EVLA Memo 71

Mathematic basis of an array configuration optimization minimizing side lobes.

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Abstract

Some times ago the algorithm of optimization of an array configuration minimizing side lobe was designed ([1]). The math basis of the algorithm, although being correct, still was not completely apparent in details. The purpose of this memo is to clarify the math basis of the algorithm of an array configuration optimization minimizing side lobe.

The purpose of an array configuration optimization minimizing side lobe is minimizing the worst side lobe of the Point Spread Function (PSF) - the response of the antenna array to the point source. Having found the direction to the worst side lobe ($\vec{e}$) we should want to diminish the value of $PSF(\vec{e})$ finding the optimal position of the array elements. To do that we have to answer on the following questions:

Q1. What direction should each of the N antennas of the array be moved to decrease the value of $PSF(\vec{e})$

Q2. How much should each of the N antennas of the array be moved along the found direction to decrease the value of $PSF(\vec{e})$?

Having known the answer on these two questions, we can move all N antennas of the array to the new positions and as a result get lower value of $PSF(\vec{e})$. But another side lobe (may be even bigger) can appear at other direction. So we should move all N antennas of the array to the new positions again following the known answers on the questions 1,2. To make this iteration process smoother we recommend to move the antennas at small portion of the recommended move. Such an approach makes a fast convergence to a lowest level of the side lobes at the sky area of optimization. Practically the algorithm produce PSF with equal low side lobes at the area of optimization.

Now lets find the answer on the questions 1,2 formulated earlier.

$PSF(\vec{e})$ is described by the following equation:

$$PSF(\vec{e}) = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{n=1}^{N}\exp(-j2\pi(\vec{r}_k - \vec{r}_n)\vec{e}) = \frac{1}{N}\sum_{k=1}^{N}\exp(-j2\pi\vec{r}_k\vec{e})\cdot\frac{1}{N}\sum_{n=1}^{N}\exp(j2\pi\vec{r}_n\vec{e}) = |U(\vec{e})|^2 \quad (1)$$

where $\vec{e}$ is vector directed to the point at the sky;

$\vec{r}_n, \vec{r}_k$ are vectors determining the position of antennas n and k;

$\vec{r}_k - \vec{r}_n$ is the baseline vector from antenna k to antenna n;
N is number of antennas in the array;
$U(\vec{e}) = \frac{1}{N}\sum_{n=1}^{N} \exp(j2\pi\vec{r}_n\vec{e})$ is voltage beam pattern of the array;
the scalar product is the vector operation at the exponent degrees.

The splitting of the double sum to the product of the single sums at the equation 1 is possible only if all baselines including zero baselines are used. If zero baselines are not used then the bias has to be added (see [2]).
For simplicity we'll consider the plane array. Then it is clear from the equation 1 that only components of vectors $\vec{r}_k, \vec{r}_n$ along the projection of vector $\vec{e}$ on the array plane (x_n, x_k at figure 1) determine the value of the PSF function at the direction $\vec{e}$. Any antenna shift at the direction perpendicular to the projection of vector $\vec{e}$ on the array plane ($\vec{e}_{xy}$ at the figure 1) does not change the $PSF(\vec{e})$ at all. Taking into account this statement we can simplify the equation 1:

$$PSF(e) = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{n=1}^{N} \exp(-j2\pi(x_k - x_n)e) \qquad (2)$$

where e is the length of the projection of the vector directed to the sky point on the array plane or sin of the angle between the sky point and the perpendicular to the array plane;
x_k and x_n are projections of the antenna k and antenna n on the projection of vector $\vec{e}$ on the array plane (vector $\vec{e}_{xy}$ at figure 1).

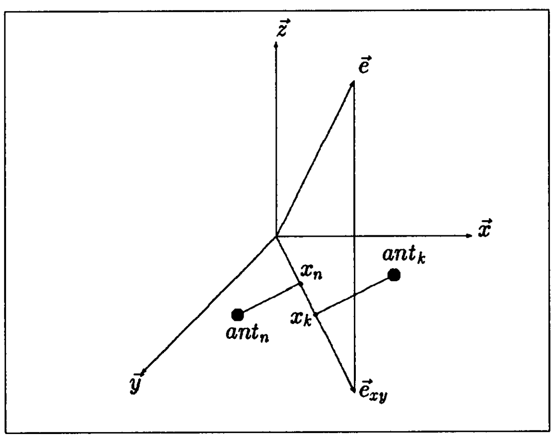

Figure 1: Geometry of the array. The PSF value at the direction $\vec{e}$ is completely determined by the antenna projections on the vector $\vec{e}_{xy}$ - projection of the vector $\vec{e}$ on the array plane. Therefore, if we want to minimize side lobe at the direction of vector $\vec{e}$, **we need to move array antennas exclusively along the vector $\vec{e}_{xy}$. Move at the perpendicular direction does not change the PSF value at the direction $\vec{e}$.**

