For the Best Performance, the Active Surface Is and Must Remain Astigmatic

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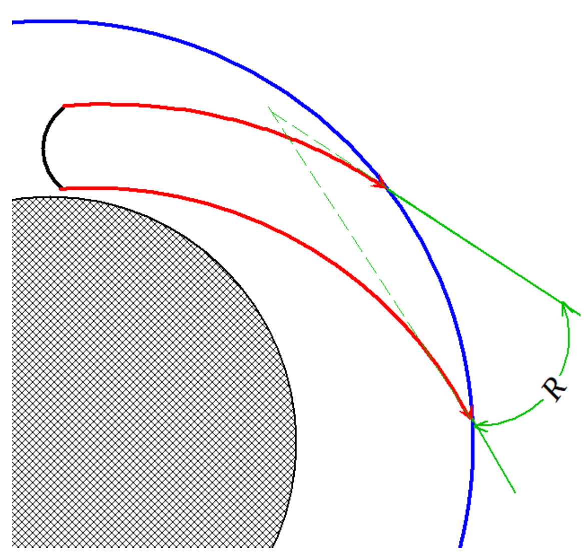
The large diameter of the GBT and the telescope's low elevation limit conspire with high-frequency observations and the Earth's atmosphere to produce refraction that is not constant across the telescope's aperture, which in turn produces an effective optical aberration that is similar to astigmatism. I provide arguments for the existence and magnitude of the effect and some potential consequences. For example, the atmosphere-induced aberration (AIA) will elongate in elevation the images of point sources by ~4.3" when observing at an elevation of 10° under median refraction conditions, and a few times this on not-too-rare occasions and at lower elevations. The range in magnitude of the effects of AIA is largest when conditions are suitable for high frequency observing. Metrology systems that measure the surface directly, like photogrammetry, should not set the surface to a perfect paraboloid at low elevations as that would be detrimental to performance. I provide models on how to adjust the surface to counter AIA as well as the expected gain loss and change in beam shape if the surface is not adjusted. For example, when observing below 20°, the surface needs to deviate from a paraboloid by over 1 mm 75% of the time. The current gravitational Zernike model, determined from observations through the atmosphere of astronomical sources, probably removes much of the effects of AIA but only for typical refraction conditions. Since the magnitude of AIA is weather dependent, but the current gravitational Zernike terms are not, we may want to develop a weather-dependent active surface to improve performance at high frequencies and low elevations.

Introduction

Consider a telescope that is transmitting when pointing at a low elevation as in this exaggerated sketch. A perfect paraboloid produces rays which leave the dish parallel to each other. The rays leaving the bottom of the dish travel through more atmosphere than the rays leaving the top. Since the atmosphere's index of refraction is dependent upon air density, which falls with height, rays leaving the bottom of the dish go through layers of air with the highest refraction. The lower rays are thus typically bent more than the upper rays. Thus, rays which leave the dish parallel to each other exit the atmosphere no longer parallel, diverging by the angle R.



Consider the principle of reciprocity: rays from distant, point-like objects are parallel to each other when they hit the Earth's atmosphere. Yet, due to the different distances and refraction the rays experience, rays will no longer be parallel when they hit the dish. This is equivalent to an elevation-dependent phase gradient across the aperture plane, similar to that for an astigmatic surface, that will elongate a point source in the elevation direction.

To correct for this phase gradient, the telescope's surface could be altered from that of a perfect paraboloid. In essence, the GBT's active surface needs to make the dish astigmatic, in the reverse sense of what the atmosphere introduces, in order to produce a point-like image from a point source. Since refraction is greatest at low elevations, the magnitude of the required surface adjustments is highest for low elevation observing. And, since the change in index of refraction across the aperture is dependent upon the gradient of dry air and water-vapor density near the Earth's surface, the magnitude of the surface adjustments is also weather dependent.

