ON THE REDUCTION OF DIGITAL INTERFEROMETER RECORDS

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The problem of finding the phase and amplitude of digitally recorded interference fringes is complicated by the fact that the record length is not an integer number of fringes. Clearly, a sine wave interrupted after a non-integer number of fringes has a D.C. component, and in revenge a D.C. level has a component at that frequency. Following Wade's memo of a few weeks ago, we can investigate the response of his procedure to a constant level.

From the equations at the bottom of Wade's memo, page 7, if $S_k = S = \text{constant}$

$$\Gamma(R_o) = \frac{\Delta t}{2\pi} S \sum_{k=-L}^{L} e^{-j\,2\pi\,R_o\,k\,\Delta t} \tag{1}$$

$$= \frac{\Delta t}{2\pi} S \left(2R_e \left\{ \frac{1-e^{-j\,2\pi\,R_o(L+1)\,\Delta t}}{1-e^{-j\,2\pi\,R_o\,\Delta t}} \right\} -1 \right)$$

which, after taking the real part of the expression in brackets and applying some trigonometric identities, becomes

$$\Gamma(R_o) = \frac{\Delta t}{2\pi} S \frac{\sin \pi\,R_o\,T}{\sin \pi\,R_o\,\Delta T} \tag{2}$$

where $T = K\,\Delta t = (2L+1)\,\Delta t$.

The amplitude corresponding to this transform is given by Wade's Eq.(5), and in this case $I_m\Gamma = 0$ and $\phi = 0$ so this simplifies to

$$A = \frac{4\pi}{T} \frac{\Gamma}{1+p}$$

$$= \frac{2}{K} S \frac{\sin \pi R_o T}{\sin (\pi R_o \Delta t) \left(1 + \frac{\sin 2\pi R_o T}{K \sin 2\pi R_o \Delta t}\right)} \qquad (3)$$

To get an idea of the size of this effect, let us consider $K \gg 1$, $\pi R_o \Delta t \ll 1$

$$A = \frac{2S}{\pi R_o T} \sin \pi R_o T$$

$$= \frac{\frac{2}{\pi} S}{\text{number of fringes in interval}} \sin \pi R_o T \qquad (4)$$

This, in the first approximation, will increase the scatter in the cosine Fourier transform by $\frac{S}{\pi R_o T}$.

With the numbers used to date, $R_o T \approx 30$ fringes, $S = 500$ counts, the scatter is increased by 5.2 counts which is about 0.3 flux units, three times the predicted noise scatter. This situation may be corrected in at least four different ways.

1. 500 units may be subtracted from each S_k at the beginning of the computer program, or S_1, the first signal bit, may be subtracted.

2. $\overline{S} = \frac{\Sigma S_k}{K}$ may be subtracted early in the program.

3. Γ may be corrected by Equation (2) using $S = 500$, $S = S_1$, or $S = \overline{S}$.

4. A least squares solution may be made to fit the S_k by a formula of the form

$$A \cos 2\pi R_o k \Delta t + B \sin 2\pi R_o k \Delta t + C \qquad (5)$$

Methods 1 and 2 have little to recommend them save the simplicity of the programming logic. Method 3 will take the least computer time if $S = 500$ or $S = S_1$ is taken. Method 4 has the advantage that it will give correct answers when the fringe rate is very small. It will be formulated in detail below.

The general form of the least squares solution is given by the vector equation

$$\begin{pmatrix} \Sigma \cos^2 \theta_k & \Sigma \sin \theta_K \cos \theta_k & \Sigma \cos \theta_k \\ \Sigma \sin \theta_k \cos \theta_k & \Sigma \sin^2 \theta_k & \Sigma \sin \theta_k \\ \Sigma \cos \theta_k & \Sigma \sin \theta_k & K \end{pmatrix} (A,B,C)$$

$$= \begin{pmatrix} \Sigma S_k \cos \theta_k \\ \Sigma S_k \sin \theta_k \\ \Sigma S_k \end{pmatrix} \qquad (6)$$

where $\theta_K = 2\pi R_o k \Delta t$ and all sums run from $-L$ to L. The sums on the left may be evaluated explicitly by means of the formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin \theta \cos \theta = \frac{\sin 2\theta}{2}$$

$$\sum_{k=-L}^{L} \sin k \Delta\theta = 0$$

and $\sum_{k=-L}^{L} \cos k\,\Delta\theta = R_e \sum_{k=-L}^{L} e^{jk\Delta\theta} = R_e \frac{1-e^{j(L+1)\Delta\theta}}{1-e^{j\Delta\theta}}$

$$= \frac{\sin K \frac{\Delta\theta}{2}}{\sin \frac{\Delta\theta}{2}}$$

The equation then becomes, after division by K

$$\begin{pmatrix} \frac{1+p}{2} & 0 & Q \\ 0 & \frac{1-p}{2} & 0 \\ Q & 0 & 1 \end{pmatrix} (A,B,C)$$

$$= \begin{pmatrix} \frac{1}{K}\sum S_k \cos \theta_k \\ \frac{1}{K}\sum S_k \sin \theta_k \\ \frac{1}{K}\sum S_k \end{pmatrix}$$

where $p = \frac{\sin 2\pi R_o K \Delta t}{K \sin 2\pi R_o \Delta t}$ as in Wade,

and $Q = \frac{\sin \pi R_o K \Delta t}{K \sin \pi R_o \Delta t}$

This equation is solved by

$$B = \frac{2}{K(1-p)} \sum S_k \sin \theta_k$$

$$A = \frac{1}{\frac{1+p}{2} - Q^2} \left(\frac{1}{K} \sum S_k \cos \theta_k - \frac{Q}{K} \sum S_k \right)$$

$$C = \frac{1}{\frac{1+p}{2} - Q^2} \left(\frac{1+p}{2K} \sum S_K - \frac{Q}{K} \sum S_K \cos \theta_k \right)$$

Since the matrix is nearly diagonal, that is, the constant term interacts very little with the sinusoidal term, it costs negligibly little in terms of signal to noise to solve for this third parameter C.

The extra time required for this procedure over the present procedure is about 15 milisec/min to form $\sum S_k$ and about 3 milisec/min to form Q and the combinations of p and Q, totaling perhaps 20 milisec/min = 30 sec/day.

This procedure yields essentially the same equations as Wade's when $\sum S_k = 0$, as $Q^2 \sim \frac{1}{K^2}$ is very small.

This procedure suffices if there is more than about half a fringe in the interval, though the signal to noise ratio will go down, essentially as

$$\sqrt{\frac{1+p}{2} - Q^2} \, , \qquad (7)$$

as more and more of the available information goes to determine C. For very slow fringes, perhaps C can be fixed to be the value derived from adjacent integrations (this is analogous to what is done in reducing the analogue records). This correction can conveniently be done after the initial reduction if C is printed as well as A and B and one may choose an average C, say C^1, to find

$$A = \frac{2}{1+p}\left(\frac{1}{K}\sum S_k \cos \theta_k - Q\, C^1\right)$$

if these quantities are printed out. This is the same as method 3 of correction above; it might occasionally be worthwhile to apply it to the ten minutes nearest crossover.

The philosophical justification of this procedure of using the form (5) instead of a single sine law, is that the zero displacement is much larger than the noise components at other frequencies, and so deserves special treatment. If it is found that the baseline wanders with a period about a minute, the net signal to noise ratio might go down slightly if a term

$$D \cos \frac{2\pi k}{600} + E \sin \frac{2\pi k}{600}$$

is fitted as well, but it is very unlikely that it would improve enough to justify the extra machine time.