So if we want to minimize the value of the $PSF(\vec{e})$ moving antenna n we have to move it **exclusively along the projection of the vector directed to the sky point on the array plane** (vector $\vec{e}_{xy}$ at figure 1).
This is the answer on the question 1, pointed out early.

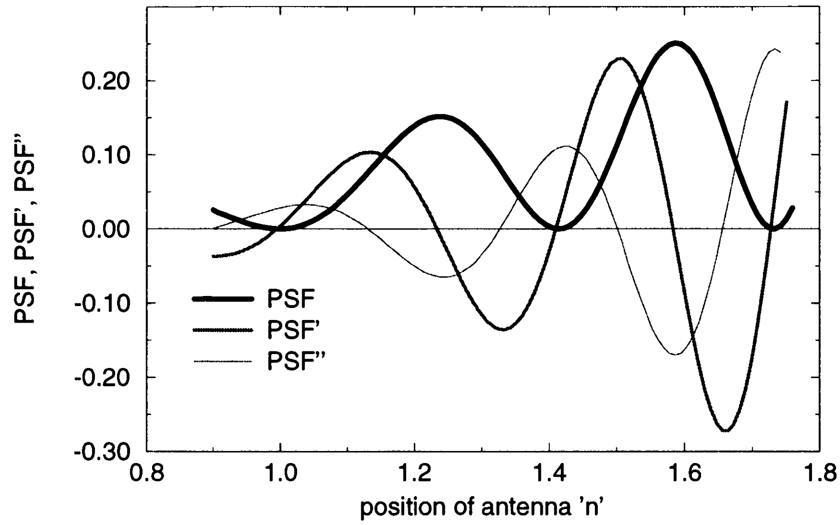

Figure 2: Example of PSF dependence (black) on position of antenna n. The relevant first(red) and second(blue) derivatives are shown also. To have the derivatives at the same plot, the first derivatives are divided by 10 and the second derivatives are divided by 300. So to restore the actual values of the first and second derivatives we need to multiply the plots values by 10 and 300 respectively.

Now question 2: How much should each of the N antennas of the array be moved along the found direction to decrease the value of $PSF(\vec{e})$?

The function PSF as it is determined by equation 1 is real and positive for any configuration and any direction $\vec{e}$. So the PSF(x_n) for fixed $\vec{e}_{xy}$ looks like parabola at some vicinity of the optimum value of x_n. At the optimum value of x_n the function PSF(x_n) has a minimum which in many cases is equal zero. The derivative of PSF by x_n determines how far is the given antenna n located from the optimum. In particular this derivative is equal zero at the optimal position of the antenna n. Indeed if the antenna n locates at Δx_n from the optimal position, then the value of PSF and its first two derivatives are determined by the following expressions at some vicinity of the optimum:

$$\begin{aligned} PSF(\Delta x_n, e) &\simeq A \cdot \Delta x_n^2 \, e^2 \\ \frac{dPSF}{d(\Delta x_n)} &= 2A \cdot \Delta x_n \, e^2 \\ \frac{d^2 PSF}{d(\Delta x_n)^2} &= 2A \, e^2 \end{aligned} \tag{3}$$

The question is how close the expressions (3) describes the actual function $PSF(\Delta x_n)$ and its derivatives at the vicinity of the minimum ($\Delta x_n = 0$). Intuitively clear that $PSF(\Delta x_n)$ looks like a sequence of maxima/minima (figure 2).

Using equations 3, we can find the two expressions for required shift of the antenna **n** to minimize the PSF side lobe at the direction of vector $\vec{e}$:

$$\Delta x_n = -\frac{dPSF}{d(\Delta x_n)} \frac{1}{2A \, e^2} \tag{4}$$

$$\Delta x_n = -\frac{\frac{dPSF}{d(\Delta x_n)}}{\frac{d^2PSF}{d(\Delta x_n)^2}} \tag{5}$$

I have calculated the ratio of $\frac{dPSF}{d(\Delta x_n)}$ and $\frac{d^2PSF}{d(\Delta x_n)^2}$ using the plots at figure 2 for different position of the antenna relatively the minimum at the $PSF(\Delta x_n)$ and found the expected result: the closer the point to the minimum the better the ratio of the derivatives describes the deviation of the antenna of the minimum at $PSF(\Delta x_n)$.