Approximate Magnitude of AIA

To estimate the magnitude of AIA, I will use Liebe and Hopponen (1977) formula for the index of refraction, n_0:

$$(n_0-1)=10^{-6}\cdot\frac{103.49\cdot P_{dry}}{T}+\frac{86.26\cdot P_w}{T}+\frac{4.958\,x\,10^5\cdot P_w}{T^2}\ . \qquad (1)$$

T is the air temperature in K, P_w is the partial pressure of water vapor (in mmHg), and P_{dry} is the total barometric pressure minus the partial pressure of water. A typical value is $n_0\text{-}1 \sim 3\cdot10^{-4}$.

At low elevations, one cannot use the plane-parallel approximation for the atmosphere to derive the total refraction. Instead, I use the model in Maddalena (1994) and refined in Maddalena et al (2002).

$$\Delta E=E_{obs}-E_{True}=C\cdot(n_o-1)\cdot g(E_{True})\ . \qquad (2)$$

$C \sim 2.35\cdot10^5$'' and g is equation 4 from Maddalena et al (2002). For typical weather conditions $\Delta E \sim$ 550" for E=5°and $\Delta E \sim$ 320" for E=10°, with 15% variations due to weather. To determine a rough, theoretical approximation for R, the difference in angle between rays hitting the bottom and top edges of the aperture, I will assume the impossible case for Green Bank of P_w=0. Substitute equation 1 into equation 2, and take the derivative to derive:

$$R=\Delta E\cdot(\Delta P_{dry}/P_{dry}-\Delta T/T)\ . \qquad (3)$$

ΔP_{dry} and ΔT are the difference in P and T between the bottom and top of the GBT when it is tilted to an observing elevation of E. As an example, for E=10°, the change in height from the lower to upper edge of the GBT aperture is 0.1 km·$cos(E)$ = 0.098 km. Using the ground-level pressure lapse rate (90 mmHg/km) and temperature lapse rates (6.5 K/km) for the 'standard' model of the atmosphere, $\Delta P_{dry} \approx$ 9 mmHg and $\Delta T \approx$ 0.65 K. I will use for P_{dry} and T the average ground-level values (660 mmHg and 285 K, respectively) for Green Bank. Thus, $R \sim 0.011\cdot\Delta E$. Since $\Delta E \sim$ 320 for E=10°, rays at the lower edge of the dish will arrive roughly R≈3.5" higher in elevation than those arriving at the vertex. Repeating this estimate for E=5° gives R≈6.1". If the GBT were a perfect paraboloid, at low elevations, point sources would be noticeably elongated in elevation when observing at high frequencies.

Variations Of AIA Due To Refraction Conditions

So far I have ignored water vapor and weather variations. The CLEO forecast application I developed and its associated web pages have provided information for *R* for about two decades (Maddalena, 2008 and 2022). The National Weather Service provides P_{dry}, *T*, and P_w in ~70 layers above the observatory, with about three layers spanning the 100 m height of the GBT. CLEO derives an estimate of the derivative of n_0-1 with height from the lowest layers using the high-accuracy equation of Froome and Essen (1969). The application then ray traces through a non-plane-parallel atmosphere using equations 19 and 20 from chapter 3 of Smart (1977). The CLEO application can generate tables of the forecasted value of *R* for any elevation over any range of dates from April 2004 (which is when I started the forecast archive) to a few days into the future.

