Fitting Source Models to Interferometer Observations, I.

C. M. Wade

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I. Introduction

The interferometer fringes for a radio source can be described by their amplitude and phase as a function of hour angle. If A is the fringe visibility amplitude normalized to unity at zero baseline length and Φ is the phase of the pattern referred to the centroid of the source, the complex fringe visibility is

$$\underset{\sim}{V}(u,v) = A\,(u,v)e^{-j\Phi(u,v)} \tag{1}$$

where u and v are respectively the east-west and north-south components of the projection of the baseline on a plane normal to the source direction (see [1]). $\underset{\sim}{V}(u,v)$ is the Fourier transform of the surface brightness distribution over the source:

$$C\,\underset{\sim}{V}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(x,y)e^{-j2\pi(ux + vy)}\,dxdy \tag{2}$$

where

$$x = (\alpha - \alpha_o)\cos\delta,$$

$$y = \delta - \delta_o,$$

(α_o, δ_o) = source centroid coordinates,

$T(x,y)$ = brightness temperature at (x,y).

It is assumed that x and y are in radians and that u and v are in wavelengths. The normalizing constant is

$$C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(x,y)\,dxdy.$$

Then, because of the Rayleigh-Jeans law,

$$C = \frac{\lambda^2 S}{2k}$$

where S is the flux density of the source at wavelength λ and k is Boltzmann's constant.

It is convenient to replace T(x,y) by the dimensionless variable

$$t(x,y) = \frac{2k}{\lambda^2 S} T(x,y).$$

Then (2) can be written as

$$\underset{\sim}{V}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t(x,y) e^{-j2\pi(ux + vy)} \, dxdy. \tag{3}$$

The fact that t(x,y) is real has a useful consequence which is evident from (3):

$$\underset{\sim}{V}(-u,-v) = \underset{\sim}{V}^*(u,v). \tag{4}$$

Given $\underset{\sim}{V}(u,v)$ for all u and v, one could obtain a map of the source simply by Fourier inversion:

$$t(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underset{\sim}{V}(u,v) e^{j2\pi(ux - vy)} \, dudv \tag{5}$$

In view of (1) and (4), this can be written in terms of the fringe parameters as

$$t(x,y) = 2\int_{-\infty}^{\infty}\int_{0}^{\infty} A(u,v) \cos 2\pi\{uv + vy - \frac{\Phi(u,v)}{2\pi}\} \, dudv \tag{6}$$

The Fourier inversion method is direct and involves no <u>a priori</u> assumptions about source structure. It has been applied to one-dimensional observations by Lequeux [2]. The main practical limitation has been that the Fourier transform can be sampled only within a finite range of antenna spacings, so that the restored source distribution is smoothed by an effective beamwidth corresponding to the greatest spacing. In two dimensions, the problem is far more difficult, since the sampling of $\underset{\sim}{V}$ is never continuous within the accessible range of u and v. With the NRAO interferometer, the sampling follows widely separated elliptical tracks in the Fourier transform plane; a crossed array operating as a meridian transit instrument would sample a rectangular grid of points. In any event, there are considerable regions in the transform plane which are never sampled. As a result, if one wishes to use equation (6) to recover a brightness distribution, he either must accept spurious quasi-periodicities in the solution, or he must interpolate between the observed points, thereby making implicit assumptions about the solution.

When one has only partial information about the Fourier transform of the source distribution, the most satisfactory approach is to assume a plausible solution and compare its Fourier transform with the observed $\underset{\sim}{V}(u,v)$. If the parameters of the assumed solution can be adjusted to give a reasonable agreement between the transforms, it is probably a fairly good representation of the source. If not, the model is clearly wrong and must be abandoned.

The present report is concerned with the latter approach. Section II discusses some simple one-dimensional models in order to show how gross properties of the source structure can be inferred directly from the Fourier transform. Section III deals with two-dimensional models. Section IV treats the representation of arbitrary brightness distributions by samples taken at points on a Cartesian grid.

II. Simple One-Dimensional Models

A. Single Gaussian Source

Consider the one-dimensional brightness distribution

$$t(x) = \frac{1}{\beta}\sqrt{\frac{a}{\pi}}\exp\left[-a(x/\beta)^2\right] \tag{7}$$

where β is the half-intensity width and

$$a = 4\log_e 2 = 2.7726.$$

Its Fourier transform is

$$\boxed{\underset{\sim}{V}(u) = e^{-\frac{\pi^2\beta^2u^2}{a}}} \tag{8}$$

This again is a gaussian. Since $\underset{\sim}{V}(u)$ is real, $A(u) = \underset{\sim}{V}(u)$ and $\Phi(u) = 0$ for all u. The widths of the source and its Fourier transform are related by

$$\beta = \frac{a}{2\pi u_{1/2}}$$

where $u_{1/2}$ is the point at which the visibility is 0.5.

