A Method for Finding the Phase and Amplitude of Interferometer Fringe Patterns

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I. Introduction

The problem is to find the amplitude and phase of a digitally recorded interferometer fringe pattern. The digital record is a sequence of numbers representing the correlator output integrated over times short compared to the fringe period.

The method of analysis described in the present report is based on two fundamental assumptions:

(a) The fringe parameters (frequency, amplitude, phase) remain almost constant for the duration of any single record.

(b) The duration of the record is long enough to contain a large number of fringe periods.

In the case of the NRAO interferometer, each record covers one sidereal minute and the fringe periods are typically in the range of one to three seconds. Hence assumption (b) is met except near the crossover points where the fringe rate passes through zero. The fringe frequency changes very little in one minute at any part of the sky. On the other hand, certain types of source structure lead to very rapid changes in phase and amplitude at certain times; when these occur, they take place in a fairly short time, leaving most of the record unaffected. Thus assumption (a) will be satisfied except at certain times on some sources.

With these two assumptions, we can derive the amplitude and phase from the value of the complex Fourier transform of the digital record at the fringe frequency; this is equivalent to cross-correlating the interferometer output with a cosine wave at the fringe frequency. The fringe frequency is always a known quantity since it is fixed by the length and orientation of the baseline, the local oscillator frequency, and the hour angle and declination of the source.

We shall not deal here with methods for absolute calibration of the derived amplitudes and phases. The method described below gives the amplitude in the units of the digital record; the phase is referred to the fringe pattern expected for a point source located at the centroid of the actual source, given the instrumental parameters.

We shall treat the signal and noise components of the digital record separately. The signal component is discussed in Section II. Section III describes the actual procedure for extracting the amplitude and phase from the digital record. Section IV deals with the errors introduced by the presence of random noise in the digital record.

II. The Signal Component of the Digital Record

The correlator output is a continuous, nearly sinusoidal function with random noise of some level superimposed. This output is integrated for a time Δt and recorded on magnetic tape; then a new integration period begins. This process continues until a large number of integrations have been recorded. In the case of the NRAO interferometer, $\Delta t = 0\overset{s}{.}1$ and the record length is 1^m; thus each record consists of a sequence of 600 data points.

The incremental nature of the integration and recording has a distorting effect on the sinusoidal component of the correlator output.

Let this component be

$$S_0(t) = A_0 \cos\left(2\pi R_0 t - \phi\right)$$

where A_o is the amplitude of the sinuosidal component, R_o is the fringe frequency, φ is a phase angle, and t is time. Then the average value of S during an interval Δt is

$$S(t) = \frac{A_0}{\Delta t} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \cos\left(2\pi R_0 t - \phi\right) dt$$

$$= A_0 \frac{\sin \pi R_0 \Delta t}{\pi R_0 \Delta t} \cos\left(2\pi R_0 t - \phi\right)$$

Hence the integration does not influence the phase of the signal component, but it does reduce the recorded amplitude by the factor

$$\frac{\sin \pi R_0 \Delta t}{\pi R_0 \Delta t}$$

In our case the reduction is always small, amounting to 1.6 per cent at $R_o = 1 \text{ sec}^{-1}$.

The signal component of the digital record then consists of the sequence

$$S_k \Delta t = A \cos\left(2\pi R_0 k \Delta t - \phi\right) \qquad (1)$$

$$\left(|k| \leq \frac{K-1}{2}\right)$$

where

$$A = A_0 \frac{\sin \pi R_0 \Delta t}{\pi R_0 \Delta t}$$

and K is the total number of data points in the sequence. We assume for convenience that K is an odd number and that $t = 0$ when $k = 0$ (i.e., at the center of the digital record).

The Fourier transform of the above sequence is

$$\Gamma_s(R) = \frac{\Delta t}{2\pi} \sum_{k=-L}^{L} \int_{-\infty}^{\infty} S_k e^{-j 2\pi R t} \delta(t - k\Delta t)\, dt$$

where we use the abbreviation

$$L = \frac{K-1}{2} .$$

Evaluation of the transform (see appendix) gives the result

$$\Gamma_s(R) = \frac{AK\Delta t}{4\pi} \left[e^{j\phi} \frac{\sin K\pi(R+R_0)\Delta t}{K \sin \pi(R+R_0)\Delta t} + e^{-j\phi} \frac{\sin K\pi(R-R_0)\Delta t}{K \sin \pi(R-R_0)\Delta t} \right] . \qquad (2)$$