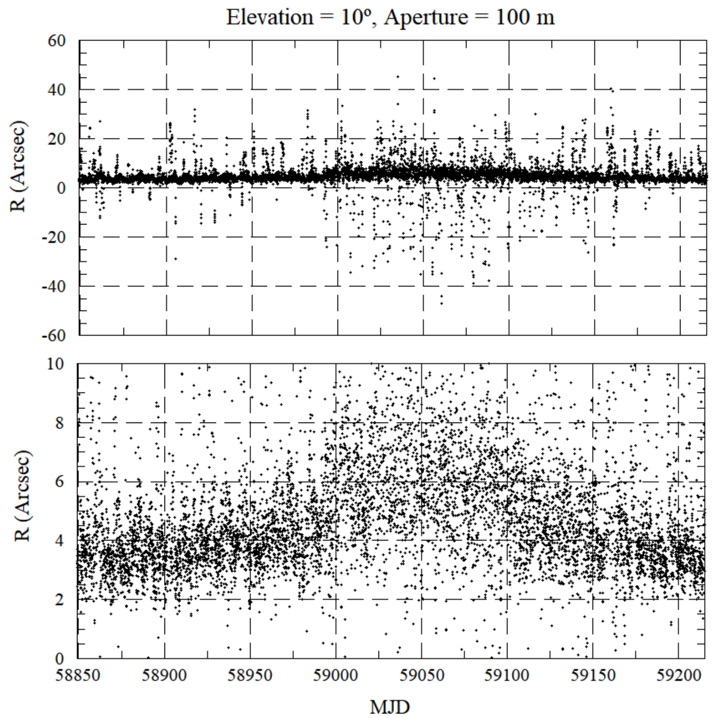


Figure 1: Values of R between the vertex and lowest panel of the GBT at an elevation of 10° for the year 2020. Lower panel is an expanded version of the upper.

The top panel of Figure 1 shows the results from CLEO for a 100 m aperture and for *E*=10° for the full year of 2020. The bottom panel is the same as the upper but with the y-axis expanded around 5". Figure 1 shows that there are times when weather conditions produce very large and even sometimes negative values for *R*. The extreme values of *R* are at times when the temperature and water density lapse rate near the ground is much steeper, much shallower, or even inverted from that of a standard atmosphere. (If we think of the GBT as a transmitter, *R*<0 indicate that the rays leaving the bottom of the dish will eventually cross the rays leaving the top of the dish.)

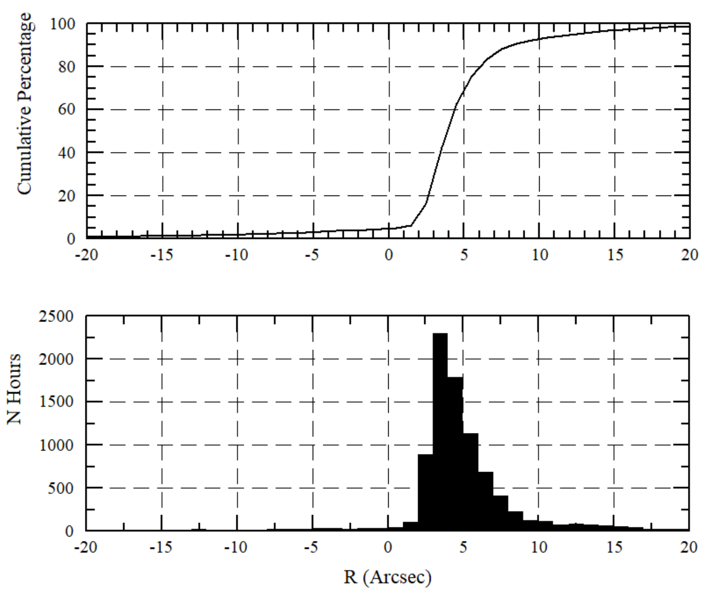


Figure 2: Statistics for R between the vertex and lowest panel of the GBT for E=10° for 2020.

Figure 2 shows the statistics for *R* for the year 2020 and *E*=10°. The dry-air estimate from the previous section, 3.5", lies just below the median value. Note that very few instances have *R* ≈ 0, which is the only times when the GBT should be a perfect paraboloid.

The scatter diagrams in Figure 3 (for 2020 and E=10°) illustrate that large variations in R occur mostly in weather conditions that are relevant for high-frequency observing. High-frequency observing is best performed when winds are low, when there is little probability of precipitation, under low cloud cover, and when the precipitable water is low. Figure 3 show that $|R|$ tend to be highest when there are low winds and low probability of precipitation. R does not seem to be related to the presence or absence of clouds. The extreme positive values for R is the same regardless of the value of precipitable water.