Equation (9) can be used to find the source width directly from the Fourier transform. Expressing β in minutes of arc instead of radians, we have

$$\beta' = \frac{1517}{u_{1/2}} .$$

More generally,

$$\beta' = \frac{1822}{u} \sqrt{\log_e \frac{1}{A(u)}}.$$

Example: Lequeux's [2] east-west visibility curve for 3C 58 is approximately gaussian in shape, with $u_{1/2} = 300$. Then, if the source is gaussian, its half-brightness width is 5!06 in the east-west direction.

B. Double Gaussian Source

Consider a source consisting of two gaussian components separated by an angular distance γ, each having the same half-brightness width β. Let the component at negative x contribute a fraction ρ_1 of the combined flux, and let the component at positive x contribute a fraction $\rho_2 = 1 - \rho_1$. Then the source model is

$$t(x) = \frac{1}{\beta}\sqrt{\frac{a}{\pi}} \left[\rho_1 \, e^{-a\left(\frac{x+\rho_2\gamma}{\beta}\right)^2} + \rho_2 \, e^{-a\left(\frac{x-\rho_1\gamma}{\beta}\right)^2}\right] \tag{10}$$

The model is shown in Figure 1.

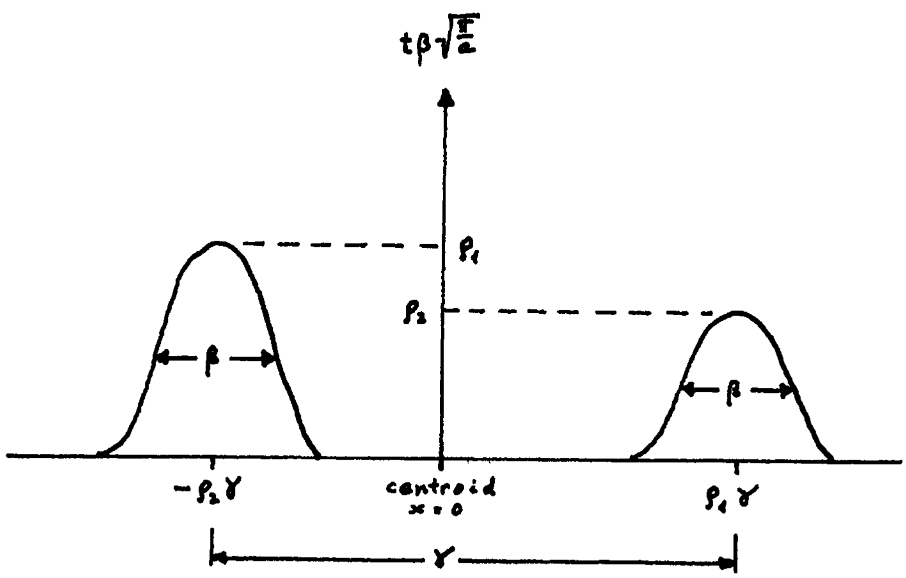

Figure 1

The Fourier transform of t(x) is

$$\boxed{\underset{\sim}{V}(u) = e^{-\frac{\pi^2\beta^2u^2}{a}} \left\{ \rho_1 \, e^{j2\pi\rho_1\gamma u} + \rho_2 \, e^{-j2\pi\rho_1\gamma u} \right\}} \tag{11}$$

The corresponding amplitude and phase are respectively

$$A(u) = e^{\frac{-\pi^2\beta^2u^2}{a}} \sqrt{\rho_1^2 + 2\rho_1\rho_2 \cos 2\pi\gamma u + \rho_2^2} \tag{12}$$

$$\Phi(u) = \text{arc tan} \left[\frac{\rho_2 \sin 2\pi\rho_1\gamma u - \rho_1 \sin 2\pi\rho_2\gamma u}{\rho_2 \cos 2\pi\rho_1\gamma u + \rho_1 \cos 2\pi\rho_2\gamma u} \right] \tag{13}$$

These relations have several consequences which are useful in making a quick interpretation of a Fourier transform in terms of the corresponding brightness distribution:

1. The amplitude has sharp minima when $\cos 2\pi\gamma u = -1$. The n^{th} minimum occurs when

$$u = \frac{2n-1}{2\gamma} \ . \tag{14}$$

   Using this, one can find the component separation directly from the locations of the amplitude minima.