Note that the total time represented by the data record is $T = K\Delta t$. Then in the limit as $K \to \infty$ (or as $\Delta t \to 0$),

$$\Gamma_s(R) \longrightarrow \frac{AT}{4\pi} \left[e^{j\phi} \frac{\sin \pi(R+R_0)T}{\pi(R+R_0)T} + e^{-j\phi} \frac{\sin \pi(R-R_0)T}{\pi(R-R_0)T} \right] .$$

This is an excellent approximation to (2) when K is large, i.e. when $\Delta t/T \ll 1$. It also provides an easy visualization of the nature of $\Gamma_s(R)$, for it shows that this function is approximated by the sum of two sin x/x functions centered at $R = \pm R_o$. These two functions are identical except that they are rotated out of the real plane in opposite directions, as shown in Fig. 1. The angle through which they are rotated is the phase angle φ.

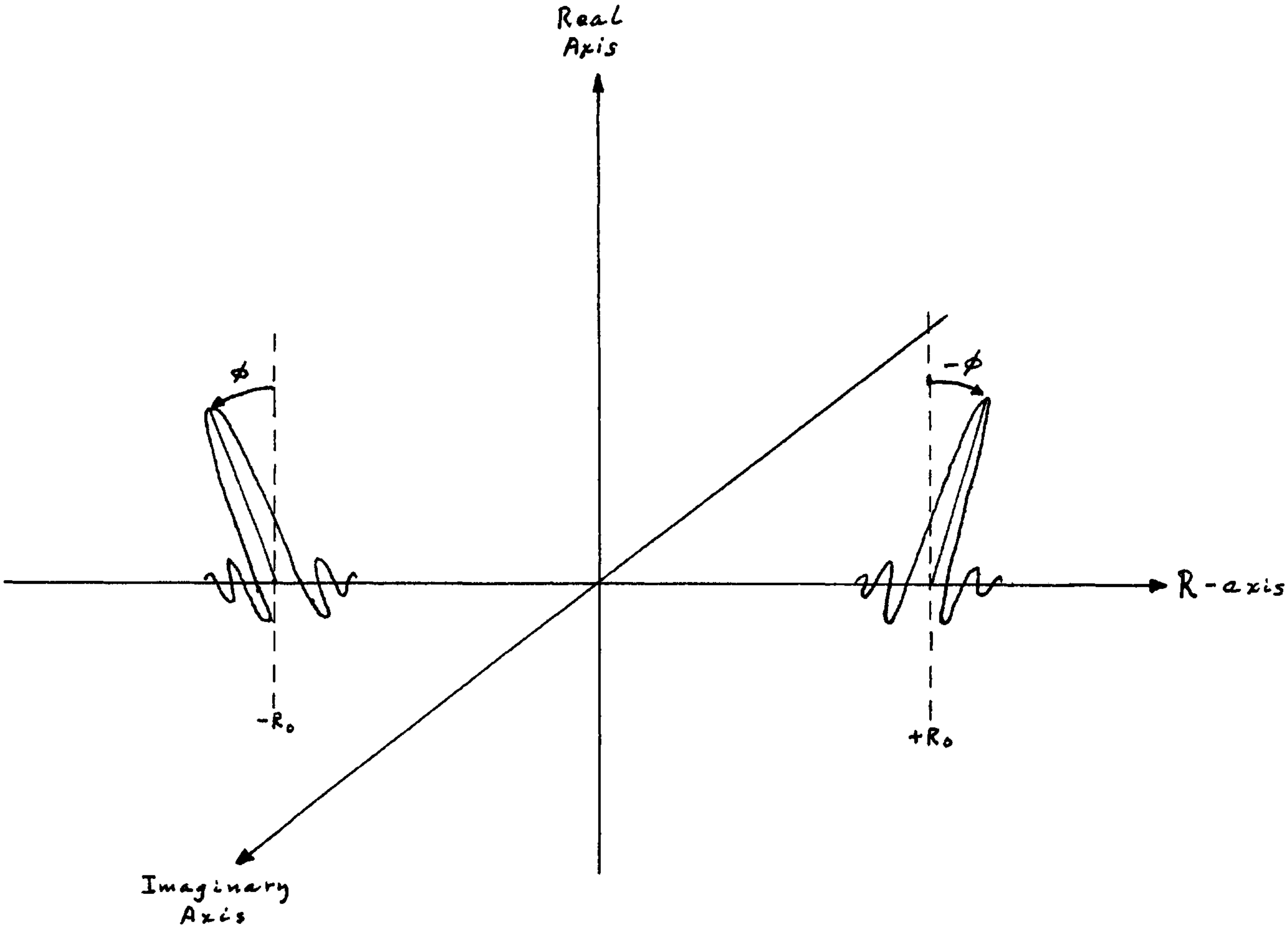

Fig. 1.