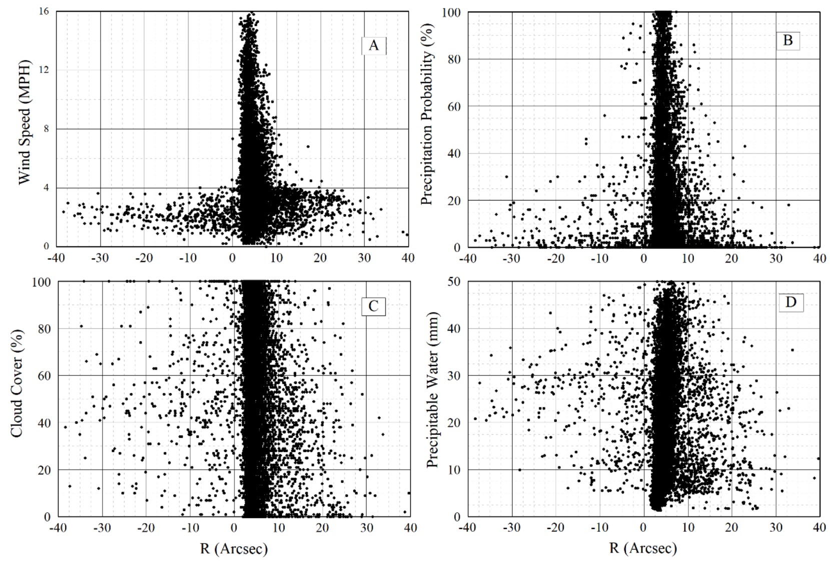
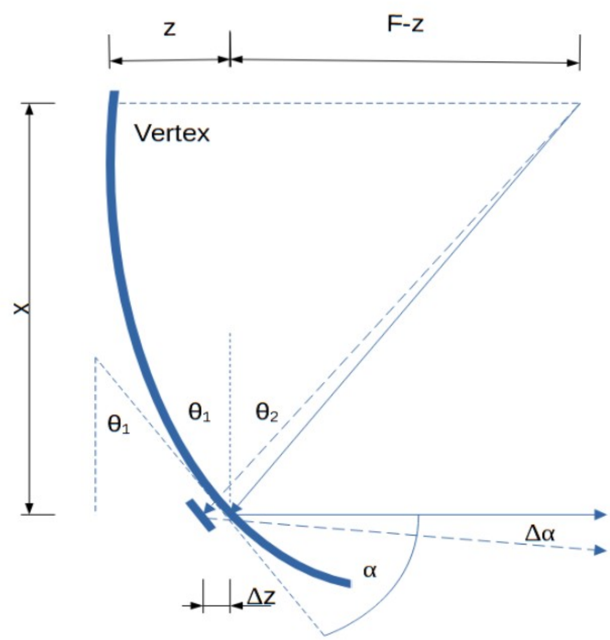


Figure 3: Relationship between R at E=10° for 2020 versus weather parameters that are typically used to determine whether high-frequency observing should be scheduled. The range of values for R is largest when conditions are suitable for high frequency observing.

Magnitude of the Surface Corrections Needed to Compensate for AIA

If a panel is adjusted in the z direction in the sketch to the right, the angle of incidence ($\alpha=\theta_1+\theta_2$) changes. Since the angle of reflection equals the angle of incidence, shifting a panel in the z direction changes the direction of the exiting ray. To correct for AIA, we would need to change the direction of the exiting ray by the appropriate angle for the observing elevation, weather conditions, and panel location on the dish. Thus, we need the relationship between a change in the setting in the z direction of any panel with a change in α. The equation for a parabola is:



$$z=\frac{x^2}{4F}$$

F is the telescope's focal length (60m). Since we want to alter only the elevation profile of the surface, leaving the cross-elevation profile a perfect paraboloid, *x is the distance from the vertex projected onto the axis of symmetry of the dish. It is not the linear distance of a panel from the vertex.* The slope of the above equation and trigonometry give the angles in the sketch:

$$\theta_1 = \arctan\left(\frac{x}{2F}\right) = \arctan\left(\sqrt{z/F}\right); \quad \theta_2 = \arctan\left(\frac{F-z}{x}\right) = \arctan\left(\frac{F-z}{2\sqrt{Fz}}\right)$$
$$\alpha = \theta_1 + \theta_2 = \arctan\left(\sqrt{z/F}\right) + \arctan\left(\frac{F-z}{2\sqrt{Fz}}\right)$$