2. The sign of $d\Phi/du$ is constant for any one model. It is related to the sense of the asymmetry in the model, as shown by the table below:

$\rho_1 < \rho_2$	$d\Phi/du > 0$
$\rho_1 > \rho_2$	$d\Phi/du < 0$
$\rho_1 = \rho_2$	$d\Phi/du = 0^+$

+Except at amplitude minima, where $\Phi(u)$ is discontinuous.

3. The visibility phase varies most rapidly at the amplitude minima and most slowly halfway between them. Thus it exhibits a stepwise variation with u. Let $\Delta u = 1/\gamma$ be the interval between successive minima. Then

$$\Delta\Phi = \Phi(u + \Delta u) - \Phi(u)$$

is a constant whose magnitude depends only on the relative strengths of the two source components.

The relationship is found easily by letting $u = 0$. Since $\Phi(0) = 0$, we have simply

$$\tan \Delta\Phi = \frac{\rho_2 \sin 2\pi\rho_1 - \rho_1 \sin 2\pi\rho_2}{\rho_2 \cos 2\pi\rho_1 + \rho_1 \cos 2\pi\rho_2}$$

Because $\rho_1 + \rho_2 = 1$, this reduces to

$$|\Delta\Phi| = \begin{cases} 2\pi\rho_2 , & \rho_1 \geq \rho_2 \\ 2\pi\rho_1 , & \rho_1 \leq \rho_2 . \end{cases}$$

Therefore

$$\rho = \frac{\Delta\Phi}{2\pi} \tag{15}$$

where ρ now refers to the weaker component of the source.

4. The depths of the amplitude minima depend on $|\rho_1 - \rho_2| = 1 - 2\rho$ and the ratio β/γ. The amplitude in the n^{th} minimum is

$$A_n = (1-2\rho) \exp \left[-\frac{1}{a} \left\{ \frac{(2n-1)\pi}{2} \frac{\beta}{\gamma} \right\}^2 \right]$$

In view of (14), this can be written

$$A_n = (1-2\rho) \exp \left[-\frac{1}{a} \{\pi\beta u_n\}^2 \right] \tag{16}$$

where u_n is the value of u at the n^{th} minimum.

Solving (16) for β, we get

$$\beta = \frac{1}{\pi u_n} \sqrt{a \log_e \left(\frac{1-2\rho}{A_n}\right)} \tag{17}$$

Figure 2 illustrates A(u) and Φ(u) for a double gaussian source.

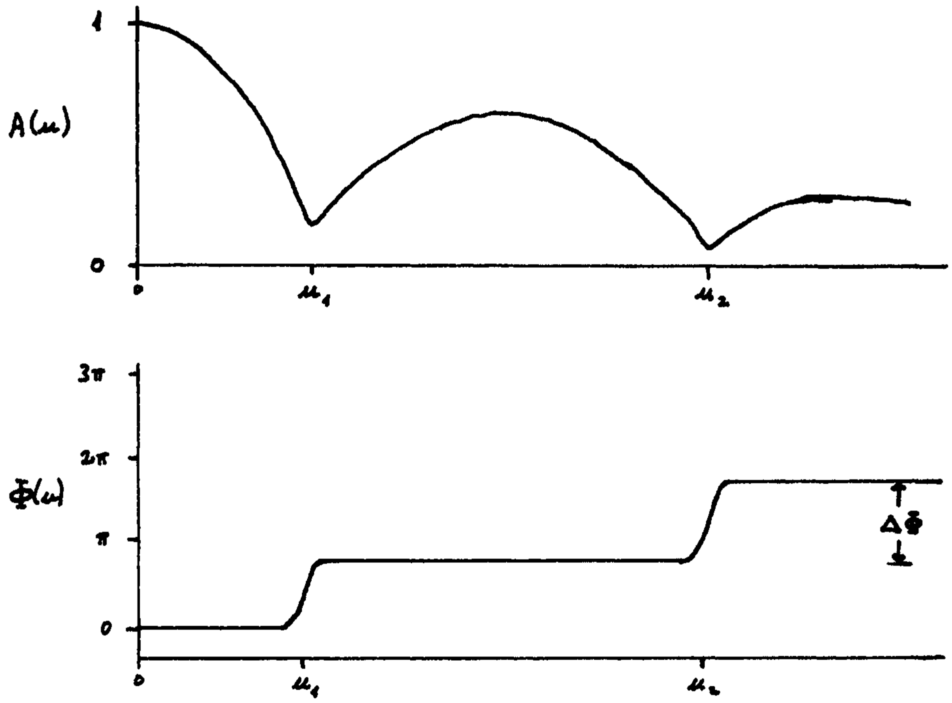

Figure 2.