It should be noted that the principal maxima are quite narrow when $R_0 \gg T^{-1}$, for the width between first zeros is $\Delta R = 2/T$. Furthermore, the peak height is proportional to T for any given value of A, so the height/width ratio goes as T^2.

It is clear that the information describing the signal component of the record is concentrated about $R = \pm R_o$. The modulus of the transform at these points is proportional to the fringe amplitude, while the angle of rotation out of the real plane gives the phase. We shall pursue these points further in the next section.

III. The Determination of Phase and Amplitude

Let us consider the Fourier transform at $R = R_0$. At this point (2) becomes

$$\Gamma_s(R_o) = \frac{AT}{4\pi}\left[e^{j\phi} P + e^{-j\phi} \right]$$

where

$$P = \frac{\sin 2\pi K R_o \Delta t}{K \sin 2\pi R_o \Delta t} \quad .$$

The real and imaginary parts of $\Gamma_s(R_o)$ are respectively

$$R_e(\Gamma) = \frac{AT}{4\pi}(1+P)\cos\phi \tag{3a}$$

$$I_m(\Gamma) = -\frac{AT}{4\pi}(1-P)\sin\phi \tag{3b}$$

From these we have directly

$$\phi = -\arctan\left[\frac{(1+\rho)\,Im(\Gamma)}{(1-\rho)\,Re(\Gamma)}\right] , \tag{4}$$

$$A = \frac{4\pi}{T}\left[\frac{\{Re(\Gamma)\}^2 + \{Im(\Gamma)\}^2}{1 + 2\rho\cos 2\phi + \rho^2}\right]^{1/2} . \tag{5}$$

It should be noted that equation (4) as it stands is ambiguous, since $\tan(x + \pi) = \tan x$. The ambiguity is resolved if we replace φ by $\varphi + \pi$ when $R_e(\Gamma) < 0$.

Equations (4) and (5) enable us to find the phase and amplitude of the fringe pattern from the Fourier transform of the digital record at $R = R_O$. $Re(\Gamma)$ and $Im(\Gamma)$ are respectively the cosine and sine transforms of the $\{S_k \Delta t\}$ sequence:

$$Re(\Gamma) = \frac{1}{2\pi}\sum_{k=-L}^{L}\{S_k \Delta t\}\cos 2\pi R_o k \Delta t$$

$$Im(\Gamma) = \frac{1}{2\pi}\sum_{k=-L}^{L}\{S_k \Delta t\}\sin 2\pi R_o k \Delta t$$

The phase angle φ given by (4) is referred to $t = 0$, i.e. to the center point on the record. We desire, however, the displacement Φ

between the actual fringes and the fringes which would be expected for a hypothetical point source at the centroid of the actual source, as shown in Fig. 2. We can compute ψ from the instrumental parameters, the source position, and the sidereal time corresponding to t = 0. Then we have simply

$$\Phi = \varphi - \psi.$$

Note that we use the convention that Φ is reckoned positive when the actual fringes lag behind those for the hypothetical point source.