Taking the derivative of α with respect to z gives, after some work:

$$\Delta z = -x\left(1 + \frac{x^2}{4F^2}\right)\Delta\alpha$$

The output of the CLEO forecast application, r, is in units of arcsec per meter of aperture, projected vertically. So, $\Delta\alpha = x \cdot R/D = x \cdot r \cdot \cos(E_{True})$ and in units of meters for x and F and mm for Δz:

$$\begin{aligned} \Delta z(mm) &= -x(m)^2\left(\frac{10^3}{206265}\right)\left(1 + \frac{x^2}{4F^2}\right) \cdot \cos(E) \cdot r(''/m) \\ \epsilon(mm) &= -x(m)^2\left(\frac{10^3}{206265}\right) \cdot \cos(E) \cdot r(''/m) \end{aligned} \qquad (4)$$

Here Δz is an offset in the surface in the axial direction while $\epsilon = \Delta z/(1 + x^2/(4F^2))$ is the offset in the aperture plane that, for example, one uses in the modeling of beam profiles (equation 12a, Ruze, 1966).

As an alternative to using forecasts, two high-accuracy weather stations (Maddalena 1994), one mounted at the vertex, the other at the lower edge of the dish, could determine accurately the difference in index of refraction between the vertex and lower edge. Then, $\Delta\alpha = x \cdot C \cdot g(E_{True}) \cdot (n_0 - n_v)/D$ and:

$$\begin{aligned} \Delta z(mm) &= -\frac{x^2}{D}\left(\frac{10^3}{206265}\right)\left(1 + \frac{x^2}{4F^2}\right) \cdot C \cdot g(E_{True}) \cdot (n_0 - n_v) \\ \epsilon(mm) &= -\frac{x^2}{D}\left(\frac{10^3}{206265}\right) \cdot C \cdot g(E_{True}) \cdot (n_0 - n_v) \end{aligned} \qquad (5)$$

With sufficient accuracy, one can use the constant C and function g from Maddalena et al (2002) and either equation 10 or 11 from Maddalena (1994) to derive from weather station data the indices of refraction at the lower edge (n_0) and vertex (n_v).

I can now estimate the range of surface adjustments for different observing elevations and weather conditions. The CLEO application provided for all of the year 2020 values of r at various elevations. From these I determined the median r values as well as values for 10%, 25%, 75%, and 90% percentile refraction conditions. I then converted these to ε for x=100 (i.e., the lower edge of the dish) using equation 4 to produce Table 1. For example, to remove the effects of AIA below 20° and under almost all weather conditions, the aperture plane will need to be adjusted from a paraboloid by over 1 mm. The numbers in parenthesis for E=15° are for instances when weather conditions were suitable for high-frequency observing (precipitable water < 15 mm, winds < 5 MPH, cloud cover and precipitation probability < 20%) and indicate that the required range of surface adjustments will be larger for high-frequency observations than for low frequencies. The last column, the hourly change in ε, might be useful for determining the rate for altering the surface to maintain a desired surface accuracy. One can also use the table to estimate the derivative $d\varepsilon/dE$ to determine if the surface needs to be adjusted during scans as sources rise or set or when making large on-the-fly maps.

Table 1: ε in mm for lower edge of the GBT. Multiply by $1.694 = 1 + 100^2 / (4 \cdot 60^2)$ *to derive Δz. The values in parenthesis are for conditions suitable for high-frequency observing.*

E(°)	Percentile Refraction Conditions					*dε/dt*
	10%	25%	Median	75%	90%	(mm/hr)
5	-2.30	-3.00	-3.84	-5.30	-7.79	0.20
10	-1.25	-1.63	-2.09	-2.88	-4.24	0.11
15	-0.83 (-0.83)	-1.14 (-1.13)	-1.39 (-1.78)	-2.31 (-3.08)	-3.39 (-6.30)	0.07
20	-0.60	-0.78	-1.00	-1.38	-2.02	0.05
30	-0.35	-0.45	-0.58	-0.80	-1.19	0.03
45	-0.17	-0.21	-0.28	-0.38	-0.55	0.01

The historical goal of metrology systems (e.g., photogrammetry and laser-ranging systems) is to set the surface to a perfect paraboloid at all elevation. Table 1 illustrates the weather- and elevation-dependent range of adjustments by which metrology systems need to set the aperture plane different from perfect to produce a surface plus atmosphere that gives the best performance. Even at the rigging angle, the GBT surface should deviate from a perfect paraboloid by ~0.2 mm.