The above relations are given below in a form convenient for practical use. As stated, they refer specifically to observations with an east-west interferometer, but this places little restriction on their usefulness.

<u>Component separation</u>: The east-west separation of the components, in minutes of arc, is

$$\gamma'_{EW} = 1719\ (2n-1)/u_n$$

where u_n is the location of the n^{th} amplitude minimum.

Component ratio and sense of asymmetry: The contribution of the fainter component to the total flux is

$$\rho = \Delta\Phi°/360$$

where $\Delta\Phi°$ is in degrees. If $d\Phi/du$ is positive, the stronger component is to the east.

Component Diameter: In minutes of arc, this is

$$\beta'_{EW} = \frac{1822}{u_n} \sqrt{\log_e \frac{1-\Delta\Phi°/180}{A_n}}$$

Example: Moffet's [3] observations of 3C 40 with an east-west baseline give:

$$\begin{aligned} u_1 &= 400 \\ A_1 &= 0.29 \\ d\Phi/du &> 0 \\ \Delta\Phi &\approx 130°. \end{aligned}$$

Using the above formulae, one finds

$$\gamma'_{EW} = 4'.3$$

ρ = 0.36. Stronger component to the east, so $\rho_1 = 0.36$, $\rho_2 = 0.64$.

$$\beta'_{EW} = 1'.4.$$

III. Two-Dimensional Models

A. Point Sources

Consider a complex source made up of N point sources. The i^{th} component source has the coordinates (x_i, y_i) and contributes a fraction ρ_i of the total flux. Then

$$\sum_{i=1}^{N} \rho_i = 1,$$

and the brightness distribution is

$$t(x,y) = \sum_{i=1}^{N} \rho_i \, \delta(x-x_i,\ y-y_i). \tag{18}$$

Its Fourier transform is

$$\boxed{\underset{\sim}{V}(u,v) = \sum_{i=1}^{N} \rho_i \, e^{-j2\pi(ux_i + vy_i)}} \tag{19}$$

The real and imaginary parts of $\underset{\sim}{V}(u,v)$ are respectively

$$R(u,v) = \sum_{i=1}^{N} \rho_i \cos 2\pi(ux_i + vy_i) \tag{20}$$

$$I(u,v) = -\sum_{i=1}^{N} \rho_i \sin 2\pi(ux_i + vy_i) \tag{21}$$

The corresponding amplitude and phase are

$$A(u,v) = \sqrt{\{R(u,v)\}^2 + \{I(u,v)\}^2} \tag{22}$$

$$\Phi(u,v) = \arctan \frac{-I(u,v)}{R(u,v)} \,. \tag{23}$$

These relations are the basis of Section IV B.

B. Gaussian Sources

Instead of points, let the component sources be circularly symmetrical gaussians. The i^{th} component has half-brightness width β_i . In this case,

$$t(x,y) = \frac{a}{\pi} \sum_{i=1}^{N} \frac{\rho_i}{\beta_i^2} \, e^{-\frac{a}{\beta_i^2}\{(x-x_i)^2 + (y-y_i)^2\}} \tag{24}$$

and

$$\boxed{\underset{\sim}{V}(u,v) = \sum_{i=1}^{N} \rho_i \, e^{-\frac{\pi^2\beta_i^2}{a}(u^2+v^2)-j2\pi(ux_i + vy_i)}} \qquad (25)$$

We shall not proceed further with this model. It is included here because it can be used in discussing double or multiple sources with components of different sizes.

IV. Representation of General Brightness Distributions by Arrays of Point Sources

An arbitrary brightness distribution can be described by its values at discrete points. The precision of the description depends on how closely the points are spaced. This is the natural way to specify an arbitrary distribution when using a computer to find its Fourier transform, since it is then necessary to replace the integration by a series anyway.