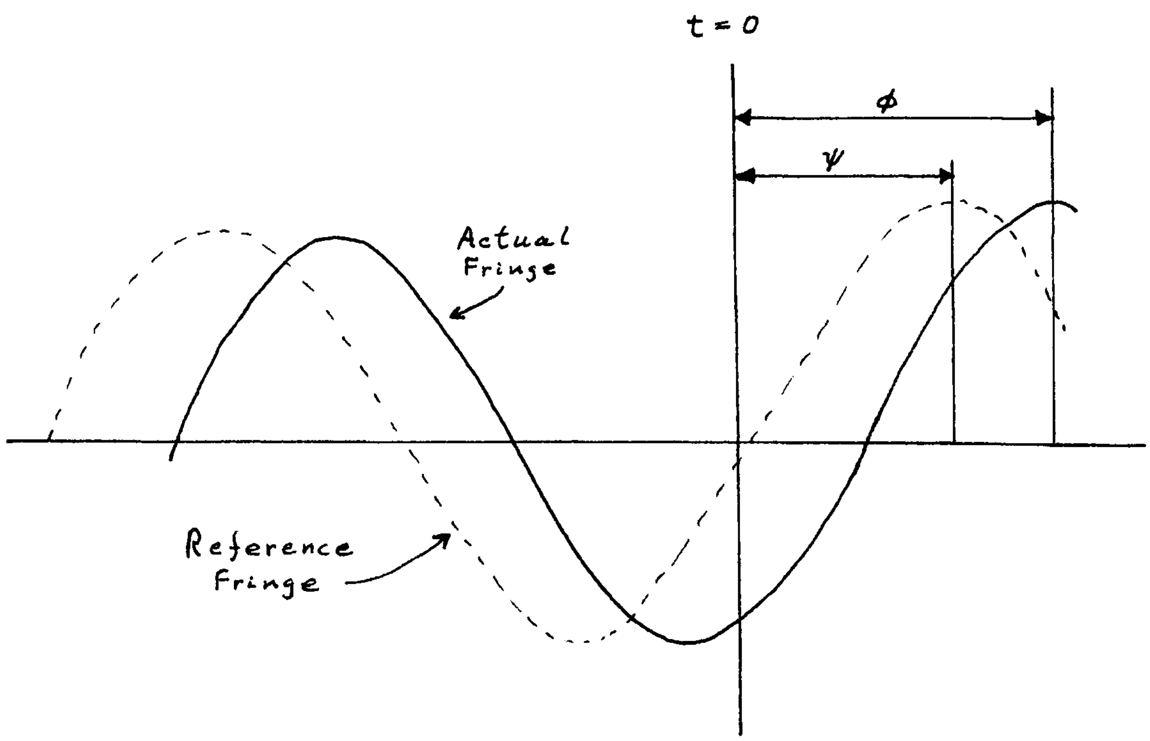

Fig. 2.

IV. The Effects of Noise

In practice each recorded data point will include a noise term $N_k \Delta t$ which is added to $S_k \Delta t$. We assume that this is a gaussian random variable with a variance σ_N^2. The noise component of the data record is the sequence $\{N_k \Delta t\}$, $|k| \le \frac{K-1}{2}$, which has the Fourier transform

$$\Gamma_N(R) = \frac{\Delta t}{2\pi} \sum_{k=-L}^{L} \int_{-\infty}^{\infty} N_k e^{-j2\pi Rt} \delta(t - k\Delta t)\, dt$$

$$= \frac{\Delta t}{2\pi} \sum_{k=-L}^{L} N_k e^{-j2\pi R k \Delta t} \quad . \tag{6}$$

In this we can think of N_k as the average of the noise in the k^{th} recording interval.

Equation (6) shows that $\Gamma_N(R)$ is the sum of K vectors with an r.m.s. length of $\frac{1}{2\pi} \sigma_N$. These vectors have several properties which should be noted:

(a) They are perpendicular to the R-axis.

(b) The length of the k^{th} vector is proportional to N_k.

(c) All the vectors lie in the real plane when R = 0.

(d) The tip of each vector describes a helix of constant radius about the R-axis. The sense and rate of rotation depend on the sign and magnitude of k, respectively. A vector completes one rotation about the R-axis in the interval

$$\Delta R = 1/|k|\Delta t.$$

(e) The value of any vector at -R is the complex conjugate of its value at +R.

These considerations have two consequences which are of immediate importance in the present discussion. First, (e) above implies that the resultant noise vector at $R = -R_0$ is simply the complex conjugate of the resultant noise vector at $R = +R_0$. Therefore no improvement in signal-to-noise ratio for the derived phase and amplitude values can be achieved by making solutions at both $+R_0$ and $-R_0$; one is sufficient. The second point follows from (d). When $R > [(K-1)\Delta t]^{-1} \approx T^{-1}$, the vectors will be spread fairly uniformly over 360°; the greater the value of R, the more nearly uniform the distribution of orientations. Since we have $R_0 \gg T^{-1}$ except near the cross-over points, the constituent noise vectors are effectively oriented at random. This greatly facilitates the discussion of the effects of noise on the phase and amplitude.

We can rewrite (6) as the sum of its real and imaginary parts; setting $R = R_0$, we have

$$\Gamma_N(R_0) = \frac{\Delta t}{2\pi}\left[\sum_{k=-L}^{L} N_k \cos 2\pi R_0 k \Delta t - j\sum_{k=-L}^{L} N_k \sin 2\pi R_0 k \Delta t\right]$$

Clearly the statistics of the real and imaginary parts are identical if the individual noise vectors have random orientations. Therefore the statistics of the <u>length</u> of the resultant noise vector must be the same as those for one of the above components. Letting $\vartheta_k = 2\pi R_0 k \Delta t$, we can write the real part of the resultant noise vector as

$$\Gamma = \frac{\Delta t}{2\pi}\sum_{k=-L}^{L} N_k \cos\vartheta_k$$

Then

$$\Gamma^2 = \left(\frac{\Delta t}{2\pi}\right)^2 \left(\sum_{k=-L}^{L} N_k \cos \vartheta_k\right)^2 .$$