The large values in Table 1 suggest that AIA is severely reducing the gain of the GBT for many weather conditions and at moderately low elevations. However, there are two reasons why this is not the case.

First, as described in Maddalena et al (2014), we used AutoOOFs taken over a wide range of elevations and weather conditions to derive the gravity terms to the surface model. Since the measurements included the effects of AIA, the resulting Zernike coefficient for terms with a shape similar to elevation-oriented astigmatism are compensating for AIA under 'typical' weather conditions. If enough weather conditions were included, then one should use Table 2, which gives the difference between the individual columns in Table 1 and the column for median values.

Table 2: Approximate ε in mm for the lower edge of the dish, assuming the current gravity Zernike model has altered the aperture for 'median' weather conditions. The values in parenthesis are for conditions suitable for high-frequency observing.

E(°)	Percentile Refraction Conditions				
	10%	25%	Median	75%	90%
5	1.54	0.84	--	-1.45	-3.94
10	0.84	0.46	--	-0.79	-2.15
15	0.56 (0.95)	0.25 (0.65)	--	-0.92 (-1.30)	-2.01 (-4.52)
20	0.41	0.22	--	-0.37	-1.02
30	0.24	0.13	--	-0.22	-0.60
45	0.11	0.06	--	-0.10	-0.28

Second, AIA introduces an effective large-scale shape to the aperture plane, which renders the Ruze equation unsuitable for determining the loss in gain. Instead, one must create theoretical beam profiles. The standard equations for modeling the normalized, inner beam profile, P, and bore-sight gain, G/G_0, when there are small errors in the aperture plane, are:

$$P(\Delta xE,\Delta E)=\left|\int_0^{2\pi}\int_0^1 E_a(\rho,\Theta)\, e^{2\pi\, i\,[\epsilon+\rho D(\Delta xE\cdot\cos\Theta+\Delta E\cdot\sin\Theta)]/\lambda}\,\rho\, d\rho\, d\Theta\right|^2 \div \left|\int_0^{2\pi}\int_0^1 E_a(\rho,\Theta)\,\rho\, d\rho\, d\Theta\right|^2$$

$$G/G_0=\left|\int_0^{2\pi}\int_0^1 E_a(\rho,\Theta)\, e^{2\pi\, i\,\epsilon/\lambda}\,\rho\, d\rho\, d\Theta\right|^2 \div \left|\int_0^{2\pi}\int_0^1 E_a(\rho,\Theta)\,\rho\, d\rho\, d\Theta\right|^2 \qquad (6)$$

λ is the observing wavelength, ΔxE and E are the sky coordinates in the cross elevation and elevation directions, ε is from either equations 4 or 5, E_a is the square root of the feed illumination pattern (zero beyond the edge of the aperture), and Θ is the azimuthal coordinate and ρ is radial coordinate (normalized to 1) across the aperture plane from its center. Since the results are weakly dependent on the feed illumination pattern, and the range of illumination for the current suite of receivers is limited, I assume the common value for Gaussian edge taper of -13 db. Figure 4 gives P, G/G_0, and the elongation of the beam in elevation for a range of values of $|\varepsilon_{LowerEdge}|/\lambda$ models.