A. The Sampling Interval

The first problem is to decide how closely to space the points where the brightness distribution is sampled. If they are too far apart, they will act as individual point sources in the solution rather than as samples of a continuous distribution. On the other hand, if one tries to avoid this problem by using excessively close grid spacings, useless information (i.e., detail invisible to the instrument) may be added to the Fourier transform. While this does not impair the quality of the solution, it means that unnecessary labor must be expended in specifying the model and that excessive computer time is used for the transformation. Thus we should determine how coarse a sampling grid we can use without losing accuracy in the observable part of the Fourier transform of the model. The problem is analogous to finding the "peculiar interval" for observations with a pencil-beam antenna.

It is sufficient to treat the problem in one dimension. Assume that we have a brightness distribution $t(x)$ which we wish to represent by N samples taken at intervals Δx, and that we wish to find its Fourier transform up to a maximum antenna spacing u_{max}. The source brightness distribution then is replaced by the sequence

$$t_n = t(x)\ \delta\ (x - n\Delta x)\ , \quad n = -k,\ldots,0,\ldots,k.$$

where

$$k = \tfrac{1}{2}(N-1).$$

The Fourier transform of the sequence is

$$\underset{\sim}{V}(u) = \sum_{n=-k}^{k} t_n \, e^{-j2\pi n\Delta x u}$$

The result is periodic, with a period $\Delta u = 1/\Delta x$. This is not unexpected, since the sequence t_n is, in effect, a Fourier series representation of $\underset{\sim}{V}(u)$ over the range $-u_{max} \leq u \geq u_{max}$. If $V(u)$ is to be calculated correctly over this range, it is clearly necessary that we have

$$\Delta u \geq 2\, u_{max},$$

or

$$\boxed{\Delta x \leq \frac{1}{2\, u_{max}}} \quad . \tag{25}$$

The largest value of Δx consistent with this result is $(2u_{max})^{-1}$, and this is what we adopt as the grid spacing. The largest antenna separation attainable with the NRAO interferometer is 24300 λ at 11.1 cm wavelength. Thus a grid spacing of

$$(2 \times 24300)^{-1} \text{ radians} = 4''.244$$

should be adequate under all circumstances for our instrument at its present operating wavelength.

B. Organization of the Problem for the Computer

Assume that we have prepared a source model consisting of a grid of samples at intervals $p = \Delta x = \Delta y$, and that we wish to evaluate its Fourier transform at specified values of u and v. In order to avoid the labor of referring the coordinates to the centroid of the model and normalizing the intensities, we use an arbitrary coordinate system (x',y') which is translated but not rotated with respect to the (x,y) system, and express the intensities on an arbitrary scale. Thus the model is given as $t'(x',y')$ instead of $t(x,y)$. The necessary adjustments are left to the computer. Otherwise the method below follows Section III A.

Let

$$\begin{aligned} x' &= m\Delta x = mp \\ y' &= n\Delta y = np \\ t'(x',y') &= t'_{mn} . \end{aligned}$$

The normalizing constant is

$$C = \left[\sum_{m,n} t'_{mn} \right]^{-1}$$

The fraction of the total flux due to sample mn is

$$\rho_{mn} = Ct'_{mn} \ .$$

The coordinates of the source centroid in the (x',y') system are

$$x'_c = pC \sum_m \{m \sum_n t'_{mn}\}$$

$$y'_c = pC \sum_n \{n \sum_m t'_{mn}\}$$

The real and imaginary parts of the Fourier transform are

$$R(u,v) = \sum_{m,n} \rho_{mn} \cos 2\pi \left[(mp - x'_c)u + (np - y'_c)v\right],$$

$$I(u,v) = \sum_{m,n} \rho_{mn} \sin 2\pi \left[(mp - x'_c)u + (np - y'_c)v\right].$$

Finally, the amplitude and phase are given by (22) and (23).

References

1. Wade and Swenson, "Geometrical Aspects of Interferometry", NRAO Report, December 1964.

2. Lequeux, An.d'Ap. 25, 221, 1962.

3. Moffet, Ap.J. Suppl. 7, 93, 1962.

4. Bracewell and Roberts, Austr. J. Phys. 7, 615, 1954.

Corrections to "Fitting Source Models to Interferometer Observations, I.", C. M. Wade, Frebruary 1965.

1. Last equation on p. 1 should be

$$C = \frac{\lambda^2 S}{2k} .$$

2. First equation on p. 2 should be

$$t(x,y) = \frac{2k}{\lambda^2 S} T(x,y).$$

3. In the next to last equation on p. 10,

$$n = -k, \ldots, 0, \ldots, k.$$

C. M. Wade
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