Since the different values of $N_k \cos \vartheta_k$ are uncorrelated, we have

$$\overline{\Gamma^2} = \left(\frac{\Delta t}{2\pi}\right)^2 \sum_{k=-L}^{L} \overline{N_k^2 \cos^2 \vartheta_k} = \frac{1}{2}\left(\frac{\Delta t}{2\pi}\right)^2 \sum_{k=-L}^{L} \overline{N_k^2} .$$

Now

$$\sigma_N^2 = \frac{1}{K} \sum_{k=-L}^{L} \overline{N_k^2} .$$

Therefore the mean square length of any one component of the resultant noise vector is

$$\sigma^2 = \overline{\Gamma^2} = \frac{K\,(\Delta t\, \sigma_N)^2}{8\pi^2} = \frac{T^2 \sigma_N^2}{8\pi^2 K}$$

and the r.m.s. length is

$$\sigma = \frac{T\, \sigma_N}{2^{3/2}\pi \sqrt{K}} . \tag{7}$$

The relative r.m.s. error in the derived value of A is

$$\frac{\Delta A}{A} = \frac{\sigma}{\Gamma_s} = \frac{\sigma_N\, T}{2^{3/2}\pi\sqrt{K}} \cdot \frac{4\pi}{A\, T}$$

when P << 1 (see (3a,b)). This reduces to

$$\frac{\Delta A}{A} = \sqrt{\frac{2}{K}}\,\frac{\sigma_N}{A} \quad . \tag{8}$$

The corresponding r.m.s. phase error is

$$\Delta\Phi = \arctan\frac{\Delta A}{A} \quad . \tag{9}$$

Now let us consider the r.m.s. uncertainties due to noise, given a particular interferometer and the method of reduction described above. If A and σ_N are expressed in units of antenna temperature, we have

$$\sigma_N = \frac{T_R}{\sqrt{B\tau}} \tag{10}$$

$$A = \frac{A_{eff}\,V\,S}{2k} \tag{11}$$

where

T_R = system noise temperature

B = effective i.f. bandwidth

τ = digital integration time

A_{eff} = effective area of a single antenna

V = fringe visibility

S = flux density of the source

k = Boltzmann's constant.

Substituting (10) and (11) into (8), we get

$$\frac{\Delta A}{A} = \frac{2 k T_R}{A_{eff} V S} \sqrt{\frac{2}{K B \tau}} \quad . \tag{12}$$

The parameters of the present NRAO system are

$T_R \approx 250$ °K

$A_{eff} \approx 260$ m^2

$K = 599$

$B \approx 25$ Mc/s

$\tau = 0.1$ sec.

With these values, we get

$$\frac{\Delta A}{A} \approx \frac{0.1}{VS}$$

where the flux density is expressed in flux units. The table below shows the performance attainable at different flux densities for a point source ($V = 1$).

S	ΔA/A	ΔΦ
0.5 flux unit	20 per cent	11°.3
1 " "	10 " "	5°.7
5 " "	2 " "	1°.2
10 " "	1 " "	0°.6

Appendix: The Fourier Transform of the Sequence $\{S_k \Delta t\}$.

We have

$$\Gamma_s(R) = \frac{A\Delta t}{2\pi} \sum_{k=-L}^{L} \int_{-\infty}^{\infty} \cos(2\pi R_0 k\Delta t - \phi) e^{-j2\pi R t} \delta(t - k\Delta t)\, dt$$

$$= \frac{A\Delta t}{4\pi} \sum_{k=-L}^{L} \left\{ e^{j\phi} e^{-j2\pi(R+R_0)k\Delta t} + e^{-j\phi} e^{-j2\pi(R-R_0)k\Delta t} \right\} .$$

It is easy to prove the identity

$$\sum_{n=-a}^{a} e^{\pm jnx} = \frac{\sin (2a+1)x/2}{\sin x/2} .$$

Using this, we have

$$\Gamma_s(R) = \frac{A\Delta t}{4\pi} \left\{ e^{j\phi} \frac{\sin K\pi(R+R_0)\Delta t}{\sin \pi(R+R_0)\Delta t} + e^{-j\phi} \frac{\sin K\pi(R-R_0)\Delta t}{\sin \pi(R-R_0)\Delta t} \right\} .$$

Multiplication of the numerator and denominator by K gives equation (2).