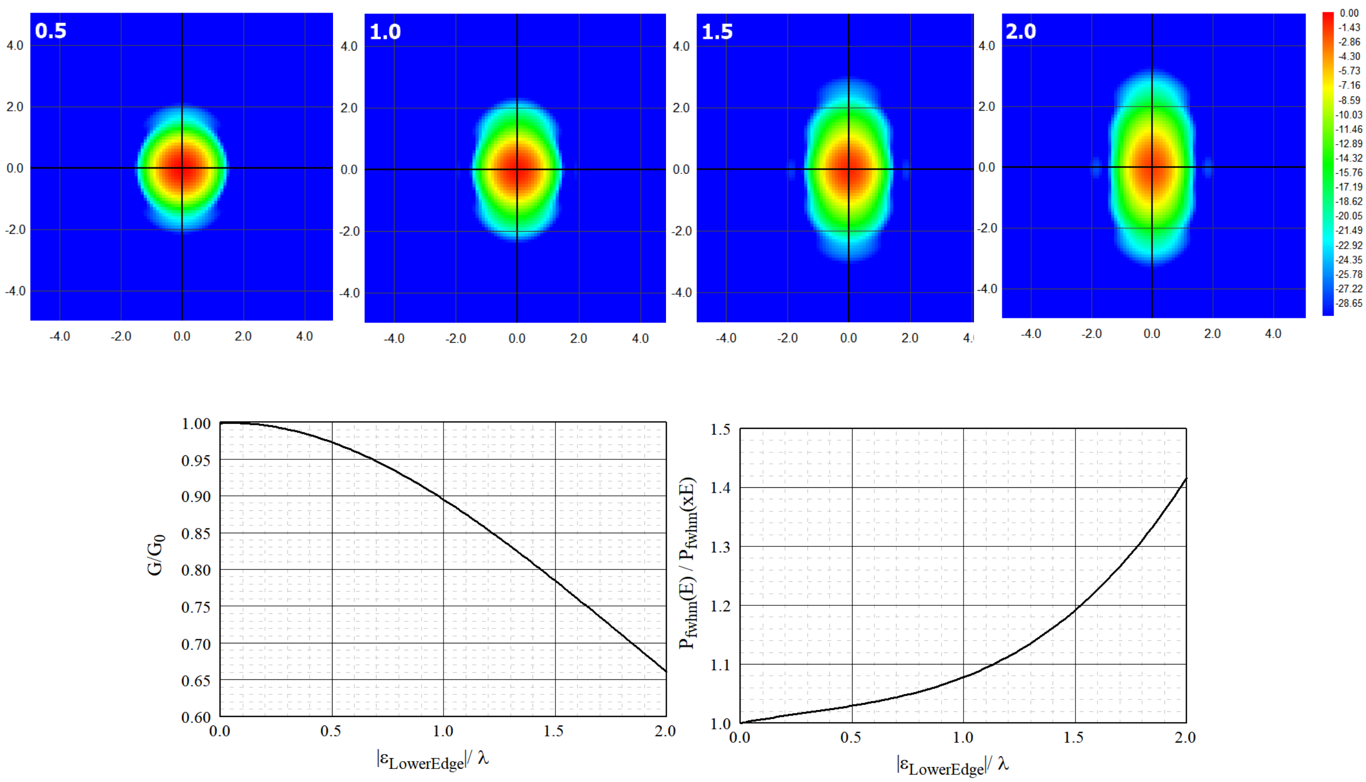


Figure 4: Top row are theoretical beam maps for $|\varepsilon_{LowerEdge}|/\lambda$=0.5, 1.0, 1.5, and 2.0. The axes are in units of λ/D radians and the range in the color scale is -30 to 0 dB, relative to the bore sight gain of a perfect aperture. The lower graphs show the loss in gain and broadening of the beam profile at the half-power points in the elevation direction. If a metrology system sets a surface to be a perfect paraboloid and does not take into consideration AIA, then the definition of $\varepsilon_{LowerEdge}$ is that of Table 1. Or, if the gravity Zernike model alters the surface for median refraction conditions, but not the weather-dependent changes in ε, then the definition of $\varepsilon_{LowerEdge}$ is that of Table 2.

Consider observations at λ=2.6 mm and E=15° (e.g., the Galactic Center), and with an AutoOOF source at a high-enough elevation that its results were not appreciably influenced by AIA. If the observations can tolerate no more than a 5% gain loss, then Figure 4 and equation 4 give maximum allowable values of $|\varepsilon_{LowerEdge}|/\lambda$<0.7, $|\varepsilon_{LowerEdge}|$< 1.8 mm, and $|R|$<3.7". Assuming the Zernike gravity model compensates for the median refraction conditions, the statistics that went into Table 2 for E=15º indicate that one would have an unacceptably high loss in gain ~35% of the times when conditions were otherwise suitable for high-frequency observing.