Taking into account that both baselines $x_k - x_n$ and $x_n - x_k$ present at the equation 2, we can infer that complex exponents can be substituted by cosines, and we can obtain the following simple expressions for PSF and its derivatives.

$$\begin{aligned} PSF(e) &= \frac{2}{N^2}\sum_{k=1}^{N}\sum_{n=1}^{N}\cos(2\pi(x_k - x_n)\,e) \\ \frac{dPSF}{dx_n} &= \frac{2}{N^2}2\pi e\sum_{k=1}^{N}\sin(2\pi(x_k - x_n)\,e) \\ \frac{d^2PSF}{dx_n^2} &= -\frac{2}{N^2}(2\pi e)^2\sum_{k=1}^{N}\cos(2\pi(x_k - x_n)\,e) \end{aligned} \tag{6}$$

Substituting expressions for the derivatives from 6 to 4, 5 we finally find the two expressions for the value of shift of the **n** array element to minimize the side lobe at the direction $\vec{e}$.

$$\Delta x_n = -\frac{2\pi}{N^2A}\frac{1}{e}\sum_{k=1}^{N}\sin(2\pi(x_k - x_n)\,e) \tag{7}$$

$$\Delta x_n = \frac{1}{2\pi}\frac{1}{e}\frac{\sum_{k=1}^{N}\sin(2\pi(x_k - x_n)\,e)}{\sum_{k=1}^{N}\cos(2\pi(x_k - x_n)\,e)} \tag{8}$$

Actually it is not recommended to shift the array element n on the full value calculated by equations 7, 8 for the two reasons:
Reason 1. Shifting the n'th array element on the large distance changes the configuration and calculation of the other array element shift has to use the new position of the n'th array element.
Reason 2. Shifting the array elements following equation 7, 8 will definitely suppress the value of the side lobe at the direction $\vec{e}$ (quite possible to zero). But another side lobe (possibly bigger) can appear at another direction.
Instead it is recommended to use multi iteration process of optimization the configuration minimizing side lobes. At each iteration each array element is shifted by the small portion of the shift determined by equation 7, 8. The value of this portion is determined by so called **gain** $= G \ll 1$. Then the algorithm suppresses the given side lobe for several iteration and go to the next side lobe until there is no more reduction of the worst side lobe value. The factors $\frac{2\pi}{N^2A}$ at the equation 7 and $\frac{1}{2\pi}$ at the equation 8 do not depend on neither position of the corrected antenna n nor the direction of the worst side lobe. Therefore those factors can be included in the **gain** and equations 7, 8 can be rewritten:

$$\Delta x_n = -G\,\frac{1}{e}\sum_{k=1}^{N}\sin(2\pi(x_k - x_n)\,e) \tag{9}$$

$$\Delta x_n = G\,\frac{1}{e}\frac{\sum_{k=1}^{N}\sin(2\pi(x_k - x_n)\,e)}{\sum_{k=1}^{N}\cos(2\pi(x_k - x_n)\,e)} \tag{10}$$

Comparing the two expressions (9, 10) for the recommended shift of the antenna n we find some advantage using the expression 9. In particular the expression 9 gives the right direction of the correction at all range between the two maxima of PSF function edged the minimum of the PSF function. Although the value of the shift near the maxima is decreased in comparison with the desired shift (figure 2). The expression 10 can be used only if the initial position of the array element is closer to the PSF minimum than the inflection point. Outside of this range the second derivative changes the sign to negative and the algorithm of the expression 10 will pull the array element to a maximum instead of the minimum. The algorithm of the expression 10 will pull the array element to indefinite when the array element is located near the inflection point where the second derivative is equal zero. Therefore the expression 10 can be used only at the last stage of optimization when almost all antennas are close to the optimum and therefore the PSF as a function of the nth antenna position is described by parabola.

The algorithm is realized at AIPS as a task **CONFI**. The expressions 9 for the recommended shift of the antenna n is chosen.

References

[1] L. Kogan, Optimizing a Large Array Configuration to Minimize the Side lobes, IEEE Transactions on Antennas and Propagation, vol 48, NO 7, July 2000, p 1075

[2] L. Kogan, Level of Negative Side Lobes in an Array Beam, Publications of the Astronomical Society of the Pacific, **111**, April 1999, p 510-511