma}$, we consider the case where the calibration solution process assumes it to be unpolarized. It is not limiting the generality of the result by using a source describable with a single spinor – we are just picking a convenient coordinate system. Nor is it limiting that we model it as unpolarized rather than some incorrect polarization state – this assumption can be readily relaxed, but at a cost of added complexity which does not provide any significant additional insight. Ultimately it is the magnitude of the error in the assumed polarization properties of the calibrator that is important.

So the true calibrator coherency matrix is $\mathbb{1} + \Delta\alpha\boldsymbol{\sigma}$, but it is assumed to be $\mathbb{1}$. Given this, it is relatively easy to see that each antenna k, the antenna Jones matrices estimated by the calibration process ($\boldsymbol{J}'_{\mathrm{a},k}$) are related to the true antenna Jones matrices, ($\boldsymbol{J}_{\mathrm{a},k}$) by

$$\boldsymbol{J}'_{\mathrm{a},k} = \boldsymbol{J}_{\mathrm{a},k}\boldsymbol{J}_{\mathrm{C}}^{-1}.$$

Here $\boldsymbol{J}_{\mathrm{C}}$ is as defined above (except now using $\Delta\alpha$ rather than α). As noted there, $\boldsymbol{J}_{\mathrm{C}}$ can be thought of as the inverse square root of the (true) calibrator coherency matrix:

$$\boldsymbol{J}_{\mathrm{C}}^{-1}\boldsymbol{J}_{\mathrm{C}}^{-1H} = (\cos 2\psi)^{-1}(\mathbb{1} + \Delta\alpha\boldsymbol{\sigma})$$

($\psi = \frac{1}{2}\sin^{-1}\Delta\alpha$).

If the true coherency matrix of the target is $\boldsymbol{C}$, the apparent coherency matrix with this calibration will be

$$\boldsymbol{C}' = \boldsymbol{J}_{\mathrm{C}}\boldsymbol{C}\boldsymbol{J}_{\mathrm{C}}^{H}$$

If the true target source is unpolarized (i.e. $\boldsymbol{C} = \mathbb{1}$), after calibration it will appear as

$$\begin{aligned}\boldsymbol{C}' &= \boldsymbol{J}_{\mathrm{C}}\mathbb{1}\boldsymbol{J}_{\mathrm{C}}^{H}, \\ &= (\cos 2\psi)^{-1}(\mathbb{1} - \Delta\alpha\boldsymbol{\sigma}).\end{aligned}$$

That is, the error in assumed calibrator polarization has led to the opposite of this polarization state being imprinted onto the calibrated target source. For a general error in the calibrator polarization state, some analysis shows

$$\boldsymbol{C}' = a(\mathbb{1} - (\Delta\alpha_1\boldsymbol{\sigma}_1 + \Delta\alpha_2\boldsymbol{\sigma}_2 + \Delta\alpha_3\boldsymbol{\sigma}_3)),$$

where

$$a = \big((\cos 2\psi_1)(\cos 2\psi_2)(\cos 2\psi_3)\big)^{-1}$$

That the opposite polarization state is imprinted onto the calibrated data is an intuitively appealing result. In terms of the Poincaré sphere, for the small error approximation, the error can be thought of as a shift of the sphere.

For an equatorial telescope or for a snapshot observation, the polarimetric calibration will not limit the dynamic range of any resultant images. There will be no noticeable closure errors of the data. However the images will not be polarimetrically pure: they will be a sum of the different Stokes parameters. There will be leakage between Stokes I and the $\boldsymbol{\sigma}$ state at the level of α (the two polarization states other than $\boldsymbol{\sigma}$ are not affected). If there is an error in the assumed polarization of the calibrator of 0.5%, Stokes I will typically leak through to the $\boldsymbol{\sigma}$ polarization state at $\sim 5 \times 10^{-3}$. For a target source polarization of 5%, the fractional error in Stokes I will be $\sim 2.5 \times 10^{-4}$.

For an observation with parallactic rotation, if there was an error in the assumed Stokes Q and U of the calibrator, and the target source is linearly polarized, then there will be a time-varying error introduced into Stokes I that will affect the dynamic range achievable in Stokes I. For an error in the assumed calibrator linear polarization of 0.5%, and a target source linear polarization of 5%, the errors will become important for dynamic ranges in Stokes I greater than 4,000.

Summary

The following table gives a rough guide to the expected magnitude of the errors in target source images (Stokes I and polarized emission) resulting from the different polarimetric calibration uncertainties. These errors are all expressed as a fraction of the target source Stokes I. α and β are the calibrator and target source fractional polarization respectively, $\Delta\alpha$ is the error in the assumed calibrator polarization divided by I, and θ represents the unsolved for instrumental parameter (e.g. 'leakage offset' or cross phase error).

	Stokes I	Stokes Q, U, V
Instrument based: Unpolarized calibrator	0	$\theta\beta$
Instrument based: Polarized calibrator	$2\alpha\theta\beta$	$2\alpha\theta$
Calibrator based:	$\Delta\alpha\beta$	$\Delta\alpha$

These are rules of thumb and should be treated with caution. They tend to exaggerate the case in that they assume two negative effects add up. Factors of several less than these error levels can be expected in practice. Also the errors may be quite different between Q, U and V depending on the particular calibrator and target source. The errors may not affect the science of interest. The averaging effect of parallactic rotation will likely beat down the error levels by a significant factor. When there is no parallactic rotation, the errors will be constant and will not limit dynamic range, but rather fidelity or polarimetric purity. If one has a weak target of interest in a field with a strong confusing source, dynamic range may be more important than fidelity.

Acknowledging these caveats and using typical values of $\alpha = 0.1$, $\theta = 10^{-2}$, $\Delta\alpha = 5 \times 10^{-3}$ and $\beta = 5 \times 10^{-2}$ (which may not be typical at all!), the following table gives some feel for typical magnitudes of the errors.

	Stokes I	Stokes Q, U, V
Instrument based: Unpolarized calibrator	0	5×10^{-4}
Instrument based: Polarized calibrator	10^{-4}	2×10^{-3}
Calibrator based:	2.5×10^{-4}	5×10^{-3}