Discussion and Summary

The aim of metrology systems like photogrammetry are to set the surface to a perfect paraboloid at all elevations and all conditions. However, to remove AIA requires the elevation profile of the surface to become less parabolic as the telescope observes close to the horizon while maintaining the cross-elevation profile. As Table 1 shows, the deviation from a paraboloid is weather dependent and is many mm at low elevations.

We might consider adding a term to the gravity model that matches exactly the way in which the aperture plane needs to be adjusted. If we instead want to use Zernike terms, then a multi-regression least-squares fit for the 18 terms in the Zernike gravity model shows that equation 4 is fit perfectly when the coefficient for vertical astigmatism equals $0.05103 \cdot \varepsilon_{LowerEdge}$ and the coefficient for focus equals $-0.03608 \cdot \varepsilon_{LowerEdge}$.

We could increase the time and productivity of high-frequency, low-elevation observations if we were to develop a way to use the measured or forecasted value of $\varepsilon_{LowerEdge}$ to adjust the surface. Instead, we could use $\varepsilon_{LowerEdge}$ as a criteria for when to safely schedule such observations. However, the statistics of Figure 3 suggest that using $\varepsilon_{LowerEdge}$ as a scheduling criteria might reduce significantly the number of hours these kinds of projects could otherwise be scheduled.

There are a number of observable consequences of AIA:

- Beam profiles will appear to be elongated in the elevation direction, with the elongation increasing with decreasing elevation and increasing $|R|$.
- Some of the current changes in aperture efficiency with elevation may be due to AIA. Since the magnitude of AIA is also weather-dependent, low-elevation efficiencies will also be weather dependent beyond what opacity produces.
- Since the aberration produces a phase gradient across the aperture, at low-elevation the telescope's pointing might be weather dependent beyond how refraction changes pointing.
- If someone were to AutoOOF at a low elevation under weather conditions far from typical, AutoOOF would correct the surface for AIA for that elevation. However, point sources at high elevations would elongate and high-elevation aperture efficiencies would decrease. The opposite is also true – a high elevation AutoOOF, which will render the surface close to a perfect paraboloid as AIA is then small, should not be used for low-elevation observations.

There are a few reasons why observers may never have noticed the effects of AIA. In other telescopes, the dish diameter is typically not as large, the elevation limit not as low, and the observing frequency not as high as that of the GBT. Observer's seldom take data at very low elevations at high frequencies and, if they do, they may not have noticed that the E widths from AutoPeaks tend to be larger than the xE widths. Few observers measure aperture or beam efficiencies at low elevations, and, when they do, the loss of efficiency may be falsely attributed to an atmospheric opacity that is higher than expected.

More importantly, I believe that something close to the median value of Figures 1 and 2, and its elevation dependence (Table 1), are already incorporated into the GBT control system. Since the measurements that went into the current gravity model (see Maddalena et al, 2014) included the effects of AIA for different weather conditions, the resulting Zernike coefficients for terms with a shape similar to elevation-oriented astigmatism may be compensating for AIA under something close to median refraction conditions. However, the current gravity surface model does not take into consideration the significant weather-related changes in AIA, and Maddalena et al (2014) probably may not have included enough weather conditions that averaged out to median refraction conditions. I recommend continuing the practices of Maddalena et al (2014) when updating the Zernike gravity model to use the results of AutoOOFs taken under a wide range of conditions.

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Acknowledgment

Soon after the publication of Maddalena (1994), Richard Simon asked in the GBT email exploder whether the change in the index of refraction across the large diameter of the GBT aperture would be important. This led to a number of exchanges where we clarified the theoretical magnitude of the affect of AIA on low-elevation beam shapes. In 2001, Artis Maciolek demonstrated the availability on the Internet of easily-interpreted vertical weather forecast profiles, which soon after led me to develop the CLEO software used here to determine the magnitude and variability of the AIA phenomena as well as the data stream that the GBT dynamical scheduling system relies upon.